Langmuir oscillations against a single-ion pulse or cavity background

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Nonlinear plasma oscillations against a background supporting an ion pulse or cavity are considered. We treat two physical cases. The first is that of a cold plasma, the second of a cold-ion–warm-electron plasma. The ion background includes a hyperbolic-secant-squared pulse or cavity but is otherwise uniform. An exact solution is obtained for the cold-plasma case. Infinite electric-field gradients result after some time, followed by a three-valued electric field, a physical impossibility. Thus some modification is called for. When the electrons are assumed to be thermal these infinite gradients can be limited. This will be the case if the ratio of pulse strength to background density is less than $\gamma^{1/3}(\lambda_D/l)^{2/3}$, where λ_D is the Debye length and l a characteristic width of the pulse. Applications are briefly discussed.

I. INTRODUCTION

This paper addresses the problem of how the presence of a localized ion pulse or cavity influences electron Langmuir oscillations. Single-ion density cavities appear in the stellar atmosphere, in Q-machine experiments, and in double and triple plasma devices.[1] There is also a possibility of observing conditions described here, with the hyperbolic-secant-squared ion profile, when Langmuir oscillations become coupled with ion acoustic solitons. Finally, we are considering adapting our results to describe oscillations near a grid. Exact solutions found here indicate that after some time, regions of very steep electric-field gradients should appear in the immediate vicinity of the pulse. Ways of proceeding once these infinite gradients are achieved are reviewed briefly. One of these procedures, that is, introducing a finite electron temperature, is then made on the basis of an approximate calculation. The density bursts can then be finite and their dynamics are easily described. They will move away from the pulse with a velocity that grows from zero and becomes uniform after a while.

Mathematically, the tool that will enable us to find these exact solutions (and the approximate one for thermal electrons) is that of introducing Lagrangian variables. Several plasma physics problems have already been solved by doing this.[2–6] However, the number of exact solutions known to date is not yet so large that another solution should not be considered with some interest. Importantly, the situation treated here is fundamental when considering a range of plasma physics experiments. The more realistic solution in which the electron temperature is nonzero and the resulting density maxima are finite may also be of some practical significance. Extensions, such as two pulses or a pulse and a cavity side by side, are envisaged.

II. SOLUTION FOR COLD ELECTRONS

In the calculation we take the ion density to be time independent and peaked at $x=0$. We take

$$n_i(x,t)=n_0[1+\alpha\,\mathrm{sech}^2(kx)],\quad \alpha>-1\ . \tag{2.1}$$

Thus $\alpha>0$ describes a pulse and $\alpha<0$ describes a cavity. The initial electron density is uniform:

$$n_e(x,0)=n_0\ . \tag{2.2}$$

This will give an initial electric field. To find it we use Poisson's equation:

$$\frac{\partial E}{\partial x}=4\pi e(n_i-n_e)\ , \tag{2.3}$$

yielding

$$E(x,0)=4\pi e n_0\alpha k^{-1}\tanh(kx)\ . \tag{2.4}$$

The initial electron velocity $v_e(x,0)$ is taken to be zero. The field (2.4) created by the charge imbalance around the pulse or cavity will drive the system in a nonlinear mode.

The equations for the electron density n_e and velocity v_e are, in the fluid model of a cold plasma,

$$\frac{\partial n_e}{\partial t}+\frac{\partial(n_e v_e)}{\partial x}=0\ , \tag{2.5}$$

$$\frac{\partial v_e}{\partial t}+v_e\frac{\partial v_e}{\partial x}=\frac{-eE}{m}\ . \tag{2.6}$$

Equations (2.3), (2.5), and (2.6) give a complete description of the plasma. We now introduce the Lagrangian coordinates (x_0,τ) and an auxiliary function ψ

$$\psi=\int_0^{\tau}v_e\,d\tau,\quad x_0=x-\psi,\quad \tau=t\ . \tag{2.7}$$

The coordinate x_0 thus follows a fluid element in its motion. Importantly, convective derivatives reduce to partial derivatives. Our equations simplify in the new coordinates to

$$\frac{\partial}{\partial \tau}\left[n_e\left[1+\frac{\partial \psi}{\partial x_0}\right]\right]=0\ , \tag{2.8}$$

$$\frac{\partial v_e}{\partial \tau}=\frac{-eE}{m_e}\ , \tag{2.9}$$

$$\frac{\partial E}{\partial \tau}=4\pi e n_i v_e\ . \tag{2.10}$$

The first equation can be integrated to give

$$n_e=n_0\Big/\left[1+\frac{\partial \psi}{\partial x_0}\right]\ , \tag{2.11}$$

and the second and third combine to give

$$\frac{\partial^3 \psi}{\partial \tau^3}+\omega_{pe}^2\{1+\alpha\,\mathrm{sech}^2[k(x_0+\psi)]\}\frac{\partial \psi}{\partial \tau}=0\ . \tag{2.12}$$

This equation can be integrated:

$$\frac{\partial^2 \psi}{\partial \tau^2}+\omega_{pe}^2\{\psi+\alpha k^{-1}\tanh[k(x_0+\psi)]\}=0\ , \tag{2.13}$$

the constant of integration being zero according to (2.4) since $\psi(0)=0$. Upon integrating once again and using initial conditions, we obtain

$$\psi_\tau^2=-\omega_{pe}^2\psi^2-2\omega_{pe}^2\alpha k^{-2}\ln\times\{\cosh[k(x_0+\psi)]/\cosh(kx_0)\}\ . \tag{2.14}$$

We now rescale the variables, introducing $\bar{\psi}=k\psi$, $\bar{\tau}=\omega_{pe}\tau$, $\bar{x}=kx_0$ to obtain an exact solution in parametric form. This amounts to, in terms of n_e and Eulerian variables x,t, interpreting the integral in (2.16) such that t increases monotonically:

$$x=(\bar{x}+\bar{\psi})/k\ , \tag{2.15}$$

$$t=\omega_{pe}^{-1}\int^{\bar{\psi}}d\bar{\psi}'\{-2\alpha\ln[\cosh(\bar{x}+\bar{\psi}')/\cosh\bar{x}]-\bar{\psi}'^2\}^{-1/2}\ , \tag{2.16}$$

$$n_e=n_0\Big/\left[1+\frac{\partial \bar{\psi}}{\partial \bar{x}}\right]\ , \tag{2.17}$$

$$\frac{\partial \bar{\psi}}{\partial \bar{x}}=\alpha\{-2\alpha\ln[\cosh(\bar{x}+\bar{\psi})/\cosh\bar{x}]-\bar{\psi}^2\}^{1/2}\times\int\frac{\tanh(\bar{x}+\bar{\psi}')-\tanh\bar{x}}{\{-2\alpha\ln[\cosh(\bar{x}+\bar{\psi}')/\cosh\bar{x}]-\bar{\psi}^2\}^{3/2}}d\bar{\psi}'\ . \tag{2.18}$$

This is an exact solution in parametric form: $x=x(\bar{x},\bar{\psi})$, $t=t(\bar{x},\bar{\psi})$, $n=n(\bar{x},\bar{\psi})$. However, some features of the solution are more simply seen by methods other than using (2.15)–(2.18) to plot $n(x,t)$.

The following two particular features of the above solution will now be needed. (i) The motion of each individual fluid element, labeled by x_0, is periodic; (ii) The period of the motion T is a function of x_0, in contradistinction to the cold plasma, uniform ion background case, for which $T=2\pi/\omega_{pe}$. Both features are seen from the figures. Figure 1 shows curves in phase space corresponding to our solution for chosen x_0 and both $\alpha>0$ (pulse) and $\alpha<0$ (cavity). We see that, as $x_0\rightarrow\infty$, we obtain perfect plasma oscillations (circles), but for finite x_0 the curves are not circles. To follow the motion of one fluid element labeled by a given x_0 through one period, we go around the corresponding phase curve once. Both ψ_τ and ψ are thus periodic functions of time provided T is finite. We now simply calculate $T(x_0)$ numerically by

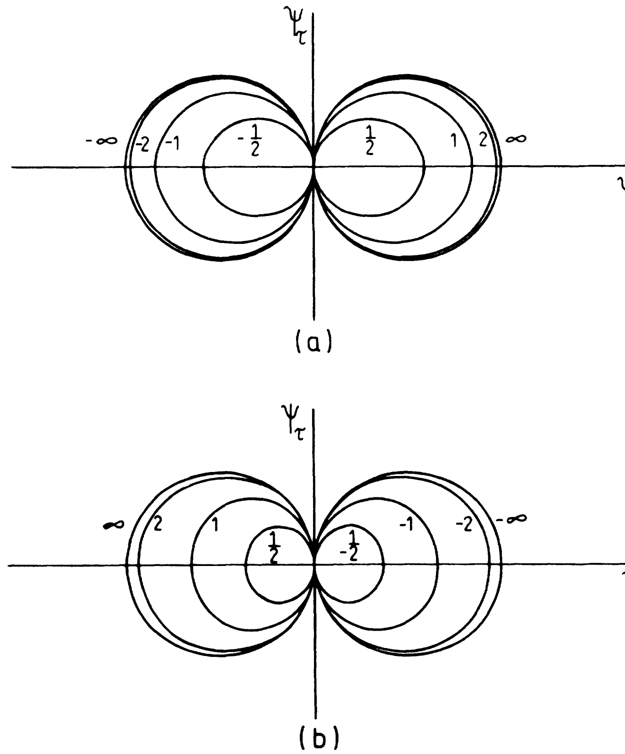

FIG. 1. Phase plane solution curves (ψ_τ,ψ) for chosen values of kx_0. (a) $\alpha=0.3$, pulse; (b) $\alpha=-0.3$, cavity.

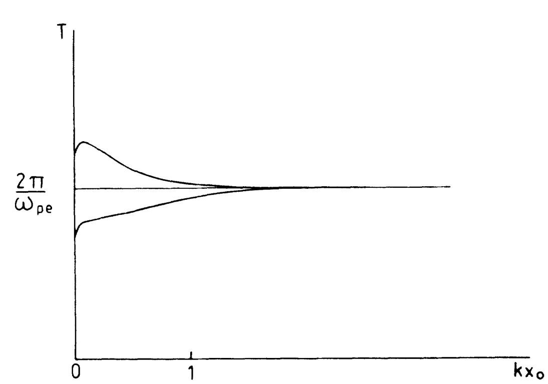

FIG. 2. Each electron fluid element, labeled by x_0, oscillates with its own period T. Dependence of T on kx_0 for $\alpha=0.3$ and -0.3.

taking two times (2.16) up to the zero of the denominator. The result, both for the pulse and the cavity, is shown in Fig. 2, where T is seen to be a function of x_0 tending to $2\pi/\omega_{pe}$ as $x\to\infty$ in both cases:

$$T(x_0\to\infty)=2\omega_{pe}^{-1}\int_0^{2|\alpha|}d\psi'(2\alpha\psi'-\psi'^2)^{-1/2}=2\pi/\omega_{pe} .$$

So ψ is always a periodic function of τ, but the period depends on x_0. We can write this as

$$\psi=\psi(x_0,\tau/T) ,$$

where the period of the second variable is now 1. Thus

$$\frac{\partial\psi}{\partial x_0}=\frac{\partial\psi}{\partial x_0}-(\tau/T^2)\psi'\frac{\partial T}{\partial x_0} ,$$

where the prime denotes differentiation with respect to the second argument. As ψ' takes both signs, the secular component will sooner or later cause the denominator in (2.17) to vanish (later on we will, in fact, calculate $\partial T/\partial x_0$ explicitly for small α). Thus, for some finite time τ, n_e becomes infinite, and then three-valued in Eulerian coordinates (without the ion pulse or cavity, $\partial T/\partial x_0$ is zero and no explosive behavior is expected). This general effect was mentioned by Dawson[7] and a corresponding exact solution for a *periodic* ion background was given by Infeld, Rowlands, and Torvén.[8] This knowledge is now seen to yield an exact, explosive solution for a single ion pulse or cavity background.

III. TREATMENT OF INFINITE DENSITIES

Once the solution gives infinite density and subsequently three beams competing for the same space (same x), it must be abandoned in its present form. This is primarily because it gives three different values for E, the three values for n not necessarily being quite so damning. What actually happens?

One school of thought[9,10] argues that the infinite density appearing in the model is not in itself a serious problem, as the *integrated* density is finite, whereas three plasma beams can coexist. However, as the field E must of course be single valued, Poisson's equation (2.3) should be altered so as to include $n_{e1}+n_{e2}+n_{e3}$ and indeed after a while five, seven, and so on beams may appear. Instead of investigating an exact solution, one now proceeds numerically. For example, a theory of pump energy conversion to a plasma has been obtained by doing this.[9] All this is based on the collisionless model (no friction between beams) and we will come back to the problem of its applicability in a moment.

A second *simple* scenario assumes we still have just one electron beam at any given point and *discontinuities* separate the beams. As there is only one n_e at a given point, Poisson's equation need not be altered. This is the collision dominated plasma model. (There are plasma mechanisms other than collisions for keeping streams apart that we will not discuss.)

Whether one of these two extreme models is applicable or not will depend on the slowing down time τ_s of an electron of one beam inside a second beam (we forget about the third beam in these simplistic considerations). If we denote the relative velocity for two beams by $\Delta v=v_{e1}-v_{e2}$ and the temperature of the second beam by T, then

$$\tau_s\sim(\Delta v)^3/n_{e2}\ln\Lambda$$

at low temperatures T, and

$$\tau_s\sim T^{3/2}/n_{e2}\ln\Lambda$$

at high T.[11] Here $\ln\Lambda$, the Coulomb logarithm, does not vary appreciably over a wide range of plasmas, and is usually roughly between 10 and 20. The collisionless model will apply if $\tau_s\gg L/\Delta v$, where L is a characteristic length for the experiment. This *is* the case for high, and *can be* for low T.

Only the above two limits are at all tractable by fluid methods. Thus we can so cope with total interpenetration of the beams and total opaqueness. The former is usually assumed in plasma physics, the latter in gas dynamics.[12] Intermediate situations are difficult, so these two have had more than their share of attention.

In what follows we will concentrate on situations such that infinite densities are never reached, leaving models based on the above two cases to a later paper. We must of course introduce more physics, namely a thermal spread for the electrons from the beginning. We will see that when this thermal spread exceeds a critical value the bursts will be finite (finite maxima of n). Below this critical value of T infinite densities persevere as they did in our $T=0$ model (a real plasma is never zero temperature). Some of the ideas of this calculation were used in a recent paper by Infeld, Rowlands, and Torvén,[8] but the following is the first systematic exposition. We will just consider the pulse case $\alpha>0$.

If the electrons are thermal with temperature T_e and adiabatic exponent γ and ψ is assumed small, we generalize (2.13) to (first reference of 5):

$$\frac{\partial^2\psi}{\partial\tau^2}+\omega_{pe}^2\psi=\beta\frac{\partial^2\psi}{\partial x^2}-\alpha k^{-1}\omega_{pe}^2\tanh[k(x_0+\psi)] ,$$
$$\beta'=\omega_{pe}^2\gamma\lambda_D^2=\gamma KT_e/2m_e . \tag{3.1}$$

This is more tractable in the form

$$\frac{\partial^2\psi}{\partial\tau^2}+\psi=\beta\frac{\partial^2\psi}{\partial x^2}-\alpha k^{-1}\tanh[k(x_0+\psi)] ,$$
$$\bar{\tau}=\omega_{pe}\tau,\quad \beta'=\omega_{pe}\beta . \tag{3.2}$$

In our calculation α is assumed small and $\beta\ll\alpha$. A lowest order calculation in which β is neglected (small ψ; cold electron calculation; a limit of the exact solution of Sec. II) yields

$$\frac{\partial^2\psi_1}{\partial\bar{\tau}^2}+\psi_1=-\alpha k^{-1}\tanh(kx_0),\quad \psi_1(0)=\psi_{1\bar{\tau}}(0)=0 , \tag{3.3}$$

solved by

$$\psi_1=-\alpha k^{-1}\tanh(kx_0)[1-\cos(\tau+\chi)]=A_1(1-\cos\theta_0)\ . \tag{3.4}$$

This suggests we look for the finite temperature solution in the form $A(1-\cos\theta)$ and also introduce multiple time:

$$\bar{\tau}=\bar{\tau}_0+\bar{\tau}_1+\cdots\ .$$

We have, by separating terms proportional to $\sin\theta$ and to $\cos\theta$, in next order

$$\frac{\partial A}{\partial\bar{\tau}_1}-(\beta/2)\left[A\frac{\partial^2\chi}{\partial x_0^2}+2\frac{\partial A}{\partial x_0}\frac{\partial\chi}{\partial x_0}\right]=0\ , \tag{3.5}$$

$$A\frac{\partial\chi}{\partial\bar{\tau}_1}+(\beta/2)\left[\frac{\partial^2 A}{\partial x_0^2}-A\left[\partial\chi/\partial x_0\right]^2\right]-(\alpha/2)A/\cosh^2(kx_0)=0\ . \tag{3.6}$$

If $\beta=0$, $\partial A/\partial\bar{\tau}_1=0$ and from (3.6)

$$\begin{aligned}\chi&=\alpha\bar{\tau}_1/2\cosh^2(kx_0)\\ \frac{\partial\chi}{\partial x_0}&=-\alpha\bar{\tau}_1\tanh(kx_0)/k\cosh^2(kx_0)\ .\end{aligned} \tag{3.7}$$

Thus, from (2.11), dropping the subscript in $\bar{\tau}_1$,

$$\begin{aligned}n&=\frac{n_0}{1-[A_1\alpha k\tanh(kx_0)\cosh^{-2}(kx_0)\sin\theta]\bar{\tau}+\dfrac{\partial A_1}{\partial x_0}(1-\cos\theta)}\\ &=\frac{n_0}{1+[\alpha^2\tanh^2(kx_0)\cosh^{-2}(kx_0)\sin\theta]\bar{\tau}-\alpha\cosh^{-2}(kx_0)(1-\cos\theta)}\ .\end{aligned} \tag{3.8}$$

This is the small-α limit of (2.17). Note the secular term in the denominator.

We notice that (3.8) implies that after a while, when $\bar{\tau}$ exceeds α^{-1}, the third term (3.6) dominates the second. Equation (3.6) can thus be approximated by

$$\frac{\partial\chi}{\partial\bar{\tau}}-(\beta/2)\left[\frac{\partial\chi}{\partial\chi_0}\right]^2-\alpha/2\cosh^2(kx_0)=0\ . \tag{3.9}$$

We now take $\chi'=-\chi$, $\tau=\beta\tau$ to obtain

$$\partial\chi'/\partial\tau+\tfrac{1}{2}(\partial\chi'/\partial x_0)^2=-\alpha/2\beta\cosh^2(kx_0)\ . \tag{3.10}$$

Differentiating by x_0,

$$\frac{\partial}{\partial\tau}\left[\frac{\partial\chi'}{\partial x_0}\right]+\left[\frac{\partial\chi'}{\partial x_0}\right]\frac{\partial^2\chi'}{\partial x_0^2}=\alpha k\tanh(kx_0)/\beta\cosh^2(kx_0)\ . \tag{3.11}$$

Upon introducing $\rho=\partial\chi'/\partial x_0$, this takes the more familiar form

$$\frac{\partial\rho}{\partial\tau}+\rho\frac{\partial\rho}{\partial x_0}=\alpha k\tanh(kx_0)/\beta\cosh^2(kx_0)\ . \tag{3.12}$$

We now perform a "second Lagrangianization," a transformation from coordinates x_0, τ to ξ,τ, where $\xi=x_0-\int_0^\tau\rho\,d\tau'$, to obtain

$$\begin{aligned}&\frac{\partial\rho}{\partial\tau}=\alpha k\tanh[k(\xi+p)]/\beta\cosh^2[k(\xi+p)]\ ,\\ &p=\int_0^\tau\rho\,d\tau,\quad \frac{\partial p}{\partial\tau}=\rho=\frac{-\partial\chi}{\partial x_0}\ .\end{aligned} \tag{3.13}$$

Thus, expressing all in terms of p,

$$\begin{aligned}&\frac{\partial^2 p}{\partial\tau^2}=\alpha k\tanh[k(\xi+p)]/\beta\cosh^2[k(\xi+p)]\ ,\\ &p(0)=p_\tau(0)=0\ ;\end{aligned} \tag{3.14}$$

and this integrates to

$$\left[\frac{\partial p}{\partial\tau}\right]^2=(\alpha/\beta)\{\mathrm{sech}^2(k\xi)-\mathrm{sech}^2[k(\xi+p)]\}\ . \tag{3.15}$$

We are now in a position to draw $\partial\chi/\partial x_0$ as a function of x_0 (this is just $-p_\tau$ as a function of $\xi+p$) (see Fig. 3) by using this result and (3.4); we could also draw $n(x,t)$:

$$\begin{aligned}n&=n_0/[1+(A\sin\theta)\chi_{x_0}+A_{x_0}(1-\cos\theta)]\ ,\\ x&=x_0+A(1-\cos\theta)\ .\end{aligned} \tag{3.16}$$

Once again, we will try to see what is happening without doing this.

Importantly, the denominator in (3.16) now remains finite if $|A\,\partial\chi/\partial x_0|<1$ for all trajectories labeled by x_0. In terms of the original, unscaled variables, this yields the condition

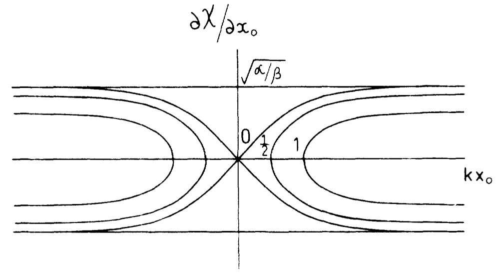

FIG. 3. Schematic of phase plane solution curves ($\partial\chi/\partial x_0$, kx_0) for pulse, $\alpha=0.3$, and chosen values of $k\xi$.

$$\alpha^{3/2}/(\gamma^{1/2}k\lambda_D)<1 \ . \qquad (3.17)$$

Infinite densities will be avoided if $T_e > 2m_e\omega_{pe}^2\alpha^3/\gamma k^2 K$, which is just (3.17) in terms of the temperature. When this condition is satisfied, densities rise to a finite maximum value and this maximum is initially stationary at $x_0=[\cosh^{-1}(\frac{3}{2})^{1/2}]/k$, the maximum of $\tanh(kx_0)/\cosh^2(kx_0)$. For large τ, most phase points will have reached the asymptotic regions where phase curves in Fig. 3 are almost straight lines and the maximum will move away from the ion cavity with velocity approaching $(\beta/\alpha)^{1/2}$ [an exact analysis could be performed from (3.15)]. Eventually $\partial\chi/\partial x_0$ will become three-valued (though not five or more) and use of our model must stop. However, quite an improvement over the cold plasma picture has been achieved at the price of some approximations and a further coordinate transformation. Densities are now finite and some interesting physics has been described before the onset of multivaluedness.

IV. SUMMARY

By introducing Lagrangian coordinates, it has been possible to give an exact solution corresponding to cold plasma, Langmuir oscillations against an ion cavity or pulse. This solution leads to infinite densities after finite time. Various ways of dealing with this problem are discussed. By introducing a second coordinate transformation that formally resembles that from Eulerian to Lagrangian coordinates in the warm-electron–cold-ion model, it has been possible to solve this extended problem and obtain density bursts limited to $n_0/(1-\alpha^{3/2}/k\lambda_D\gamma^{1/2})$. The density maxima appear at finite distances from the cavity $\pm k^{-1}\cosh^{-1}(\frac{3}{2})^{1/2}$ (one on each side of the cavity), stay there more or less immobile for a while, and then move away at velocities approaching plus minus a maximal value. All this assumes that the electron temperature exceeds a critical value. The predictions of this paper could probably be checked by simple experiments (see introduction).

From a mathematical point of view, it is interesting that a second application of the same procedure, i.e., introducing coordinates that reduce the convective to a partial derivative $\partial_t + v\partial_x$ to ∂_t in some space, should be so powerful. Perhaps calculations in which this is done as many as three or more times will be useful in the future. Here introducing a thermal spread and this approach made it possible to avoid infinities and delay multivaluedness. It is also noteworthy that *three* is now the maximum number of values for n.

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