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Nonlinear pseudospin dynamics on a noncompact manifold

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We describe the motion of an SU(1,1) pseudospin vector in the frame of the mean-field approximation induced by the variational principle on linear-plus-quadratic Hamiltonians. The dynamics of the SU(1,1) states of the Perelomov type obeys a nonlinear Bloch or torquelike equation, and each orbit can be interpreted as the intersection of two quadrics, one representing the energy surface and the other the group manifold, both in the space of the averaged algebra generators or semiclassical pseudospin. The fixed points of the flow can be also determined by resorting to strictly geometric considerations. The evolution of the phase diagram in parameter space is investigated as well for selected Hamiltonians. The bifurcation sets are constructed and the nonthermodynamic phase transitions can be clearly identified for the systems under consideration.

I. INTRODUCTION

Nonlinear dynamical systems that, in some suitable approximation, attempt to describe the motion of N interacting particles, constitute an active research field in many-body physics. Applications to nuclear dynamics in the mean-field approximation derived through Dirac's variational principle[1] have received great attention,[2] with particular emphasis on those physical configurations or channels that can be modeled by systems with spectrum-generating algebras.[3] In this spirit, n-level models of the SU(n) type are well suited and due to its algebraic simplicity and integrability properties, the SU(2) dynamical group has been the focus of several recent investigations.[4] It has been shown[5] that the variational SU(2) dynamics exhibits a collection of geometrical characteristics which greatly simplify the investigation of the phase diagram, i.e., the location of fixed points, their nature, and the appearance of topologically invariant regions of phase space. In addition, the evolution of the phase diagram in parameter space[6] can be examined and the bifurcation sets can be determined as well as the kind of phase transitions undergone by the system.

SU(2) models can be linked to a variety of objects from two-level atoms immersed in a radiation field[7] to N nucleons with two-level spectrum which interact through a two-body force.[8] In the latter case, the Hamiltonian is quadratic in the SU(2) algebra generators. Now, the essence of the mean-field approximation is the reduction to a one-body scheme and one can see that the motion is formulated in terms of a nonlinear Euler equation,[5]

$$\dot{\mathbf{J}}=\mathbf{\Omega}(\mathbf{J})\times\mathbf{J} \tag{1.1}$$

where $\mathbf{J}$ is the expectation value of the SU(2) basis vector $\hat{\mathbf{J}}=(\hat{J}_x,\hat{J}_y,\hat{J}_z)$ with respect to an SU(2) coherent state[5,7] and $\mathbf{\Omega}(\mathbf{J})$ is the gradient of the averaged or mean-field Hamiltonian in $\mathbf{J}$ space.

Equations of the form (1.1) with a $\mathbf{J}$-independent vector $\mathbf{\Omega}$—which may, however, admit some time dependence—are known as Blochlike (or torque) equations.[9] It has been recently shown that they appear in dynamical problems with SU(2), SU(1,1), and SU(n) dynamical groups[10–12] together with Hamiltonians that belong to the algebra. Furthermore, several authors have remarked[13] that in every case, these are the coherence-preserving Hamiltonians, in other words, they generate motions which map group coherent states of the Perelomov type[14] onto states of the same class. It is interesting as well to remark that in all fermion realizations of the algebras, coherent states are Slater determinants, and consequently the group coset can be parametrically mapped onto the Grassmann manifold.[5–7]

The SU(2) variational problem with quadratic Hamiltonians expressible through (1.1) is coherence-preserving as well;[4,5] in fact, Eq. (1.1) describes a J^2-conserving motion on the SU(2)/U(1) coset or Bloch sphere which is in a one-to-one correspondence with SU(2) coherent states.[7] This property appears as a characteristic of the approximation associated to the variational procedure, since the exact motion does not necessarily lie on the sphere at all times. The reason for this conservation lies in the fact that the generator of the motion in the mean field can be expressed as a linear combination of the averaged algebra operators, however, with nonlinear coefficients, in other words, with coefficients that depend on the same averages. In the Fermion realization of the algebra and the flow, such are the nonlinear one-body Hamiltonians arising from the time-dependent Hartree-Fock[2,4–6] approximation.

On the other hand, we observe nowadays an increasing interest in systems with SU(1,1) dynamical groups,[15] in view of the wide set of applications that include harmonic motion,[15,16] description of superfluid elementary excitations,[17] Coulomb problem,[18] two-photon processes,[11] degenerate parametric amplification,[19,20] s states of the Morse oscillator,[21] path integral methods,[22] anharmonic motion,[20] squeezing of states in the electromagnetic field,[23] and damped oscillatory motion.[24] Those illustra-

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tions mostly involve linear SU(1,1) Hamiltonians—with the exception of Refs. 20 and 25 where a class of anharmonic motion has been investigated. However, a number of problems with quadratic Hamiltonians remain to be considered, among them, anharmonic oscillatory motion with quartic perturbation[20,25] in the mean field and superfluid dynamics in the presence of residual quasiparticle interactions.[26] The formulation of this problem in the framework of the mean-field approximation induced by the variational principle is the purpose of the present work.

In this vein, we will describe the variational dynamics of SU(1,1) coherent states on mostly geometric grounds.[5] In particular, the orbits appear as the intersections of quadrics that represent constant-energy surfaces, with one sheet of a two-sheeted hyperboloid—the curved phase space of SU(1,1) or Lobatchevsky plane[13,22]—in the space of averaged algebra generators. We will then see that the resulting equation is of the form (1.1). This is the subject of Sec. II. In Secs. III and IV we will investigate several special applications to linear and quadratic Hamiltonians, respectively, and construct the associated phase diagram. The evolution of the flow in parameter space is examined in Sec. V, where the bifurcation sets are constructed for each characteristic Hamiltonian. The summary and conclusions are presented in Sec. VI.

II. VARIATIONAL SU(1,1) DYNAMICS WITH GENERAL QUADRATIC HAMILTONIANS

In this section we are going to show that in view of the fact that bilinear functions of the SU(1,1) algebra generators factorize, quadratic operators such as the Hamiltonian or the SU(1,1) Casimir operator, when averaged with respect to group coherent states, define corresponding quadrics in the averaged algebra space. This property is enforced to geometrically characterize the orbits.

Let us consider the SU(1,1) vector $\hat{\mathbf{K}}=(\hat{K}_1,\hat{K}_2,\hat{K}_3)$ whose components commute as

$$[\hat{K}_1,\hat{K}_2]=-i\hat{K}_3\ , \tag{2.1a}$$

$$[\hat{K}_2,\hat{K}_3]=i\hat{K}_1\ , \tag{2.1b}$$

$$[\hat{K}_3,\hat{K}_1]=i\hat{K}_2\ , \tag{2.1c}$$

and the Casimir operator,

$$\hat{C}=\hat{K}_3^2-\hat{K}_1^2-\hat{K}_2^2\ . \tag{2.2}$$

Since the group is noncompact, its unitary representations are infinite dimensional;[27] we will restrict ourselves to the positive representation $\mathcal{D}_k^{\dagger}$, k being the Bargmann index[27] related to the Casimir eigenvalue as

$$C=k(k-1)\ . \tag{2.3}$$

Let $|k,n\rangle$ be the basis,

$$\hat{K}_3|k,n\rangle=(k+n)|k,n\rangle\ , \tag{2.4}$$

and let $|z\rangle$ be a Perelomov coherent state,[14]

$$|z\rangle=\exp(z\hat{K}_+-z^*\hat{K}_-)|k,0\rangle\ . \tag{2.5}$$

Now, some algebraic manipulations, which amount to using disentangling theorems,[7] essentially identical to those performed in the SU(2) case,[5] easily give the factorization property for the anticommutator $\{\hat{K}_i,\hat{K}_j\}$ of any two algebra generators,

$$\langle z|\tfrac{1}{2}\{\hat{K}_i,\hat{K}_j\}|z\rangle=\frac{2k+1}{2k}K_iK_j-\Delta_i\delta_{ij} \tag{2.6}$$

where $K_i=\langle z|\hat{K}_i|z\rangle$ and $\Delta_1=\Delta_2=-\Delta_3=-k/2$.

It is clear then that the locus of the expectation value of the Casimir operator, which in turn is the dynamical manifold, is a two-sheeted hyperboloid in a three-dimensional space (K_1,K_2,K_3), actually,

$$C=K_3^2-K_1^2-K_2^2=k^2\ . \tag{2.7}$$

However, since $K_3\geq k\geq 0$ [cf. Eq. (2.4)], we disregard the hyperboloid sheet with negative values of K_3. It is also evident that for general quadratic Hamiltonians of the form

$$\hat{H}=\epsilon_i\hat{K}_i+\tfrac{1}{2}\beta_{ij}\hat{K}_i\hat{K}_j\ , \tag{2.8}$$

with β_{ij} a symmetric matrix, the energy of a system with a coherent state as wave function is

$$E=\epsilon_iK_j+\tfrac{1}{2}\tilde{\beta}_{ij}K_iK_j-\tfrac{1}{2}\beta_{ii}\Delta_i\ , \tag{2.9}$$

being $\tilde{\beta}_{ij}=(2k+1)\beta_{ij}/2k$. Equation (2.9) is only meaningful on the hyperboloid (2.7) whose points are in a one-to-one correspondence with the states (2.5); however, we can interpret that the orbit labeled by a given energy value E is the intersection of the quadric (2.7) with positive K_3 and the quadric defined by (2.9). This statement simply generalizes the construction of the SU(2) orbits.[5]

We may easily see that the variational description of the dynamics leads to the above kind of orbits. Dirac's variational principle specialized on SU(1,1) coherent states as trial functions demands us to set Euler-Lagrange equations for the Lagrangian,[5]

$$\mathcal{L}=\left\langle z\left|i\frac{\hat{\partial}}{\partial t}-\hat{H}\right|z\right\rangle\ . \tag{2.10}$$

Due to the analytical analogies between SU(2) and SU(1,1) coherent states, all results previously obtained for the former dynamical group[5] can be straightforwardly extended with the replacement $\theta_{\mathrm{SU}(2)}\rightarrow i\theta_{\mathrm{SU}(1,1)}$. The SU(1,1) Euler-Lagrange equations are then

$$\dot{q}=\frac{\partial\mathcal{H}}{\partial p}\ , \tag{2.11a}$$

$$\dot{p}=-\frac{\partial\mathcal{H}}{\partial q}\ , \tag{2.11b}$$

with $p=k\cosh\theta$, $q=\phi$, being $z=\tanh(\theta/2)\exp(-i\phi)$ and $\mathcal{H}=\langle z|\hat{H}|z\rangle$. Here ϕ and θ are the usual spherical coordinates, which in turn give rise to an unbounded canonical SU(1,1) space (p,q) with $p\in[1,\infty)$ and $q\in[0,2\pi]$.

As in the SU(2) problem, the Hamiltonian equations (2.11) are the statement of Ehrenfest's theorem for coherent states.[28] It is also an algebraic matter to calculate the Euler-Lagrange equations in **K** space; one obtains the vector equation

$$\dot{\mathbf{K}} = -\tfrac{1}{2}\nabla\mathcal{H}\times\nabla C \ , \tag{2.12}$$

with C given by (2.7). We then realize that, as already stated, each orbit lies at the intersection between each energy surface $\mathcal{H}(\mathbf{K})=E$ and the group manifold (2.7). We may as well notice that the validity of Ehrenfest's theorem for coherent states implies that, if the exact dynamics is confined to the SU(1,1) hyperboloid, it necessarily coincides with the variational motion. Such is indeed the case of linear Hamiltonians, which have been shown to be coherence preserving[13] and to yield Bloch-type equations.[10–12]

The nonlinear flow defined by Eq. (2.12) possesses stationary points when either $\nabla\mathcal{H}$ is proportional to ∇C or when at least one gradient is zero. The first condition in turn implies that fixed points of the flow are those where the energy and group manifolds become tangent. This is a particular geometrical property among several other ones which greatly simplify the qualitative characterization of the flow.

III. PHASE FLOW FOR LINEAR HAMILTONIANS

In this section we describe in detail the phase flow for linear Hamiltonians, both on the group manifold and on the plane canonical phase space (p,q). Let us first consider

$$\hat{H} = \epsilon\hat{K}_3 \ . \tag{3.1}$$

This is the most widely investigated Hamiltonian; it corresponds to the one-dimensional harmonic oscillator;[15,16,22] to the isotropic three-dimensional one;[16] to the group version of the Bogoliubov transformation[20] for superfluids;[17] generally speaking, to any linear combination $\epsilon_i K_i$ which can be "tilted"[22] according to some rules that define a particular approximation. Each problem here listed gives, in turn, at least one realization of the SU(1,1) algebra.

For the Hamiltonian (3.1), the energy quadrics are planes perpendicular to the K_3 axis, thus every orbit is a rotation. The only fixed point is the hyperboloid vertex $(0,0,k)$ and the canonical phase flow is a set of horizontal lines above $p=1$. This is illustrated in Fig. 1. The velocity vector can be easily computed, since $\nabla\mathcal{H}=(0,0,\epsilon)$ and $\nabla C = 2(-K_1,-K_2,K_3)$; thus, according to Eq. (2.12),

$$\dot{\mathbf{K}} = (-\epsilon K_2, \epsilon K_1, 0) \tag{3.2}$$

or $\phi = \epsilon t + \phi_0$, which is just a rigid rotation around the K_3 axis.

A slight deformation of (3.1) is

$$\hat{H} = \epsilon\hat{K}_3 - \alpha\hat{K}_1, \quad \alpha>0, \ \epsilon>0 \ . \tag{3.3}$$

This Hamiltonian appears, for example, as an SU(1,1) algebra realization of a one-quasiparticle Hamiltonian; the Bogoliubov transformation[29,30] is equivalent to a "tilt" or generalized rotation of the form[17,22]

$$\hat{K}_3' = \hat{K}_3\cosh\eta + \hat{K}_1\sinh\eta \ , \tag{3.4a}$$

$$\hat{K}_1' = \hat{K}_3\sinh\eta + \hat{K}_1\cosh\eta \ , \tag{3.4b}$$

with η chosen in such a way that the coefficient of either $\hat{K}_1'$ or $\hat{K}_3'$ vanishes after replacement of both $\hat{K}_1$ and $\hat{K}_3$ in (3.3). It is easy to verify that in any case, the Blochlike equation of motion (2.12) is invariant under the "tilt" (3.4). This property holds in spite of the fact that a vector equation of the form (2.12) is not rotationally invariant in the usual sense.

If $\alpha<\epsilon$, one chooses

$$\tanh\eta = -\alpha/\epsilon \ . \tag{3.5}$$

The phase flow of the "untilted" Hamiltonian (3.3) in the original **K** space possesses an absolute energy minimum at the tangency point where $\nabla\mathcal{H}=\nabla C$, namely,

$$\mathbf{K}_0 = \left[\frac{\xi k}{(1-\xi^2)^{1/2}}, 0, \frac{k}{(1-\xi^2)^{1/2}} \right] , \tag{3.6}$$

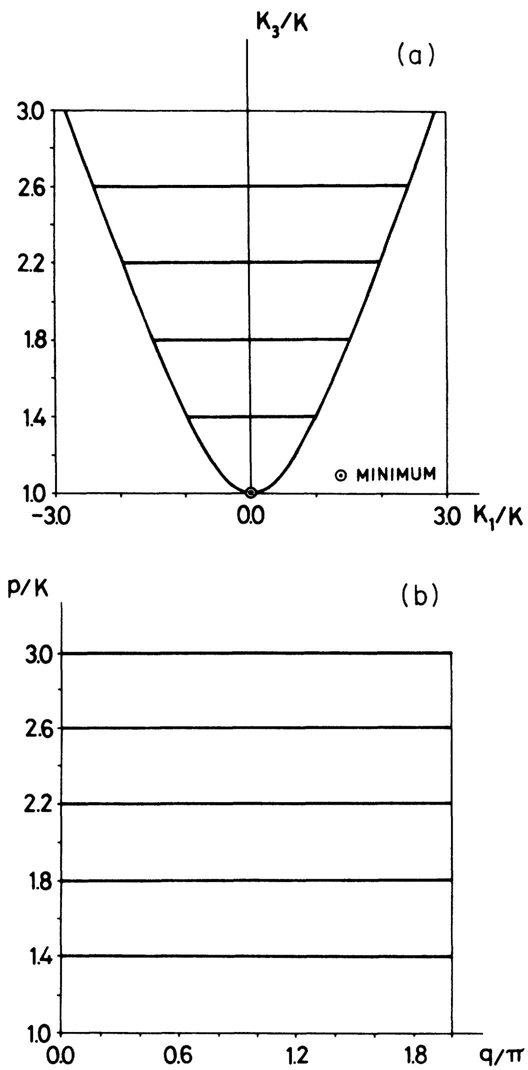

FIG. 1. (a) The phase flow of the linear Hamiltonian on the group manifold with $k=1$ and $\epsilon=1$; (b) the phase flow on canonical phase space.

with

$$E_0=\mathcal{H}(\mathbf{K}_0)=\epsilon k(1-\xi^2)^{1/2} \; . \tag{3.7}$$

This situation is illustrated in Fig. 2, where we can appreciate that all orbits are closed, however, falling into two categories, the local librations and the rotations, whose separatrix is the plane containing the hyperboloidal vertex. The geometrical meaning of the tilting operation (3.4) in the current frame becomes then clear; it just carries the point $\mathbf{K}_0$ onto the vertex of the hyperboloid in $\mathbf{K}'$ space, and eliminates the libration zones in phase space. From the physical viewpoint, in this case one lies in the region of parameters of the Bogoliubov-Valatin transformation where the energy spectrum is that of a compact operator.[17]

In the strong-coupling limit when $\alpha/\epsilon \geq 1$, we can appreciate that all orbits are open ones, or equivalently, that the canonical momentum is unbounded, and that no energy extremum appears. In the extreme strong-coupling regime $\alpha \to \infty$, which can be mimicked setting $\epsilon=0$, the energy planes become vertical ones. In this case there exists an orbit $\mathbf{K}(t)$ which can be analytically integrated, corresponding to zero energy or to the energy plane through the vertex, with velocity,

$$\dot{\mathbf{K}}=-\alpha(0,K_3,K_2) \tag{3.8}$$

from where one may compute the transit time between $K_2(0)$ and $K_2(t)$ which is

$$t=\frac{1}{\alpha}\left[\sinh^{-1}\frac{K_2(0)}{k}-\sinh^{-1}\frac{K_2(t)}{k}\right] . \tag{3.9}$$

The corresponding phase diagram is shown in Fig. 3. Now, from the geometrical viewpoint this situation corresponds to a "tilt" defined by $1/\tanh\eta=-\alpha/\epsilon$ [cf. (3.5)], leading to a Bogoliubov-Valatin Hamiltonian proportional to $\hat{K}'_1$, an unbounded operator. The phase diagram thus obtained illustrates a more general property of the flows on noncompact manifolds, namely, the fact that there is not a conserved characteristic.[5,6,31] Let us recall

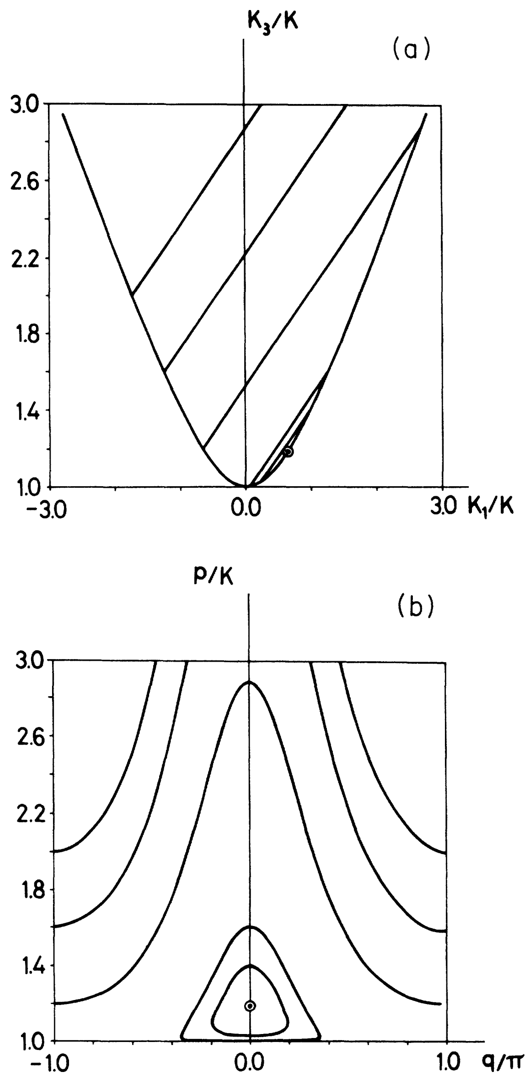

FIG. 2. Same as Fig. 1 for the perturbed linear Hamiltonian given in Eq. (3.3) with $\alpha=\frac{1}{2}$.

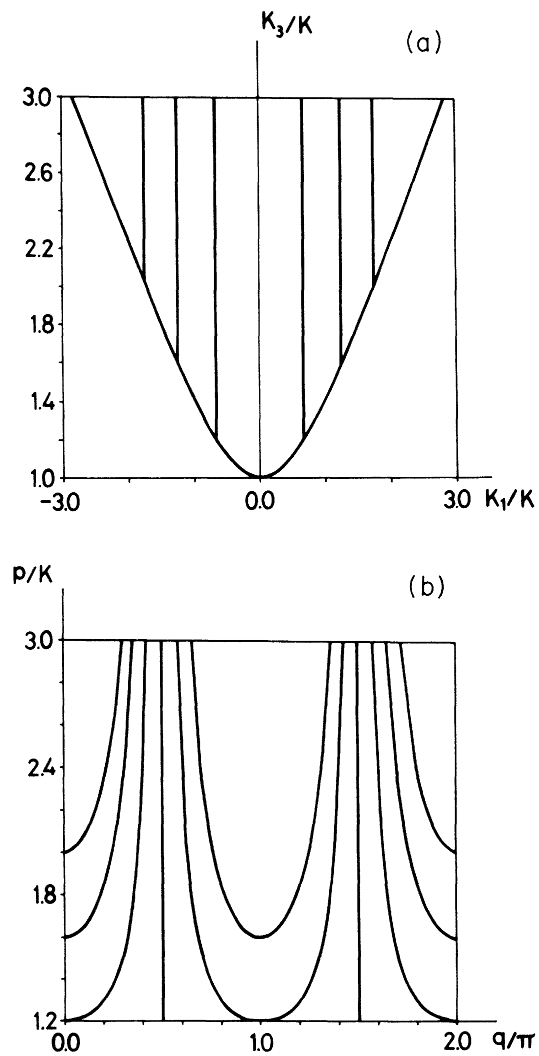

FIG. 3. Same as Fig. 2 for $\epsilon=0$ and $\alpha=1$.

that in a compact manifold, the characteristic is the sum of the indices of the singular points;[31] its conservation fixes then the number and kind of bifurcations that may occur.[5,6] In our case and in forthcoming examples, critical points can be created and destroyed as one evolves in parameter space, with a source or a sink at infinity.

IV. NONLINEAR FLOW FOR QUADRATIC HAMILTONIANS

The general expression for the quadratic Hamiltonians is given in (2.8). Physical realizations of such a Hamiltonian are, for example, listed below.

(a) *Quartic anharmonic oscillator.* We have[25]

$$\hat{H}=\tfrac{1}{2}(\hat{q}^2+\hat{p}^2)+\lambda\hat{q}^4 \; . \tag{4.1}$$

Actually there exist several realizations of the SU(1,1) algebra in terms of the canonical coordinates $\hat{q}$ and $\hat{p}$, the simplest one for the present example being

$$\hat{K}_1=i\frac{\hat{q}^2+\hat{p}^2}{4},\quad \hat{K}_2=\frac{\hat{q}^2-\hat{p}^2}{4},\quad \hat{K}_3=\frac{i\{\hat{q},\hat{p}\}}{4} \; . \tag{4.2}$$

One then finds

$$\hat{H}=-2i\hat{K}_1-4\lambda(\hat{K}_1+i\hat{K}_2)^2 \; . \tag{4.3}$$

(b) *Superfluid with residual quasiparticle interactions.* With the standard realization of the SU(1,1) algebra for the three-level superfluid,[17]

$$\hat{K}_1=-\tfrac{1}{2}(\hat{a}_+^\dagger\hat{a}_-^\dagger+\hat{a}_+\hat{a}_-) \; , \tag{4.4a}$$

$$\hat{K}_2=\tfrac{1}{2}i(\hat{a}_+^\dagger\hat{a}_-^\dagger-\hat{a}_+\hat{a}_-) \; , \tag{4.4b}$$

$$\hat{K}_3=\hat{a}_+^\dagger\hat{a}_++\hat{a}_-^\dagger\hat{a}_-+1 \; , \tag{4.4c}$$

where $\hat{a}_\pm^\dagger$ and $\hat{a}_\pm$ are the particle creation and annihilation operators, respectively, in single-particle states $|\pm\rangle$, one finds, for the two-body Hamiltonian in the Bogoliubov representation, an expression of the form (2.8) with coefficients β_{ij} that are complicated functions of the original interaction strengths and the tilting angle η [cf. Eqs. (3.4)].[29]

In what follows, we will consider two particular selections of the matrix elements β_{ij} that apply to the superfluid problem and permit a geometrical analysis of the flow.

A. Superfluid with quasiparticle interactions

Let

$$\hat{H}=(\hat{K}_3-a)^2+\hat{K}_2^2 \; . \tag{4.5}$$

It can be verified that it corresponds to a superfluid with two-quasiparticle interactions together with four quasiparticle creation and annihilation events. The energy quadric $\mathcal{H}(\mathbf{K})=E$ are cylinders with their axis through $(0,0,a)$ and parallel to the K_1 axis. Let us consider the following cases.

1. $0\leq a<k$

In this case no cylinder axis intersects the group manifold; the only fixed point is the energy minimum $E_m=(k-a)^2$ at the vertex $\mathbf{K}_m=(0,0,k)$ as illustrated in Fig. 4. It is clear that every orbit is a positive rotation around the K_3 axis and also that all orbits are nondegenerate.

2. $k<a<2k$

When $a=k$, the cylinder axis is tangent to the group manifold at the vertex and when $a>k$, the axis intersects the hyperboloid twice at degenerate relative minima $\mathbf{K}_{m\pm}=(\pm(a^2-k^2)^{1/2},0,a)$ with zero energy. In this situation, the vertex becomes a saddle point with energy $E_s=(k-a)^2$. This is illustrated in Fig. 5, where we may see that in this range of values of a three topologically invariant regions, namely, two degenerate librational zones around the minima and nondegenerate rotations with energies above E_s.

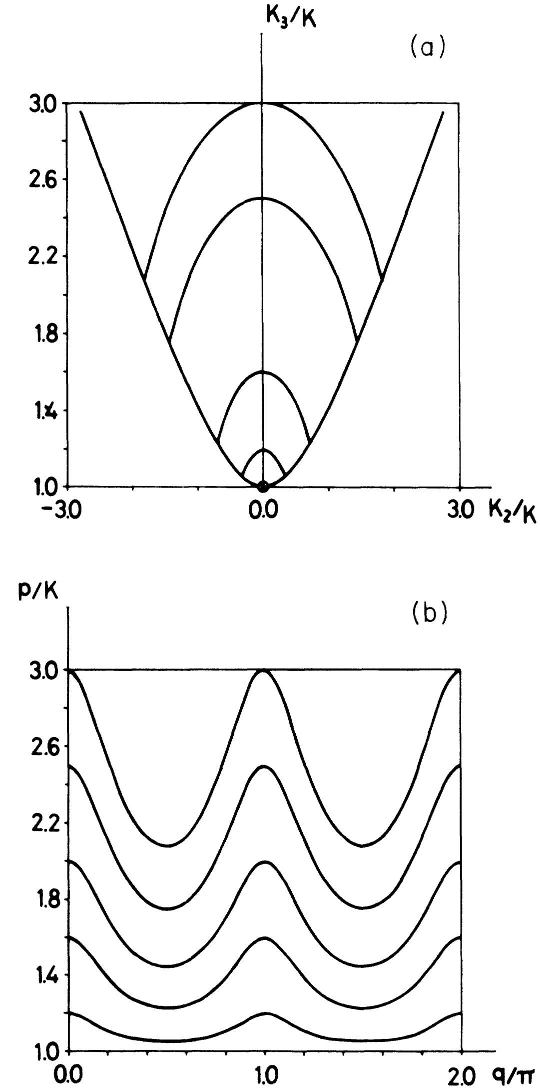

FIG. 4. (a) The phase flow of the quadratic Hamiltonian given in Eq. (3.14) for $k=1$ and $a=\frac{3}{4}$; (b) the phase flow on canonical phase space.

3. $a > 2k$

The situation is depicted in Fig. 6, where we can appreciate that the saddle point at the vertex has bifurcated into two degenerate saddle points and a relative maximum. The flow presents four invariant regions, namely, the following.

1. Rotations around the relative maximum $\mathbf{K}_M=(0,0,k)$ of energy $E_M=(a-k)^2$. They are degenerate with type-3 orbits (see below), since the cylinder intersects the group manifold twice; this is illustrated by the dashed line in Fig. 6, which represents two different orbits that arise from the same energy surface.

2. Degenerate librations around the minima, their projections on the (K_2,K_3) plane being circumferences.

3. Rotations around the K_3 axis up to energy $E_M=(a-k)^2$. They correspond to the upper intersections of cylinders whose lower intersections are rotations around $\mathbf{K}_M$; consequently, they are degenerate with type-1 rotations.

4. Nondegenerate rotations with energies higher than E_M, since in such a case the cylinders intersect the hyperboloid once.

Notice that the Hamiltonian (4.5) is invariant with respect to rotations around K_3 by an angle π. This implies, on the one hand, that the vertex is always a fixed point, and on the other hand, that critical points other than the vertex always appear in degenerate pairs. It is then interesting to examine a problem where this symmetry is broken, as in the following example.

B. Superfluid with quasiparticle coalescence

Let us now consider the Hamiltonian

$$\hat{H}=\epsilon\hat{K}_3+U\{\hat{K}_3,\hat{K}_1\} \ . \tag{4.6}$$

Physically, it corresponds to a superfluid with quasiparticle coalescence or decay processes.[26] The associated en-

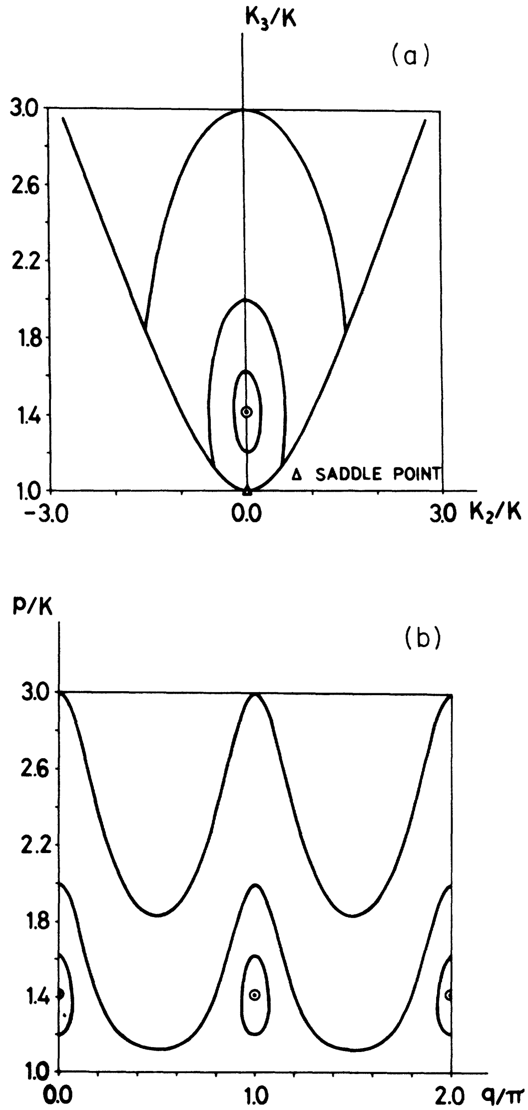

FIG. 5. Same as Fig. 4 for $a=2^{1/2}$.

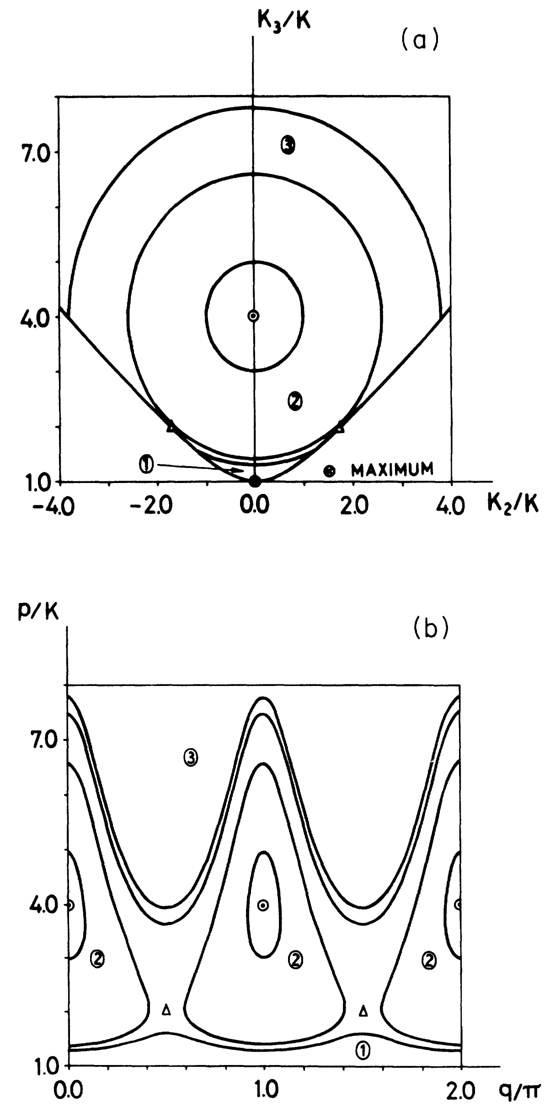

FIG. 6. Same as Figs. 4 and 5 for $a=4$.

ergy quadric is

$$\mathcal{H}(\mathbf{K})=\epsilon\left[K_3+\frac{\chi}{k}K_1K_2\right] \tag{4.7}$$

with $\chi=U(2k+1)/\epsilon$. An equivalent SU(2) Hamiltonian has been proposed and investigated in Ref. 5. As in the SU(2) case, the energy surfaces are hyperbolic cylinders with axis through $K_1=-k/\chi$; if $\chi=0$, the axis lies at infinity and the quadrics become planes as in Sec. III A. As the interaction strength increases, the minimum at the group manifold vertex evolves towards negative K_3 values, while a saddle point appears. The location of either fixed point can be found setting $\nabla\mathcal{H}$ parallel to ∇C (cf. Sec. II); we find

$$-\frac{K_1}{K_3}=\frac{K_3}{K_1+k/\chi} \text{ and } K_2=0 , \tag{4.8}$$

which in turn yields two critical points, namely,

$$\mathbf{K}_m=\left[-\frac{k}{4\chi}+\frac{k}{4\chi}(1-8\chi^2)^{1/2},0,\frac{k}{4\chi}\{2[1+4\chi^2-(1-8\chi^2)^{1/2}]\}^{1/2}\right] , \tag{4.9a}$$

$$\mathbf{K}_s=\left[-\frac{k}{4\chi}-\frac{k}{4\chi}(1-8\chi^2)^{1/2},0,\frac{k}{4\chi}\{2[1+4\chi^2+(1-8\chi^2)^{1/2}]\}^{1/2}\right] , \tag{4.9b}$$

with corresponding energies E_m and E_s.

The character of each critical point can be assigned by strictly geometrical considerations as illustrated in Fig. 7. We see in Fig. 7(a) that when $\mathbf{K}_m$ is the tangency point, the energy surface does not intersect the group manifold and thus leaves it entirely in the region of energies higher than E_m; consequently, $\mathbf{K}_m$ is a minimum. Instead, when the contact occurs at $\mathbf{K}_s$, a manifold intersection occurs that separates a libration area from three regions with open orbits. This geometrically means that $\mathbf{K}_s$ is a hyperbolic point. This is clearly illustrated in Fig. 7(b). For energies between E_m and E_s, each libration possesses a degenerate open partner in the rotation zone.

We may then realize that as the interaction parameter χ increases from zero, a bifurcation occurs at $|\chi|=1/2\sqrt{2}$ where both critical points coalesce, disappearing for higher values of χ. Notice that this situation is substantially equivalent to the SU(2) case,[5] where a saddle point bifurcation occurs for $|\chi|\geq 1$. However, the noncompact manifold does not allow energy maxima for Hamiltonian (3.16). Once again, as in Sec. III, we verify that the sum of indices of singular points of the hyperboloid[31] is not conserved.

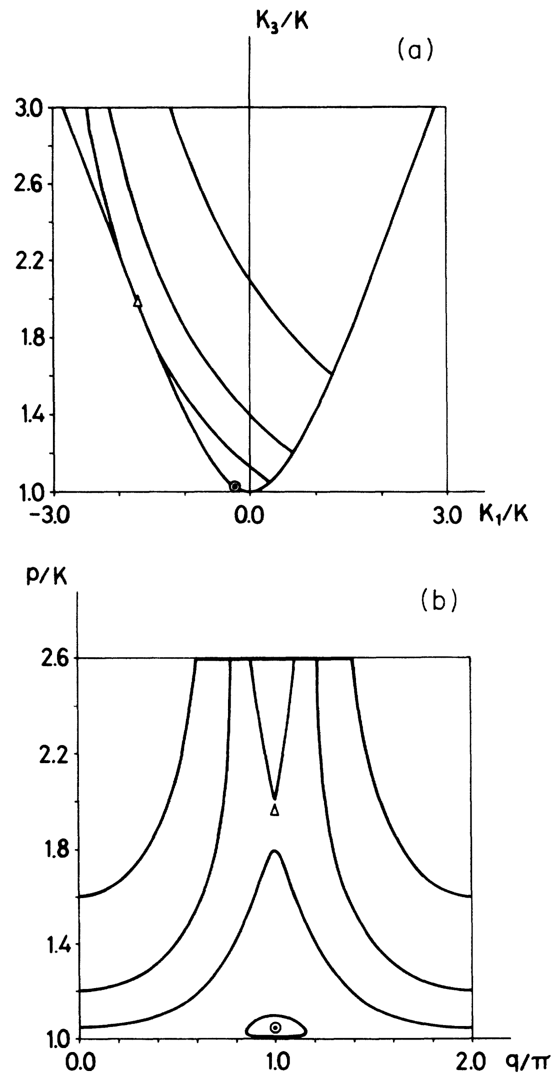

FIG. 7. (a) The phase flow of the Hamiltonian given in Eq. (3.15) for $k=1$, $\chi=\frac{1}{4}$.

V. BIFURCATION SETS AND QUALITATIVE DYNAMICS

In this section we will investigate the shapes of the bifurcation sets in parameter space for the general quadratic Hamiltonian (2.8), together with the topologically equivalent flows. We restrict ourselves to selected Hamiltonians that contain the particular ones presented in Sec. IV, since their parameter spaces are two dimensional and permit a straightforward analysis.

A. Superfluid with quasiparticle interactions

Let us consider the mean-field Hamiltonian,

$$\mathcal{H}(\mathbf{K})=(K_3-a)^2+\alpha K_1^2+K_2^2 , \tag{5.1}$$

which corresponds to the one investigated in Sec. IV A when $\alpha=0$. While the parameter a just sets the center of the energy surfaces, the interaction strength α determines the nature of the quadric as follows: (i) $\alpha<0$ gives one-sheet hyperboloids, cones, and two-sheeted hyperboloids according to the energy E being positive, zero, or negative; (ii) $\alpha=0$ gives cylinders (Sec. IV A); (iii) $\alpha>0$ gives ellipsoids, or spheres if $\alpha=1$.

The critical points of the phase flow are those which verify either $\nabla\mathcal{H}$ parallel to ∇C or $\nabla\mathcal{H}=\mathbf{0}$. For nonvanishing α, the parallelism condition gives five critical points, namely,

$$\mathbf{K}_{\pm}=\left[\pm\left[\frac{a^2}{(1+\alpha)^2}-k^2\right]^{1/2},0,\frac{a}{1+\alpha}\right], \tag{5.2}$$

if $a/(1+\alpha)\geq k, \alpha\neq-1$, with energies

$$E_{\pm}=\frac{\alpha a^2}{1+\alpha}-\alpha k^2 , \tag{5.3}$$

and

$$\mathbf{K}'_{\pm}=(0,(a^2/4-k^2)^{1/2},a/2) , \tag{5.4}$$

if $a\geq 2k$, with

$$E'_{\pm}=\frac{a^2}{2}-k^2 . \tag{5.5}$$

For $K_1=K_2=0$, we get

$$\mathbf{K}_0=(0,0,k) , \tag{5.6}$$

with energy

$$E_0=(k-a)^2 . \tag{5.7}$$

If $\alpha=-1$ and $a=0$, the intersection between $\mathcal{H}=K_3^2+K_2^2-K_1^2$ and $C=K_3^2-K_2^2-K_1^2$ is a cone; consequently, no isolated critical point appears. This intersection is the locus of zero velocity points and corresponds to $K_2=0$, $K_3^2-K_1^2=k^2$, which yield the minimum energy. We see in Eq. (5.2) that for nonvanishing a, as α approaches -1 the fixed points $\mathbf{K}_{\pm}$ depart towards infinity, leaving us only three critical points, namely, $\mathbf{K}'_{\pm}$ and $\mathbf{K}_0$ in Eqs. (5.4) and (5.6), respectively.

From these considerations one can draw the bifurcation diagram in (α,a) space that appears in Fig. 8. The bifurcation sets are defined by the existence conditions of the fixed points (5.2) and (5.4) and the regions where topologically equivalent flows belong are labeled from I to VIII as indicated in the picture. Let us now analyze in detail the type of bifurcations that show up when going from one region into the other; they are graphically illustrated in Fig. 9 and correspond to the following points.

(a) From region I into region II. An absolute minimum at $K_3=k$ bifurcates into two relative minima and a saddle point that remains at the vertex.

(b) From region II into region III. The saddle point at $K_3=k$ bifurcates into a relative maximum and two saddle points. Notice that in region III the K_2 axis of the energy ellipsoid is larger that the K_1 axis; the vertical dashed-dotted line indicates the location of the spherical energy surface. It is then clear that the flow in region IV (V) is identical to that in region II (III) interchanging the roles of the K_2 and K_1 axis.

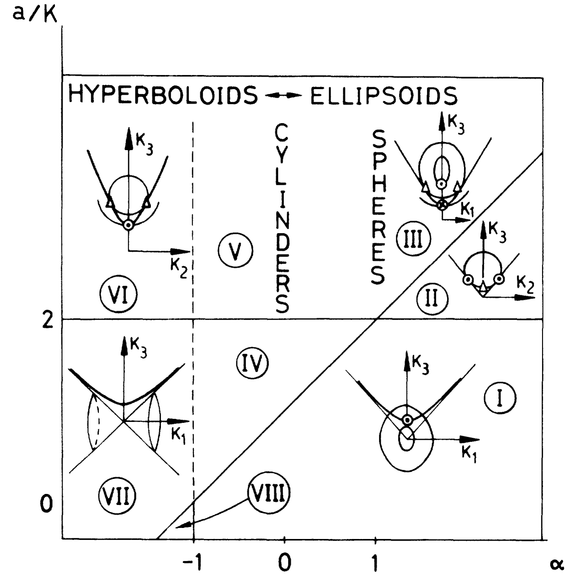

FIG. 8. The bifurcation diagram in (α,a) space for the mean-field Hamiltonian (5.1).

(c) From region I into region IV: same as (a).

(d) From region IV into region V: same as (b).

Before examining the remaining parts of Fig. 9 we need some more considerations. It should be noticed that for $\alpha=-1$ and $a<0$ there exists only one minimum at the vertex, due to the fact that the curvature of the quadrics in the (K_1,K_2) plane is higher for the energy surface than for the group manifold. When a vanishes, both quadrics intersect at a curve and for positive a the vertex becomes a saddle point, as a consequence of the inverted relation-

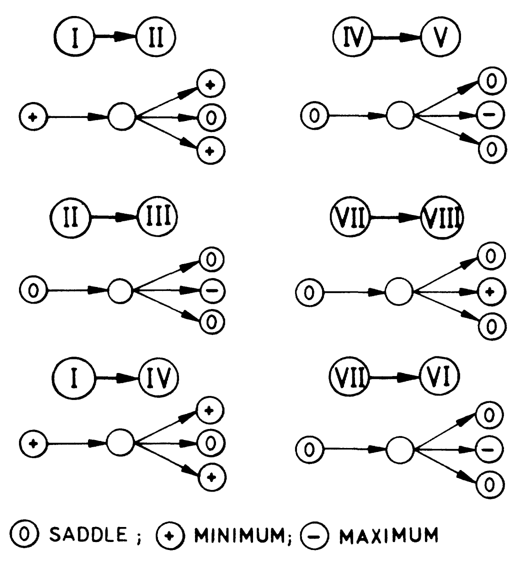

FIG. 9. The bifurcations that occur when crossing a bifurcation set in Fig. 8. See text for details.

ship between the curvatures. When a reaches the value $2k$, this saddle bifurcates as in (b). In relation to this analysis, it is especially interesting to remark that in the vicinity of $\alpha=-1$ for every value of a, as well as in the neighborhood of $a=0$ when $\alpha=-1$, it is not possible to enclose the bifurcating fixed points within a closed curve, and, consequently, the local conservation of the index[31] cannot be ensured. This arises from the fact that the Grassmann manifold is noncompact.

Therefore the bifurcations on the left of the line $\alpha=-1$ are (cf. Fig. 9) (e) from region VII into region VIII: a saddle point bifurcates into a relative minimum and two saddle points.

From region VII into VI, the same bifurcation occurs as in (b).

Finally, let us comment that the line $\alpha=-1$ separates two half planes with well-defined and conserved sum of indices of singular points[31] that correspond to the index of one minimum if $\alpha>-1$ and to one saddle for $\alpha<-1$. As discussed above, the index of a curve cannot be defined on the line itself since the definition of the critical points (5.2) there contains a divergence.

B. Superfluid with quasiparticle coalescence

A mean-field Hamiltonian that generalizes the one presented in Sec. IV B is

$$\mathcal{H}(\mathbf{K})=(K_3-b)(K_1-a)\ . \tag{5.8}$$

The energy surfaces are hyperbolic cylinders with axis parallel to K_2 in the (K_1,K_3) plane. One can easily realize that the sum of indices of singular points[31] vanishes on the complete parameter space; the bifurcation diagram is presented in Fig. 10.

In this figure we appreciate that the parameter space displays three qualitatively different regions. In region I,

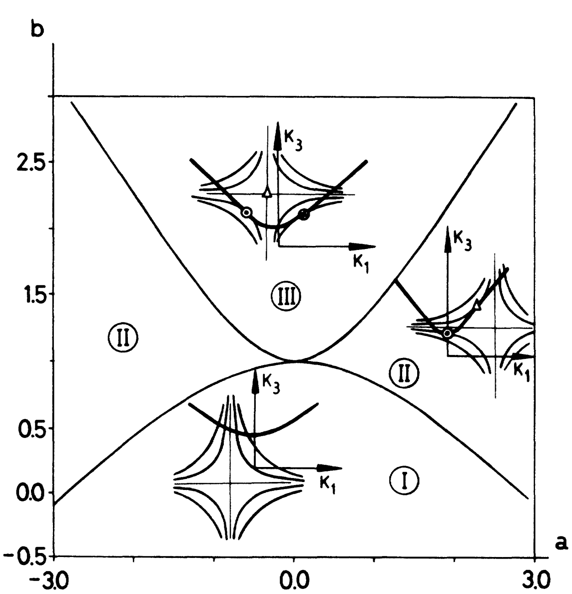

FIG. 10. Same as Fig. 8 for the mean-field Hamiltonian given in Eq. (5.8).

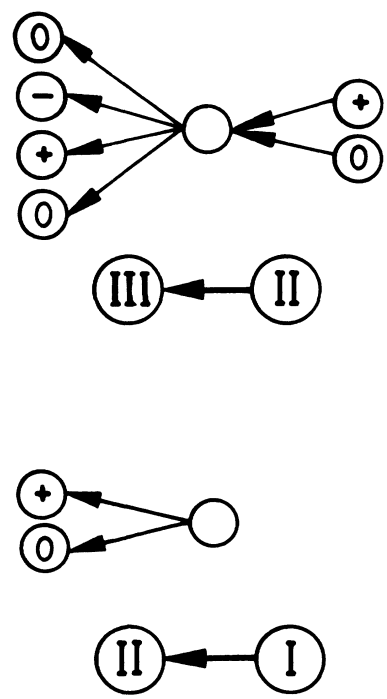

FIG. 11. Same as Fig. 9 for the bifurcation sets in Fig. 10.

no fixed point exists, either isolated or belonging to a degenerate curve. Region II exhibits one relative minimum and one saddle point as in Fig. 7 (see the discussion in Sec. IV B). In region III, the axis of the hyperbolic cylinders intersects the group manifold at two degenerate saddle points, while librations appear near one relative maximum and one relative minimum. These bifurcations are schematically indicated in Fig. 11.

VI. SUMMARY AND CONCLUSIONS

Prior to the present work, we had developed a geometrical method[5] to characterize the topology of the nonlinear variational flow on the SU(2) group coset or Grassmann manifold provoked by a linear-plus-quadratic Hamiltonian in the algebra generators. The equation of motion for the pseudospin vector is a Bloch-like equation with a nonlinear frequency. In addition, we had shown that such systems admit a straightforward way of determining their bifurcation sets in parameter space[6] together with the description of the associated phase transitions.

This work has dealt with an extension of the above techniques to linear-plus-quadratic SU(1,1) Hamiltonians, aimed at representing either quartic anharmonic motion[20,25] or two-quasiparticle interactions in boson superfluids.[26,29] We have then shown that as in the SU(2) case, a nonlinear torque equation describes the evolution of the averaged algebra vector on the noncompact group manifold; the motion can be viewed as a generalized rotation or "tilt" on a Lobatchevsky space. We have analyzed in detail the topological features of the flow for two particular choices of the Hamiltonian. The common property of both algebras resides in the geometrical identification of the fixed points of the corresponding flows; the most significant difference, however, is related to the noncompactness of the SU(1,1) group manifold and consists of the impossibility of establishing indexes of curves surrounding singular points[31] when one of these departs towards infinity.

The latter phenomena, namely the changes that the global phase flow undergoes when a fixed point bifurcates, can be generally classified as nonthermodynamic phase transitions or catastrophes.[32] We have constructed the bifurcation sets in a two-dimensional parameter space for two Hamiltonians that generalize the previously investigated generators of the motion and shown that the transitions undergone by the phase portrait can be rather easily identified on a pure geometrical basis.

The results here presented encourage further research concerning more general nonlinear flows. Fermion systems with SU(n) dynamical groups and quadratic Hamiltonians are especially interesting in view of their immediate applications to nuclear physics.[3] Work along this line is in progress and will be presented elsewhere.

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