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Canonical quantization of constrained systems

A. Bouzas, L. N. Epele, H. Fanchiotti, and C. A. Garcia Canal
Laboratorio de Fisica Teorica, Departamento de Fisica, Universidad Nacional de La Plata, Casilla de Correo No. 67, 1900 La Plata, Argentina
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The consideration of first-class constraints together with gauge conditions as a set of second-class constraints in a given system is shown to be incorrect when carrying out its canonical quantization.

I. INTRODUCTION

The quantization of constrained systems, and in particular gauge-invariant systems, has been analyzed from several points of view. In the canonical formalism developed by Dirac,[1] the so-called first-class constraints are imposed, upon quantizing, as restrictions on the state space. On the contrary, in order to deal with second-class constraints, canonical commutators are modified so that the corresponding constraint equations may be considered as operator equations. Notice that no reference to gauge conditions is made.

Systems with first-class constraints are gauge invariant.[1,2] To quantize them, it is possible to eliminate the unphysical variables, as is usually done, by fixing the gauge through the implementation of subsidiary conditions. In one of these gauges the operator ordering in the Hamiltonian is Cartesian, thereby permitting one to obtain the corresponding ordering in any other gauge by means of a gauge transformation.[3]

Alternatively, it has recently been proposed[4,5] to consider first-class constraints along with gauge conditions as a set of second-class constraints. This is justified by the fact that classically, this procedure leads to the correct equations of motion. However, it does not appear to be the appropriate way for canonical quantization, as is shown in what follows. Even in the case of a simple mechanical model the proposal presents ambiguities and may yield incorrect results.

In Sec. II a simple model allowing detailed analysis is presented and its quantization as a second-class system is performed. In Sec. III it is explicitly shown that this method of quantizing is not the correct one. In this section we also present the way to recover the proper result. In Sec. IV a different Lagrangian approach is analyzed in which nonindependent coordinates appear. Here, operators satisfy gauge conditions as strong operator equations but still gauge invariance is essential to determine the correct Hamiltonian. The final results coincide with those of Sec. III. Finally, Sec. V is devoted to a discussion of the contents of previous sections.

II. ROTATIONAL GAUGE-INVARIANT MODEL

If in the Lagrangian of a particle moving in a plane and acted upon by a central force

$$L=\frac{1}{2}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}-V(|\mathbf{x}|)$$

the time derivatives are replaced by "covariant" derivatives defined by

$$\mathcal{D}x=\dot{\mathbf{x}}-zT\mathbf{x}\ ,$$

$$T=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},$$

the resulting Lagrangian

$$L=\tfrac{1}{2}(\dot{x}_1^2+\dot{x}_2^2)-z(x_1\dot{x}_2-x_2\dot{x}_1)+\tfrac{1}{2}z^2(x_1^2+x_2^2)-V(|\mathbf{x}|) \tag{2.1}$$

is invariant under the gauge transformations

$$\mathbf{x}'=R(\theta)\mathbf{x},\quad R(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix},\qquad z'=z+\dot{\theta}\ . \tag{2.2}$$

This simple model was introduced in Refs. 3 and 6 with the purpose of exemplifying different methods of quantization. In Ref. 5, the treatment of this quantization with Dirac brackets was carried out and will be briefly reviewed here.

The Lagrangian (2.1) is singular. As a consequence, the associated Hamiltonian system is a constrained one

$$\begin{aligned}&H=\tfrac{1}{2}(p_1^2+p_2^2)+z(x_1p_2-x_2p_1)+V(|\mathbf{x}|)\ ,\\ &p_z=0\ ,\\ &x_1p_2-x_2p_1=0\ ,\end{aligned} \tag{2.3}$$

the two last equations being the constraints. It is immediate to verify that the system is first class, because

$$[H,p_z]=[H,x_1p_2-x_2p_1]=[p_z,x_1p_2-x_2p_1]=0\ . \tag{2.4}$$

In Dirac's formulation, the state space must be restricted to the subspace defined by the constraints

$$\mathcal{S}:\begin{cases}\hat{p}_z|\ \rangle=0\ ,\\ (\hat{x}_1\hat{p}_2-\hat{x}_2\hat{p}_1)|\ \rangle=0\ ,\end{cases} \tag{2.5}$$

where the eigenvalue equation takes the form

$$[\tfrac{1}{2}(\hat{p}_1^2+\hat{p}_2^2)+V(\hat{x}_1^2+\hat{x}_2^2)]|\rangle=E|\rangle ,$$
$$[\hat{x}_i,\hat{p}_j]=i\delta_{ij} . \tag{2.6}$$

Returning to the classical level, if the gauge conditions

$$z=0 ,$$
$$x_2-ex_1=0, \quad e\in\mathbb{R} \tag{2.7}$$

are imposed and considered as additional constraints, a second-class system is obtained. In this case, Dirac brackets must be defined and the constraint equations become "strong" equations.[1]

Taking into account this last property, the fundamental brackets are[5]

$$[x_1,p_1]_D=\frac{1}{1+e^2}, \quad [x_1,p_1]_D=\frac{e}{1+e^2}=[x_1,p_2]_D ,$$
$$[x_2,p_2]_D=\frac{e^2}{1+e^2}, \quad [x_1,x_2]_D=0=[p_1,p_2]_D , \tag{2.8}$$

and the classical Hamiltonian reduces to

$$H=\frac{(1+e^2)}{2}p_1^2+V((1+e^2)x_1^2) . \tag{2.9}$$

Notice that both z and p_z have been eliminated from the dynamics by virtue of conditions (2.3) and (2.7), and that the matrix of Dirac brackets is singular.

To quantize, the commutators are taken as i times the Dirac brackets

$$[\hat{x}_1,\hat{p}_1]=\frac{i}{1+e^2} , \tag{2.10}$$

etc., and the constraint equations turned into operator equations

$$\hat{p}_z=0, \quad \hat{x}_2=e\hat{x}_1 ,$$
$$\hat{z}=0, \quad \hat{x}_1\hat{p}_2=\hat{x}_2\hat{p}_1 . \tag{2.11}$$

The equality $\hat{x}_1\hat{p}_2-\hat{x}_2\hat{p}_1=0$ is obtained by applying Weyl ordering to the corresponding classical expression with the modified commutators (2.8). Then

$$H=\frac{(1+e^2)}{2}\hat{p}_1^2+V((1+e^2)\hat{x}_1^2) . \tag{2.12}$$

Introducing new variables

$$P^*=(1+e^2)^{1/2}p_1 ,$$
$$x^*=(1+e^2)^{1/2}x_1 , \tag{2.13}$$

one obtains

$$[\hat{x}^*,\hat{P}^*]=i ,$$
$$\hat{H}=\tfrac{1}{2}\hat{P}^{*2}+V(\hat{x}^{*2}) . \tag{2.14}$$

The definition of x^* should be understood as a "strong" equation, therefore, this x^* is a Cartesian coordinate.[5]

In the $|x^*\rangle$ representation, the Hamiltonian reads

$$\hat{H}=-\frac{1}{2}\frac{\partial^2}{\partial x^{*2}}+V(\hat{x}^{*2}) . \tag{2.15}$$

This expression has to be compared with (2.6), which, written in the $|x_1,x_2\rangle$ representation, results in

$$\hat{H}=-\frac{1}{2}\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right]+V(\hat{x}_1^2+\hat{x}_2^2) ,$$
$$\left[\hat{x}_1\frac{1}{i}\frac{\partial}{\partial x_2}-\hat{x}_2\frac{1}{i}\frac{\partial}{\partial x_1}\right]\psi=0 , \tag{2.6'}$$

showing explicitly our claim in the Introduction about the inadequacy of considering gauge conditions as second-class constraints.

III. PASSING OVER TO CURVILINEAR COORDINATES

The incorrect result obtained in Sec. II by consistently using the Dirac bracket formalism is just a manifestation of the fact that the gauge conditions (2.7) cannot be considered a constraint like that of Eq. (2.3) as far as canonical quantization is concerned.

Let us develop this argument: calling Γ the subspace of phase space defined by (2.3) and Γ^* that one defined by (2.3) and (2.7), it is clearly seen that (x^*,P^*) are local coordinates for Γ^*. If the dynamics of the system is to be expressed in terms of them, they should first be extended to Γ by means of a gauge transformation.

Let there be given two points of phase space

$$\gamma\in\Gamma, \quad \gamma=(x_1,x_2,z,p_1,p_2,p_z) \text{ with } p_z=0 ,$$
$$x_1p_2-x_2p_1=0 ,$$

and

$$\gamma^*\in\Gamma^*, \quad \gamma^*=(x_1',x_2',z',p_1',p_2',p_z') \text{ with } p_z'=0, \quad x_1'p_2'-x_2'p_1'=0$$
$$\text{with } z'=0, \quad x_2'-ex_1'=0 \tag{3.1}$$

connected by a gauge transformation with parameter $\theta\in[0,2\pi)$. For any γ in a neighborhood of Γ^* there exists such a γ^* and it is unique. If the coordinates

$$x^*=(1+e^2)^{1/2}x_1' ,$$
$$P^*=(1+e^2)^{1/2}p_1' ,$$

are assigned to γ^*, then the coordinates x^*,θ,P^*,p_θ are assigned to γ. It is easy to verify that in this way canonical coordinates for Γ are constructed, related with the original Cartesian coordinates by

$$x^*=\left[1+\frac{x_2^2}{x_1^2}\right]^{1/2}x_1 ,$$
$$\theta=\arctan\frac{x_2}{x_1}-\arctan e ,$$
$$P^*=x_1^{-1}\left[1+\frac{x_2^2}{x_1^2}\right]^{-1/2}(x_1p_1+x_2p_2) ,$$
$$p_\theta=x_1p_2-x_2p_1 .$$

This should be complemented, on account of the transformation (2.2), by

$$z=-\dot{\theta} .$$

From these definitions, several conclusions are drawn.

(1) The transformation to curvilinear coordinates is singular at the origin independently of the value of e.

(2) The gauge condition has two branches,[3] $x^* > 0$ and $x^* < 0$, i.e., there is a Gribov ambiguity[7] whatever the value of e is.

(3) In these coordinates, the constraint $x_1 p_2 - x_2 p_1 = 0$ reduces simply to $p_\theta = 0$. Therefore the correct Hamiltonian in the coordinate representation is

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^{*2}} - \frac{1}{2\hat{x}^*}\frac{\partial}{\partial x^*} + V(\hat{x}^{*2}) .$$

Clearly, the transformation to curvilinear coordinates (3.1) is nothing but a reformulation of the method exposed in Ref. 3.

IV. QUANTIZATION IN DEPENDENT COORDINATES

In this section we shall analyze the quantization of the Christ-Lee model in the gauge introduced in Eq. (2.7) by means of a realization of the commutator algebra (2.10) in the coordinate representation. As we shall see, it is possible to represent dynamical variables by operators which satisfy gauge conditions as strong operator equations.[1] However, to construct the desired quantum Hamiltonian, it is necessary to obtain first the metric tensor. This is achieved by resorting to another gauge, the Cartesian gauge, which amounts to violating the gauge condition as happened in Sec. III. Were the system a true second-class one, it would not be possible to ascertain the metric properties of restricted configuration space this way, since there is no gauge invariance in this case, and the answer would be different, as we discuss below.

Unlike the discussion in previous sections, the treatment here will be essentially Lagrangian in the sense that gauge invariance will not be taken over to the Hamiltonian formulation.

Although this problem has been developed in Ref. 3, we think it is worth stressing some aspects of the method in the present context. Our starting point will be to show that there exists a gauge condition, namely, $z=0$, such that the metric is Euclidean and the coordinates x_1, x_2 in the Lagrangian (2.1) are Cartesian. Moreover, the phase-space variables have canonical Poisson brackets. By gauge transforming dynamical functions from this gauge, we can find their expression in any other gauge. For example, $x_2 - e x_1 = 0$. The bracket structure is derived in this manner as Poisson brackets, but it can also be given an intrinsic expression as Dirac's brackets (see Sec. II). Such an intrinsic expression has not been given for the metric tensor yet, but in any case it cannot be the same tensor for first- or second-class systems.

Within the gauge condition

$$z=0 \tag{4.1}$$

the Lagrangian recovers its original form

$$L=\tfrac{1}{2}\dot{x}_1^2+\tfrac{1}{2}\dot{x}_2^2-V(r) \tag{4.2}$$

with the additional condition

$$x_1\dot{x}_2 - x_2\dot{x}_1 = 0 . \tag{4.3}$$

The Hamiltonian is then

$$H=\tfrac{1}{2}p_1^2+\tfrac{1}{2}p_2^2+V(r) \tag{4.4}$$

with the constraint

$$x_1 p_2 - x_2 p_1 = 0 \tag{4.5}$$

and the canonical brackets

$$[x_1,p_1]=[x_2,p_2]=1 \tag{4.6}$$

(the rest are zero).

The quantization is carried out from here in the usual fashion, with the state space restricted by the constraint equation (4.5). We see that in this gauge the quantization is formally identical to the procedure followed in Sec. II, Eqs. (2.3)–(2.6), according to Dirac's prescription for first-class systems. This justifies considering coordinates in this gauge as Cartesian. We consider now the gauge condition

$$S(x_1',x_2')\equiv x_2' - e x_1' = 0 . \tag{4.7}$$

Configuration-space trajectories $x_2'(t), x_2'(t), \xi'(t)$ are related to trajectories in the former gauge $x_1(t), x_2(t)$ through the transformation (2.2) with parameter

$$\theta = \arctan e - \arctan\frac{x_2}{x_1} . \tag{4.8}$$

Since (4.7) is also satisfied for $\theta+\pi$, we have to fix an additional condition

$$x_1' \geq 0 \tag{4.9}$$

for the gauge to be uniquely determined and in order to eliminate the Gribov ambiguity already found in Sec. III.

Condition (4.7) restricts configuration space to a straight line. Velocities and momenta which are coordinates in the tangent and cotangent space, respectively, to the restricted configuration space, will then be related by the equations

$$\dot{x}_2' - e\dot{x}_1' = 0 , \tag{4.10a}$$

$$p_2' - e p_1' = 0 . \tag{4.10b}$$

The Lagrangian expression for momenta under the condition (4.10a) results in

$$\begin{aligned} &p_i' = \dot{x}_i', \quad i=1,2 \\ &p_\xi' = 0 , \end{aligned} \tag{4.11}$$

which satisfy Eq. (4.10b) provided (4.7) holds. From (4.13), (4.8), and (2.2) we obtain

$$p'_1=\frac{1}{(1+e^2)^{1/2}}\frac{1}{r}(x_1p_1+x_2p_2)\ ,$$
$$p'_2=\frac{e}{(1+e^2)^{1/2}}\frac{1}{r}(x_1p_1+x_2p_2)\ , \tag{4.12}$$

and with (2.2), (4.8), and (4.12) we can find the fundamental brackets $[x'_i,p'_j]$ starting from the former gauge brackets (4.6). The result is (2.8), as it should be.

It is convenient to express the Lagrangian in this gauge in coordinates x'_1,x'_2,θ instead of x'_1,x'_2,ξ', before passing to the Hamiltonian formulation. This can be achieved by transforming Lagrangian (4.2) through the inverse of transformation (2.2)

$$L=\tfrac{1}{2}\dot{x}'^2_1+\tfrac{1}{2}\dot{x}'^2_2+\tfrac{1}{2}r'^2\dot{\theta}^2-V(r')\ , \tag{4.13}$$

where $r'^2=x'^2_1+x'^2_2$ and use of (4.7) has been made. From Eq. (4.3) there results an additional equation

$$p_\theta=r'^2\dot{\theta}=0\ . \tag{4.14}$$

We see that in this gauge Lagrangian (2.1) is equivalent to (4.13) with the restriction (4.14).

Proceeding as before, we find

$$p_\theta=x_1p_2-x_2p_1\ ,$$
$$[\theta,p_\theta]=1\ , \tag{4.15}$$

which together with (2.8) are the only nonzero fundamental brackets. The metric tensor is obtained in Appendix A. However, in this very simple case, it can be read off directly from the Lagrangian (4.13):

$$g_{ij}=\begin{vmatrix}1&0&0\\0&1&0\\0&0&r'^2\end{vmatrix}\ . \tag{4.16}$$

Finally, the Hamiltonian is found to be

$$H=\tfrac{1}{2}p'^2_1+\tfrac{1}{2}p'^2_2+\frac{1}{2r'^2}p^2_\theta+V(r') \tag{4.17}$$

and we are now ready to carry out quantization. In the coordinate representation the state of the system $|\psi\rangle$ is represented by a wave function

$$\langle x'_1,x'_2,\theta|\psi\rangle=\psi(x'_1,x'_2,\theta) \tag{4.18}$$

defined on the surface Σ of configuration space determined by (4.7).

In this representation the position operators are diagonal

$$\hat{x}'_k|x'_1,x'_2,\theta\rangle=x'_k|x'_1,x'_2,\theta\rangle,\quad k=1,2 \tag{4.19}$$

and satisfy (4.7) as a strong operator equation.

Momenta operator are defined as (see Appendix B)

$$\hat{p}'_1=\frac{1}{i}\frac{1}{(1+e^2)}\left[\frac{\partial}{\partial x'_1}+e\frac{\partial}{\partial x'_2}\right]\ ,$$
$$\hat{p}'_2=\frac{1}{i}\frac{1}{(1+e^2)}\left[e\frac{\partial}{\partial x'_1}+e^2\frac{\partial}{\partial x'_2}\right]\ , \tag{4.20}$$
$$\hat{p}'_\theta=\frac{1}{i}\frac{\partial}{\partial\theta}\ ,$$

which verify the commutator algebra (2.10) and the operator equation (4.10b). The Hamiltonian operator is now constructed in the usual way

$$\hat{H}=\frac{1}{\sqrt{g}}\hat{p}'_i\sqrt{g}\,(g^{-1})_{ij}\hat{p}'_j+V(\hat{r}'),\quad i,j=1,2,\theta \tag{4.21}$$

where g_{ij} is given by (4.16) and

$$g=\det g_{ij}=r'^2\ . \tag{4.22}$$

Therefore it explicitly reads, taking into account (4.14),

$$\hat{H}=\frac{1}{\hat{r}'}\hat{p}'_1\hat{r}'\hat{p}'_1+\frac{1}{\hat{r}'}\hat{p}'_2\hat{r}'\hat{p}'_2+V(\hat{r}')\ . \tag{4.23}$$

Or, using (4.10b) and (4.7),

$$\hat{H}=\frac{(1+e^2)}{\hat{x}'_1}\hat{p}'_1\hat{x}'_1\hat{p}'_1+V((1+e^2)x'_1) \tag{4.24}$$

which, upon making the transformation

$$\hat{x}^*=(1+e^2)^{1/2}\hat{x}'_1\ ,$$
$$\hat{p}^*=(1+e^2)^{1/2}\hat{p}'_1\ , \tag{4.25}$$

coincides with the Hamiltonian given in the preceding section.

It is apparent in Eqs. (4.23) and (4.24) that the kinetic energy term has the characteristic features of a two-dimensional problem. At this point we should be convinced that this is due to the constraints being first class or equivalently, to the systems being gauge invariant. In the next section, a different case that leads to a one-dimensional structure for the kinetic energy is briefly examined.

V. DISCUSSION

The model discussed so far has the property that one of the constraints, the gauge condition, is arbitrary to a large extent. It is not dynamically related to the true constraint (4.5), except for the integrability condition for their Poisson bracket to be nonzero. In particular, the parameter e in (2.7) and (4.7) can take any real value.

In order to clarify the differences with a true second-class system, let us consider an analogous two-dimensional system defined by

$$H=\tfrac{1}{2}p^2_1+\tfrac{1}{2}p^2_2+\tfrac{1}{2}(x^2_1+x^2_2)\ , \tag{5.1}$$

where x_1,x_2 are Cartesian coordinates in the plane and for the sake of argument we have chosen the potential to be harmonic. We impose now on the system the constraint

$$x_2 - e x_1 = 0 , \tag{5.2}$$

which immediately leads to the secondary constraint

$$p_2 - e p_1 = 0 . \tag{5.3}$$

It can also be written

$$x_1 p_2 - x_2 p_1 = 0 . \tag{5.4}$$

This is a true second-class system. In this case, none of the arguments discussed in the previous sections applies due to the absence of gauge invariance, despite the formal analogy among classical Hamiltonian and constraints in both cases [see Eqs. (2.11) and (2.12) and Sec. IV].

Here, once constraint equations are solved and replaced in the Hamiltonian (5.1) according to Dirac's method for second-class systems, we are left with a one-dimensional oscillator. Furthermore, coordinates in this case are Cartesian in every step of the process.

If the potential in Secs. III and IV were harmonic, we would obtain a two-dimensional harmonic oscillator restricted to s states, whose spectrum differs from that of a one-dimensional oscillator.

We see, then, the central role played by gauge invariance in the problem. The fact that the chosen gauge has to be abandoned by performing a gauge transformation in order to construct Hamilton's operator shows that it is wrong to consider the model (2.1) as a second-class constrained one.

In Sec. III the difference between constraints due to gauge invariance and gauge conditions is apparent. The extension of the coordinates of Γ^* is explained in the geometrical interpretation of first-class systems dynamics[8] as the transport of the value of x^*, P^* from Γ^* to the rest of Γ by the Hamiltonian flow generated by the true constraints of this theory. This flow is tangent to Γ and transverse to Γ^*.

In Sec. IV gauge invariance is exploited at the configuration-space level to introduce nonindependent coordinates appropriate to the quantization in the gauge chosen. It is the metric tensor in restricted configuration space which displays here the first-class (gauge-invariant) character of the system. The example at the beginning of this section illustrates this fact.

One may wonder why these problems do not show in, for example, the case of the electromagnetic field. Loosely speaking, the action of the gauge group is there merely additive, which makes the Jacobian of the transformation trivial. The appearance of additional, potential-like terms in the Hamiltonian in the non-Abelian case[3,4] has its origin in similar reasons as those discussed here. One might conclude that the method of Dirac's brackets applied to that case should meet severe difficulties. The analysis of the field-theoretic case is, however, a more delicate matter far beyond the scope of this paper.

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APPENDIX A

To obtain the metric tensor in the gauge (4.7) we start from the transformation (2.2) which together with Eq. (4.8) relates the coordinates x_1', x_2', θ with the coordinates x_1, x_2 in the $z=0$ gauge

$$\begin{aligned} x_1' &= \cos\theta x_1 - \sin\theta x_2 , \\ x_2' &= \sin\theta x_1 + \cos\theta x_2 , \\ \theta &= \arctan e - \arctan\frac{x_2}{x_1} . \end{aligned} \tag{A1}$$

Since in the coordinates (x_1, x_2), g_{ij} is Euclidean, we have for the inverse matrix

$$(g^{-1})_{ij} = \sum_{k=1}^{2} \frac{\partial q_i'}{\partial x_k}\frac{\partial q_j'}{\partial x_k}, \quad i,j=1,2,\theta \tag{A2}$$

where $q_i' = x_i'$ $(i=1,2)$ and $q_\theta' = \theta$. Then

$$(g^{-1})_{ij} = \begin{vmatrix} \frac{1}{1+e^2} & \frac{e}{1+e^2} & 0 \\ \frac{e}{1+e^2} & \frac{e^2}{1+e^2} & 0 \\ 0 & 0 & 1/r^2 \end{vmatrix} , \tag{A3}$$

which is singular as a matrix on the whole space but not on the restricted space. Its kernel is the space of multiples of the vector

$$\nabla S = \begin{bmatrix} -e \\ 1 \\ 0 \end{bmatrix} . \tag{A4}$$

The restricted space is the space of column vectors $\mathbf{v}$ such that

$$\mathbf{v}^t \cdot \nabla S = 0 .$$

Thus, within gauge conditions, (A3) is equivalent to the matrix

$$(g^{*-1})_{ij} = \left[(g^{-1})_{ij} + \frac{\mu}{(1+e^2)} \nabla_i S \nabla_j S \right] \tag{A5}$$

and

$$\det(g^{-1})_{ij}\Big|_\Sigma = \frac{1}{\mu}\det(g^{*-1})_{ij} , \tag{A6}$$

with $\mu \neq 0$ a constant at our disposal. Choosing $\mu = 1$ we find the result given in Sec. IV.

Analogously, we can define

$$g_{ij} = \sum_{k=1}^{2} \frac{\partial x_k}{\partial q_i'}\frac{\partial x_k}{\partial q_j'} ,$$

where the derivatives are derivatives under the constraint (4.7), obtained by means of a Lagrange multiplier. g_{ij} is

$$g_{ij} = \begin{vmatrix} \frac{1}{1+e^2} & \frac{e}{1+e^2} & 0 \\ \frac{e}{1+e^2} & \frac{e^2}{1+e^2} & 0 \\ 0 & 0 & r'^2 \end{vmatrix} .$$

As before, we define g_{ij}^* and the following relations hold:

$$g_{ij}^*(g^{*-1})_{jk} = \delta_{ik} ,$$

$$\det g_{ij}^* \det(g^{*-1})_{ij} = 1 .$$

For $\mu = 1$

$$g_{ij}^* = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r'^2 \end{vmatrix} .$$

APPENDIX B

Let us call Σ the subspace of configuration space defined by (4.7). A function f defined on Σ can be expressed as function of the coordinates x'_1, x'_2, θ restricted to Σ

$$f = f(x'_1, x'_2, \theta) . \quad \text{(B1)}$$

We define the derivatives of f as

$$\frac{\partial}{\partial x'_i} f = \frac{\partial}{\partial x'_i} \tilde{f}, \quad i = 1,2$$
$$\frac{\partial}{\partial \theta} f = \frac{\partial}{\partial \theta} \tilde{f} , \quad \text{(B2)}$$

where $\tilde{f}$ is a differentiable function on the whole space such that

$$\tilde{f}|_\Sigma = f .$$

Of course, f is not unique and derivatives (B2) depend on the choice of f. We then define the derivatives of f under the constraints (4.7) as

$$\partial'_1 f = \frac{1}{1+e^2} \frac{\partial}{\partial x'_1} f + \frac{e}{1+e^2} \frac{\partial}{\partial x'_2} f ,$$
$$\partial'_2 f = \frac{e}{1+e^2} \frac{\partial}{\partial x'_1} f + \frac{e^2}{1+e^2} \frac{\partial}{\partial x'_2} f , \quad \text{(B3)}$$
$$\partial_\theta f = \frac{\partial}{\partial \theta} f .$$

It is easily seen that this definition is independent of the extension f used to computate the derivatives. Indeed, if $\tilde{g}$ is a function such that

$$\tilde{g}|_\Sigma = 0 \quad \text{(B4)}$$

then on Σ we have

$$\nabla \tilde{g} = \lambda \nabla S = \lambda(-e, 1, 0) \quad \text{(B5)}$$

Therefore, on Σ

$$\partial'_1 \tilde{g} = \partial'_2 \tilde{g} = 0 , \quad \text{(B6)}$$

so that functions constant on Σ have null derivatives.

Furthermore, the combination

$$\partial'_2 - e\partial'_1 = 0 \quad \text{(B7)}$$

vanishes identically.

The vector $(\partial'_1 \tilde{f}, \partial'_2 \tilde{f}, (1/r')\partial_\theta \tilde{f})$ on Σ is the projection of $\nabla \tilde{f}$ on the space tangent to Σ.

In Sec. IV we have defined the operators

$$\hat{p}'_j \psi = \frac{1}{i} \partial'_j \psi, \quad j = 1,2 \quad \text{(B8)}$$

which satisfy (B7) and give the correct commutators with coordinate operators.

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