PHYSICAL REVIEW A VOLUME 42, NUMBER 2 15 JULY 1990

Electron macrokinetics in partially ionized gases: The hydrodynamic regime

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(Received 2 October 1989)

The concept of macrokinetic distributions is used to investigate the macroscopic dynamics of an assembly of electrons in a weakly ionized gas in the hydrodynamic regime. In this regime, the macrokinetic distribution (MKD) is shown to obey an equation that is equivalent to the Boltzmann equation in the time scale of electron-density transport. Formal, approximate solutions to this equation are obtained whose range of validity depend on the magnitude of the spatial derivatives of the density. Specific conditions on the magnitude of these derivatives have been obtained. Explicit expressions for the MDK are presented for the case of a quasi-Lorentz gas model. They have been used to evaluate the electron current density in the hydrodynamic regime and to obtain expressions for the mobility and diffusion coefficient. In the regime of large electron-density gradient, these coefficients have been found to depend on the normalized gradient. The consequences of these results are illustrated for the case of constant collision frequency.

I. INTRODUCTION

The ensemble-averaged dynamic behavior of an assembly of electrons in a background gas under the influence of a space-time varying field may be described at a microscopic level by the space-time-dependent velocity distribution function, $f(\mathbf{v},\mathbf{r},t)$ (where $\mathbf{v}$ is velocity, $\mathbf{r}$ is position, and t is time).[1] Given the initial state of the assembly, the distribution function at any other time may be obtained from either a kinetic-type equation, such as the Boltzmann equation[2] (BE), or from Monte Carlo simulations.[3] Once the distribution function is known, desired space-time-dependent macroscopic properties (which can be measured) can be calculated by velocity averaging over the microscopic distribution the corresponding microscopic properties.[1]

In general, the macroscopic variables that characterize the macroscopic dynamics of the electron assembly depend on average properties of the microscopic distribution over an extended velocity interval; moreover, their dynamical changes occur over space-time scales that are coarser than those of the microscopic distribution. This distribution has a resolution in space-time scales corresponding to those of a two-body collision, and thus contains more information than necessary to provide a characterization of the assembly in terms of macroscopic variables.[4] A distribution function, with less space-time resolution (coarser), and equivalent (as far as macroscopic properties) velocity dependence than that of the microscopic distribution, can equally serve to determine macroscopic properties of the assembly, and to obtain closed equations of evolution for the macroscopic variables that characterize the dynamics of the assembly.[4]

There exists a number of coarser distributions each characterized by a different space-time resolution.[4] This resolution is dictated by the characteristic scales of variation of the dynamical macroscopic variables. Since these distributions are velocity-dependent (i.e., kinetic variables) with macroscopic scales of resolution, they are collectively referred to as macrokinetic distributions (MKD). A procedure for obtaining MKD is to expand the microscopic distribution functions in terms of those eigenfunctions of the acceleration plus collision operators in the BE (these operators are defined in Sec. II) with eigenvalues whose real part corresponds to the desired resolution. However, for space-time-dependent accelerations this may not be possible, and alternate, less compact expansions (for example, in terms of either a local field or eigenfunctions of the collision operator only) would have to be used. Moreover, since, in general, the expansion coefficients have no physical significance, it is desirable to use alternate approaches for obtaining the MKD.

An alternate, physical, approach is to first identify the macroscopic variables that describe the dynamics of the assembly. Their equation of evolution contains the characteristic space-time scales that define the resolution of the description.[4] These scales are then used to obtain, from the BE, the equation for the MKD. Since the characteristic scales of the macroscopic equations depend on the distribution, the procedure outlined above must be carried out self-consistently (see Sec. II). Although not a unique set (rate coefficients can in principle also be used), the velocity moments of the distribution can serve to define the resolution scales of the MKD. Increasing resolution is obtained by choosing an increasing number of moments in the description, which are selected by ordering the moment equations according to their characteristic space-time scales and keeping those with less or equal resolution than desired.[4] Each finite set of moments thus selected defines a resolution scale and a corresponding MKD.

In this paper, an analysis of the macroscopic dynamics of the electron assembly in the coarsest space-time scale is presented. By analogy with neutral gas kinetics, this regime is referred to as the hydrodynamic regime. For electron in gases, it is defined by the scale of resolution of the density. In the Sec. II, the various regimes that can be defined through the use of the moments is reviewed. In Sec. III, the hydrodynamic regime is discussed in detail. An illustration is given in Sec. IV using a quasi-Lorentz gas model. Concluding remarks are given in Sec. V.

II. MACROKINETIC REGIMES AND DISTRIBUTIONS

The state of the electron assembly can be described at a microscopic level by the distribution function $f(\mathbf{v},\mathbf{r},t)$ in $(\mathbf{v},\mathbf{r})$ space. This distribution obeys the BE; namely,[1,2,4]

$$\partial_t f+\mathbf{v}\cdot\nabla f+\frac{q}{m}\mathbf{E}\cdot\nabla_v f=I(f)\ , \tag{1}$$

where $\mathbf{E}=E(\mathbf{r},t)$ is the electric field (prescribed), and $I(f)$ is the linear scattering operator.[5] No specific form for the operator I need be assumed at this time. At the macroscopic level, the assembly is characterized by a "state vector," H_N, whose components are moments of the distribution and the corresponding MKD, $f_M^{(N)}$. That is, $H_N=(S_N,f_M^{(N)})$, where $S_N=(m_j,j=1,\ldots,N)$ with m_j being a velocity moment of the distribution (a scalar, vector, or tensor). The dimension of S_N, N, and its components, m_j, are selected depending on the space-time resolution desired for the description. Alternatively, S_N and $f_M^{(N)}$ define the scale of resolution of the macroscopic description. S_N and $f_M^{(N)}$ are obtained as follows.

First, the moment equations [obtained by taking appropriately weighted integrals (in $\mathbf{v}$ space) of Eq. (1) (Ref. 1)] are ordered according to their characteristic scales.[4] This step requires *a priori* assumptions about the relative magnitude of these scales, which can be made from physical consideration. In any event, the ordering used need to be confirmed after a self-consistent description is obtained. The first three, time-scale ordered moment equations are[4]

$$\partial_t n+\nabla\cdot(n\mathbf{u})=\nu n\ , \tag{2a}$$

$$d_t(n\bar{\varepsilon})+\nabla\cdot\langle\varepsilon\mathbf{v}\rangle-q\mathbf{E}\cdot n\mathbf{u}=-\nu_\varepsilon n\bar{\varepsilon}\ , \tag{2b}$$

$$\partial_t(n\mathbf{u})+\nabla\cdot\langle\mathbf{v}\mathbf{v}\rangle-\frac{q}{m}\mathbf{E}n=-\nu_m n\mathbf{u}\ , \tag{2c}$$

where, $n(\mathbf{r},t)$, $\bar{\varepsilon}(\mathbf{r},t)$, and $\mathbf{u}(\mathbf{r},t)$ are the electron density, mean energy, and average velocity, respectively; the angular bracket implies an average over the distribution, $\varepsilon=\frac{1}{2}mv^2$, and ν, ν_ε, and ν_m are the (space-time-dependent) effective ionization, energy-exchange, and momentum-exchange frequencies, respectively. These frequencies are defined by

$$\nu n=\int I(f)d\mathbf{v}\ , \tag{3a}$$

$$-\nu_\varepsilon n\bar{\varepsilon}=\int\tfrac{1}{2}mv^2 I(f)d\mathbf{v}\ , \tag{3b}$$

$$-\nu_m n\mathbf{u}=\int\mathbf{v}I(f)d\mathbf{v}\ . \tag{3c}$$

An integral without limits implies integration over all space. Since it is difficult to ascribe physical significance to higher-order moments, their equations of evolution are seldom written down. The higher moment equations would also have to be ordered accordingly. It is assumed that their characteristic times are smaller than those defined above. Note that, in general, τ_ε $(=\nu_\varepsilon^{-1})>\tau_m$ $(=\nu_m^{-1})$ for weakly ionized gases.[6]

The first three moments, n, $\bar{\varepsilon}$, and $\mathbf{u}$ and their equation of evolution, Eq. (2), can be used to develop three levels of descriptions, each characterized by a space-time resolution scale.[4] The most coarse-grained description (i.e., lease resolution) has a time scale of the order of ν^{-1} $(=\tau)$; that is, the scale of the density equation. From Eqs. (2a)–(2c), since $\nu<\nu_\varepsilon<\nu_m$,[6] there is a time for which the mean energy and average momentum of the electrons have relaxed to a state of quasiequilibrium where their subsequent variation is in the scale of τ. In such a scale, the dynamics of the system is determined from Eq. (2a). Consequently, S_N contains one component, n; i.e., $S_1=(n(\mathbf{r},t))$. The equation for the corresponding MKD $f_M^{(1)}$ is obtained by averaging the BE over times shorter than τ. An equivalent approach is used in Sec. III to obtain this equation. By analogy with classical gas kinetics,[7,8] the time regime for which this description is valid (namely, the longest time scale) is named the hydrodynamic regime. However, in contrast to gas kinetics, the properties of this state can be derived from a single macroscopic variable (instead of three), the density. This definition of hydrodynamic is less restrictive than that used in the literature which in addition assumes a specific form for the distribution (see Sec. III).[9]

"Nonhydrodynamic" (higher resolution) descriptions can systematically be obtained by using an additional moment in S_N. Thus, the next less coarse-grained description is in terms of $S_2=(n(\mathbf{r},t),n\bar{\varepsilon}(\mathbf{r},t))$ and the corresponding MKD $f_M^{(2)}$. This description is valid for times of the order of ν_ε^{-1}. From a practical point of view, the description with most resolution is in terms of $S_3=(n(\mathbf{r},t),n\bar{\varepsilon}(\mathbf{r},t),nu(\mathbf{r},t))$ and $f_M^{(3)}$, which is valid for times of the order of ν_m^{-1}. Note that the functional dependence of $f_M^{(3)}$ corresponds to that of the normal solutions of the BE for classical gas kinetics as introduced by Chapman-Enskog.[10] However, as shown in the Sec. III, the context in which this functional dependence is used in this paper differs from that of Chapman-Enskog in that it also represents spatial gradients of the corresponding moments. These nonhydrodynamic descriptions will be discussed in a subsequent paper.

III. THE HYDRODYNAMIC REGIME

In this regime, the electron assembly is characterized (by definition) by $H_1=(S_1,f_M^{(1)})$. The evolution of the assembly is determined from Eq. (2a) with the current density given by

$$n\mathbf{u}(\mathbf{r},t)=\mathbf{J}(\mathbf{r},t)=\int\mathbf{v}f_M^{(1)}d\mathbf{v} \tag{4a}$$

and the rate ν by Eq. (3a), with $f=f_M^{(1)}$. Thus, the equation for $f_M^{(1)}$ [Eq. (6) below] and Eq. (2a) form a closed set.

It is convenient in some situations to write the current density in this regime as,

$$\mathbf{J}=\mathbf{W}n-\mathsf{D}_0\cdot\nabla n+\mathbf{J}_R\ , \tag{4b}$$

where $\mathbf{W}$ is the drift velocity, D_0 is the diffusion tensor, and $\mathbf{J}_R$ accounts for other contributions to the current density that depend on derivatives of n greater than the first ($\mathbf{W}$ and D_0 are, in general, space-time dependent). A significant amount of work has been devoted to the theoretical determination of $\mathbf{W}$, D_0, and ν;[9,11–19] equivalently, to the closure of Eq. (2a). These investigations fall into two categories according to the method used; namely, the free path[11] and the perturbed distribution function methods.[9,11,12] In all approaches, the background (or zeroth order) electron distribution has been taken to be space independent. The effects of nonequilibrium (arising from density gradients, for example) are then taken into account by introducing the concept of a free path [method (1)] or by perturbing the distribution directly [method (2)]. Approaches based on the perturbation method have, in general, yielded more accurate results. They are characterized by the expansion of either the distribution function in terms of spatial derivatives of the density[9,11,17–19] or the spatial Fourier transform of the distribution in a power series in the spatial wave number.[12] In all cases, the lowest-order solution is spatially uniform so that the expansions are valid in the limit of small density gradients. The results that have been obtained have elucidated a number of phenomena, such as, the properties of the diffusion tensor[9,12] and the effect of ionization on electron drift and diffusion.[15]

In the approach presented in this paper, an evaluation of $\mathbf{W}$, D_0, and $\mathbf{J}_R$ follows from Eq. (4a) after substituting an expression for $f_M^{(1)}$. The equation for $f_M^{(1)}$ is obtained by changing the time scale of the BE [Eq. (1)] from the fine grained to a τ scale. This can be achieved using a technique introduced by Bogoliubov.[20] Mathematically, the change can be accomplished by the following relation:

$$f(\mathbf{v},\mathbf{r},t)=f_M^{(1)}(\mathbf{v},n(\mathbf{r},t))\ .$$

That is, in the τ scale, the space-time dependence of the distribution is implicit through a dependence on the density. Physically, this is equivalent to saying that the space-time dynamics of the assembly is determined by the density. In the literature, the distribution function in the hydrodynamic regime has been further restricted to have a density dependence of the form of a (linear) expansion in terms of gradients of the density.[9,11] This regime corresponds to a subset of that presented in this paper. Thus, the changes in f can be written as

$$\partial_t f=\partial_n f_M^{(1)}\partial_t n\ , \tag{5a}$$

$$\nabla f=\partial_n f_M^{(1)}\nabla n\ , \tag{5b}$$

$$\nabla_{\mathbf{v}} f=\nabla_{\mathbf{v}} f_M^{(1)}\ . \tag{5c}$$

The equation defining $f_M^{(1)}$ can be found using Eqs. (1), (2a), and (5). The time derivative of the density may be eliminated from Eq. (5a) by using Eq. (2a). After placing Eqs. (2a) and (5) into Eq. (1), the following equation is obtained for the MKD:[4]

$$\partial_n f_M^{(1)}\left[-\int\partial_n f_M^{(1)}\mathbf{v}\,d\mathbf{v}\cdot\nabla n+\int I(f_M^{(1)})d\mathbf{v}\right]+\mathbf{v}\cdot\nabla n\,\partial_n f_M^{(1)}+\frac{q}{m}\mathbf{E}\cdot\nabla_{\mathbf{v}} f_M^{(1)}=I(f_M^{(1)}) \tag{6}$$

with the condition that $\int f_M^{(1)}d\mathbf{v}=n(\mathbf{r},t)$. Letting

$$f_M^{(1)}=n(\mathbf{r},t)f_M(\mathbf{v},n(\mathbf{r},t))\ , \tag{7}$$

where $\int f_M d\mathbf{v}=1$, Eq. (6) can be rewritten as

$$\frac{q}{m}\mathbf{E}\cdot\nabla_{\mathbf{v}} f_M+(f_M+n\partial_n f_M)\frac{\nabla n}{n}\cdot\mathbf{v}+N(f_M)=I(f_M)\ , \tag{8a}$$

where $N(f_M)$ is a nonlinear term given by

$$N(f_M)=-(f_M+n\partial_n f_M)\left[\int(f_M+n\partial_n f_M)\mathbf{v}\,d\mathbf{v}\cdot\frac{\nabla n}{n}-\int I(f_M)d\mathbf{v}\right] \tag{8b}$$

Equation (8) is an integro-differential equation in four variables $(\mathbf{v},n)$ [in contrast to seven variables $(\mathbf{v},\mathbf{r},t)$ in Eq. (1)]; however, it is nonlinear [whereas Eq. (1) is linear]. The nonlinear term arises from the changes in the shape of the distribution due to changes in the particle density at $\mathbf{r}$ as a function of time.

Formal solutions to Eq. (8) can be obtained using perturbation procedures. To simplify the algebra (for the sake of clarity), assume that the electric field is in the z direction [$E=E_0(z,t)\mathbf{a}_z$, where $\mathbf{a}_z$ is a unit vector in the z direction] and that the density gradients are also in the same direction ($\nabla n=\partial_z n\mathbf{a}_z$). It is convenient to obtain an alternate representation for Eq. (8) by explicitly displaying the nature of the n dependence of f_M. This is done primarily to assist in the physical interpretation of the various approximations to be used for the solution of Eq. (8). Since electron-electron collisions have been neglected, the functional form of f_M should not be affected by the magnitude of the electron density. Thus, its density dependence can only be through normalized derivatives of n; namely, $g_i=\partial_z^i n/n$, $i=1,\ldots,\infty$. That is,

$$f_M(\mathbf{v},n)=f_M(\mathbf{v},\{g_i\})\ ,$$

where

$$\{g_i\}=(\partial_z n/n,\partial_z^2 n/n,\ldots)\ .$$

Then,

$$\partial_n f_M=\sum_i\partial_{g_i}f_M\partial_n g_i=\sum_i\partial_{g_i}f_M\frac{\partial_z g_i}{ng_1}\ .$$

Substituting in Eq. (8a),

$$\frac{qE_0}{m}\partial_{v_z}f_M+\delta\nu f_M+\delta\nu\sum_i\partial_{g_i}f_M\frac{\partial_z g_i}{g_1}=I(f_M)\ , \tag{9a}$$

where

$$\delta\nu=g_1(v_z-u)-\sum_j\partial_z g_j\int\partial_{g_j}f_M v_z d\mathbf{v}+\int I(f_M)d\mathbf{v}\ . \tag{9b}$$

These equations can also be written in terms of the ith spatial derivative by noting that $\partial_z g_i = g_{i+1} - g_i g_1$. $\delta\nu$ has the units of frequency and represents a change in the scattering rate in and out (depending on sign) of various regions in velocity space due to modifications to the distribution resulting from density gradients [first two terms of Eq. (9b)] and from ionization [last term of Eq. (9b)].

Equation (9) is the (complete) working equation from which various approximations to f_M can be obtained. The first such approximation is the neglect in all subsequent discussions of all but the first factor in the expression for $\delta\nu$. This corresponds to a linearization of $\delta\nu$ (higher-order contributions can readily be included). Depending on the magnitude of the gradient terms in Eqs. (8a) or (9a) [last two terms on the left-hand side (lhs)] as compared to the electric field term (first term in lhs), solutions for the hydrodynamic distribution can be obtained that differ in their range of validity. These solutions are discussed in the following subsections.

General expressions for the current density, Eq. (4a), can be readily obtained by inserting these solutions in Eq. (4a). This is illustrated in Sec. IV for the case of a quasi-Lorentz gas model.

A. The density gradient expansion: small spatial derivatives

To ease the record keeping, introduce a small parameter δ into Eq. (9a) by letting $\partial_z \to \delta \partial_{z'}$, and thus $g_i = \delta^i g_i'$. Then the gradient terms are found to be of order δ, $O(\delta)$; whereas all others are of $O(1)$. Using δ as a basis for a perturbation expansion, the distribution may be expressed as

$$f_M(\mathbf{v},n)=f_M(\mathbf{v},\{g_i'\})=\sum_i \delta^i f_i \ . \tag{10}$$

[This form of expansion is used throughout this paper. However, as mentioned in the introduction to this section, various ordering for the terms in the lhs of Eq. (9a) are to be investigated.] Substituting this expression into the scaled form of Eq. (9a), and equating the coefficients of each power of δ to zero, leads to the following equations:

$$\frac{q}{m}E_0 \partial_{v_z} f_0 - I(f_0) = 0 \quad \text{for } \delta^0 \tag{11a}$$

with the condition

$$\int f_0 d\mathbf{v} = 1 \ . \tag{11b}$$

Equation (11a) has the form of a steady-state, homogeneous Boltzmann equation, and f_0 can be identified as the (zeroth-order) steady-state distribution of a homogeneous assembly of electrons in a homogeneous field defined by the value of the field at (z,t). This is the distribution that exists at (z,t) if local equilibrium with the field is assumed. From Eq. (11a), f_0 is found to be only a function of $\mathbf{v}$, $f_0(\mathbf{v})$, and

$$\partial_{g_i'} f_0(\mathbf{v}) = 0, \quad i \geq 1 \ . \tag{11c}$$

Thus,

$$\frac{q}{m}E_0 \partial_{v_z} f_1 - I(f_1) = -v_z g_1' f_0 \quad \text{for } \delta^1 \tag{12a}$$

with solution

$$f_1 = g_1' h_1(\mathbf{v}) \ , \tag{12b}$$

where

$$\frac{q}{m}E_0 \partial_{v_z} h_1 - I(h_1) = -v_z f_0 \tag{12c}$$

and

$$\int h_1(\mathbf{v}) d\mathbf{v} = 0 \ .$$

From Eq. (12), $\partial_{g_1'} f_1 = h_1(\mathbf{v})$, and $\partial_{g_i'} f_1 = 0$, $i > 1$. Thus,

$$\frac{q}{m}E_0 \partial_{v_z} f_2 - I(f_2) = -v_z g_2' h_1 \quad \text{for } \delta^2 \tag{13a}$$

with solution

$$f_2 = g_2' h_2(\mathbf{v}) \ , \tag{13b}$$

where

$$\frac{q}{m}E_0 \partial_{v_z} h_2 - I(h_2) = -v_z h_1 \tag{13c}$$

and

$$\int h_2(\mathbf{v}) d\mathbf{v} = 0 \ .$$

Equations for δ^i can be obtained similarly. To each order in δ, the solution for f_i is a product of a function of $\mathbf{v}$ only $[h_i(\mathbf{v})]$ and g_i'. This is a consequence of the assumption that the last two terms in the lhs of Eq. (9a) are of the same order. Contributions from nonlinear terms containing products $g_i' g_1'$ cancel out.

Combining Eqs. (7), (9), and (11)–(13),

$$\begin{aligned} f_M^{(1)}(\mathbf{v},n) &= n(f_0 + \delta g_1' h_1 + \delta^2 g_2' h_2 + \dots) \ , \\ &= n(f_0 + g_1 h_1 + g_2 h_2 + \dots) \ . \end{aligned} \tag{14}$$

This solution for $f_M^{(1)}$ is the density gradient expansion (DGE). It has previously been *a priori* assumed for the distribution in the hydrodynamic regime.[9,11] Explicit evaluation of the distribution function requires the solution of Eqs. (11a), (12c), (13c), etc., for the h_i's. The range of validity of this solution is determined from the condition that $g_i = O(\delta^i)$. This condition is further quantified in Sec. III C.

Within the small spatial derivative regime, it is possible to obtain a distribution with a larger range of validity than that of Eq. (14) by considering g_1 and g_2 to be of the same order, and both of $O(\delta)$ (in fact, many such distributions can be obtained by considering groups of g_i to be of the same order). In this case, the right-hand side (rhs) of Eq. (12a) for f_1 needs to be modified to include terms of $O(\delta)$ that depend on g_2 (or other g_i's). Similar modifications need to be made to the higher-order equations [Eq. (13a), etc.]. An example of this procedure is given in Sec. IV A. The resulting distribution reduces to Eq. (14) in the limit $g_2 \ll g_1$.

B. Local hydrodynamic equilibrium

For $g_{i+1}=g_i g_1$, i.e., $g_i=\text{const}$ in space, Eq. (9a) becomes,

$$\frac{qE_0}{m}\partial_{v_z} f_M^L - I(f_M^L) + f_M^L \delta\nu = 0 \; , \tag{15a}$$

where $f_M = f_M^L(\mathbf{v})$, only a function of $\mathbf{v}$. Using this result in Eq. (7),

$$f_M^{(1)}(\mathbf{v},n) = n(z,t) f_M^L(\mathbf{v}) \; . \tag{15b}$$

For conditions of local hydrodynamic equilibrium in gas kinetics, the distribution function depends only on the local hydrodynamic variables (not their spatial derivatives). Since for electrons in the hydrodynamic regime, the only hydrodynamic variable is n, the analog for the distribution in local equilibrium is that of Eq. (15b). In steady state, Eq. (15b) is the distribution function obtained by Phelps[21] for an exponentially growing (in space) discharge.

C. Large density gradient g_1

When the density gradient is sufficiently large that its effect on the distribution approaches that of the electric-field term, the second term in the lhs of Eq. (9a) approaches 0(1). In this regime, the solution obtained in Sec. III A, namely, Eq. (14), is no longer applicable. [Note that Eq. (14) represents a distribution that is obtained by perturbing about a uniform state, i.e., lowest-order term is space independent.] This occurs, for example, in the wings of a Gaussian density profile in space. Moving away from the center of the distribution (where $g_1=0$), a point is reached at which the g_1 term becomes of $O(1)$. Further out, the g_2 term also becomes of $O(1)$, and so on. Since this does not occur over the whole spatial domain being considered, it is desirable to obtain solutions that are valid over larger spatial ranges than the DGE, without the need of complicating the description by involving higher-order moments of the distribution (such as S_2 or S_3). Moreover, these solutions can be used to determine the range of validity of the DGE (Sec. III E). A solution valid in the range where the g_1 term can also be of $O(1)$ is presented in this subsection. In Sec. III D, the regime where in addition the g_2 term is of $O(1)$ is considered.

Further define the "g_1 regime" by assuming that $\partial_z g_i \partial_{g_i} f_M$ is of $O(\delta)$ for $i=1,\ldots,\infty$; thus, the third term in the lhs of Eq. (9a) is of $O(\delta)$. Proceeding as in Sec. III A [representing f_M by Eq. (10)], the following equations are obtained for the f_i's:

$$\frac{qE_0}{m}\partial_{v_z} f_0 + v_z g_1' f_0 = I(f_0) \quad \text{for } \delta^0 \; , \tag{16a}$$

with

$$\int f_0 d\mathbf{v} = 1 \; . \tag{16b}$$

This equation is of the form of Eq. (15a); however, g_1' here is not a constant. Thus, the lowest-order solution f_0 is space dependent, and Eq. (10) in this case constitutes an expansion about a nonuniform state whose functional "form" is that of a distribution in local hydrodynamic equilibrium. The solution to Eq. (16a) may be written, formally, as

$$f_0 = Q_0(\mathbf{v},g_1')\exp\left[-\frac{mv_z^2}{2qE_0}g_1'\right] , \tag{17a}$$

where

$$\frac{qE_0}{m}\partial_{v_z} Q_0 = I'(Q_0) \tag{17b}$$

with

$$I'(Q_0) = \exp\left[\frac{mv_z^2}{2qE_0}g_1'\right] I\left[Q_0 \exp\left[-\frac{mv_z^2}{2qE_0}g_1'\right]\right] . \tag{17c}$$

In the next order

$$\frac{qE_0}{m}\partial_{v_z} f_1 + v_z g_1' f_1 - I(f_1) = -v_z \partial_{g_1'} f_0 \partial_z g_1' \quad \text{for } \delta^1 \tag{18a}$$

with $\int f_1 d\mathbf{v}=0$. From Eq. (17a),

$$\partial_{g_1'} f_0 = -\frac{mv_z^2}{2qE_0} f_0 + \exp\left[-\frac{mv_z^2}{2qE_0}g_1'\right]\partial_{g_1'} Q_0 \; . \tag{18b}$$

Using this in Eq. (18a), the formal solution for f_1 can be obtained; namely,

$$f_1 = Q_1(\mathbf{v},g_1')\partial_z g_1' \exp\left[-\frac{mv_z^2}{2qE_0}g_1'\right] , \tag{19a}$$

where

$$\frac{qE_0}{m}\partial_{v_z} Q_1 - I'(Q_1) = \frac{mv_z^3}{2qE_0} Q_0 - v_z \partial_{g_1'} Q_0 \; . \tag{19b}$$

Higher-order contributions to f_M (in δ) are similarly obtained. Combining Eqs. (7), (10), (17a), and (19a),

$$f_M^{(1)}(\mathbf{v},n) = n \exp\left[-\frac{mv_z^2}{2qE_0}g_1\right](Q_0 + \partial_z g_1 Q_1 + \cdots) \; . \tag{20}$$

This is a "gradient expansion" in g_1, where the lowest solution depends on g_1. This is the equivalent of Eq. (14) in this regime. The spatial domain over which Eq. (20) is valid is greater than that of Eq. (14) since no restrictions have been imposed on the magnitude of g_1 (see Sec. III E). Explicit evaluation of the distribution requires the solution of Eqs. (17b), (19b), etc. for the Q_i's. An example is given in Sec. IV. Note that the Q_i's are space dependent, since the rhs of Eqs. (17b), (19b), etc., depend on g_1.

When Eq. (20) is used in Eq. (4a) to compute the current density, and if the current density is separated according to Eq. (4b), the expressions for $\mathbf{W}$ and D_0, are found to be space-time dependent (see Sec. IV). It may not be practical in some applications to use Eq. (20) to compute these parameters. An approximate expression for f_M in this regime can be obtained by letting $I'(Q_0) \sim I(Q_0)$ in Eq. (17b). Then, $Q_0(\mathbf{v},g_1)=Q_0(\mathbf{v})$. Using this approximation in each order, the Q_i's become space independent. The resulting distribution can then be used to evaluate the current density. If in the resulting expression for **J** it is, in addition, assumed that the g_i's are small, expressions for **W** and D_0 can be obtained that are space-time independent. These results can then be compared with those obtained from the DGE (which expands the distribution before the evaluation of the integrals). By expanding after integration, it is possible to explicitly determine the expansion parameter in terms of macroscopic variables, which variables are not available for comparison at the kinetic level. This is illustrated in Sec. IV.

D. Large g_1 and g_2

In this regime, g_1 and g_2 (or equivalently, g_1 and $\partial_z g_1$) are assumed to be of $O(1)$; whereas, all other g_i's are of $O(\delta)$. Before scaling Eq. (9a), it is desirable to carry out a change of variables that identifies part of the $\partial_z g_1$ term in Eq. (9a) with changes in the distribution that are equivalent to those caused by the field term. Let

$$\xi = v_z - \frac{a^2}{\nu_s^3} g_1 \ , \tag{21a}$$

$$w = \frac{a^2}{\nu_s^3} g_1 \ , \tag{21b}$$

where $a = qE_0/m$, and ν_s is a scaling frequency (to be chosen later). The motivation for this transformation lies with the fact that it is desired to assess the effect of the $\partial_z g_1 \partial_{g_1} f_M$ term in Eq. (9a) on f_M. When compared to the field term [first term in Eq. (9a)], it may be interpreted as a force driving the distribution away from equilibrium. Qualitatively, since the electric force tends to displace the distribution in the field direction, the density gradient force is to similarly displace the distribution in the direction of the gradient.

Using the chain rule to obtain the transformation of the derivatives, the lowest-order equation [with f_M represented by Eq. (9)] is found to be

$$(a+\eta)\partial_\xi f_0 = -\nu_e f_0 + I(f_0) \ , \tag{22a}$$

where

$$\eta(\mathbf{v}_\perp,\xi,\omega) = -v_z \frac{a^2}{\nu_s^3}\partial_z g_1 + \frac{3a^2}{\nu_s^4} g_1 \frac{\partial \nu_s}{\partial v_z} \tag{22b}$$

$$\nu_e(\mathbf{v}_\perp,\xi,\omega) = v_z g_1 \tag{22c}$$

with $(\mathbf{v}_\perp, v_z, n)$ related to $(\mathbf{v}_\perp,\xi,\omega)$ through Eqs. (21). To obtain Eq. (22a), it has been assumed that $\partial_z g_i$, $i>1$, and $\partial_\omega f_0$ are of $O(\delta)$. Equation (22a) has the form of a steady-state, homogeneous BE with an equivalent field $E_{\mathrm{eq}} = m(a+\eta)/q$. At this level of approximation, the MKD is the distribution that exist at $\mathbf{r},t$ if local equilibrium with the equivalent field is assumed. Unlike the steady-state BE, the equivalent field in Eq. (22) is velocity dependent due to the dependence of η on velocity. The origin and effect of this velocity dependence can be assessed from Eq. (21b). Let $v_a = a/\nu_s$ (where v_a is the velocity increment due to the external field in the time ν_s^{-1}) and $\gamma = v_D/\varepsilon_c = g_1 n \nu_s$. For $\nu_s = \nu_m$ (where ν_m is the average momentum exchange collision frequency), v_a is the drift velocity, and γ is the ratio of the diffusion velocity to the characteristic energy. The equivalent field can then be written as

$$E_{\mathrm{eq}} = \frac{m}{q} a \left[1 + v_z v_a \gamma^2 \left(\frac{g_2}{g_1} - 1 \right) \right] . \tag{23}$$

For given g_1,g_2, and applied field, the effect of the second term inside the parentheses in Eq. (23) is to produce an acceleration that increases with v_z. This results in an increase or decrease in the tail of the distribution, depending on the sign of the density gradient. Thus, diffusion currents not only change the local density at **r**, but also the velocity spectrum of those particles since the particle flux depends on the value of v_z. This is illustrated in Sec. IV. The numerical evaluation of the current density in this case requires the solution of Eqs. (4a) and (22a). For computational expediency, the mobility, diffusion coefficient, and Eq. (3a) can be calculated from Eq. (22a) and tabulated as a function of E_0 and η. Equation (2a) for the density can then be solved for the desired conditions using the tabulated results.

E. Range of the validity of the DGE

From the results obtained in Secs. III A and III C, the range of validity of the DGE can (quantitatively) be ascertained. At the macrokinetic level (where the only macroscopic variables are the g_i's), for

$$\left| \frac{m v_z^2}{2qE_0} g_1 \right| < 1 \tag{24}$$

(physically, when the potential energy in a distance corresponding to the scale length of the density gradient is greater than the kinetic energy associated with the z component of electron velocity over the range for which the distribution is non-negligible), the exponentials in Eqs. (20) and the equations for the W_i's [Eq. (17b), (19b), etc.,] can be expanded in a power series in $(m v_z^2/2qE_0)g_1$. That is, letting

$$\exp\left[-\frac{m v_z^2}{2qE_0} g_1 \right] \sim \left[1 - \frac{m v_z^2}{2qE_0} g_1 + \cdots \right] , \tag{25a}$$

$$Q_0 = Q_{00} - Q_{01} g_1 + \cdots \ , \tag{25b}$$

and

$$Q_1 = Q_{10} + Q_{11} g_1 + \cdots \ , \tag{25c}$$

where the equations for the Q_{ij} follow from Eqs. (17b), (19b), etc. Recalling that $\partial_z g_i = g_{i+1} - g_i g_1$, and using Eqs. (25) in Eq. (20), the resulting distribution (assuming the coefficients of the nonlinear terms to be approximately zero, see Sec. IV), is the density gradient expansion, Eq. (14). Thus, Eq. (24) must be satisfied in the range of validity of the DGE. In practical applications, since most particles fall in the energy range below the mean energy, an approximate condition is obtained by substituting $\bar{\varepsilon}$ for $\frac{1}{2}mv_z^2$ in Eq. (24). Since $\bar{\varepsilon}$ is approximately qD_0/μ (where μ is the mobility), in the DGE regime, $|g_1| < W/D_0$.

In addition to Eq. (24), the results obtained also require that $\partial_z g_i \partial_{g_i} f_i$, $i = 1, \ldots, \infty$ are of $O(\delta)$. It is possible to quantify this condition by proceeding from a formal solution of the complete equation, Eq. (9a), obtained by successive use of variation of parameters.

Let

$$f_M = W_0(\mathbf{v}, \{g_i\}) \exp\left[-\frac{mv_z^2}{2qE} g_1 \right] \tag{26a}$$

in Eq. (9a). The equation for Q_0 is found to be

$$\partial_{v_z} Q_0 - \left[\frac{m}{qE_0} \right]^2 \frac{v_z^3}{2} \partial_z g_1 Q_0 + \frac{m}{qE_0} v_z \sum_i \partial_z g_i \partial_{g_i} Q_0 = I'(f_0) \ . \tag{26b}$$

Again, let

$$Q_0 = S_0 \exp\left[\left[\frac{m}{qE} \right]^2 \frac{v_z^4}{4\times 2} \partial_z g_1 \right] \tag{27a}$$

in Eq. (26b) (the period in the expression 4×2 implies the product of the two numbers); the equation for S_0 is found to be

$$\partial_{v_z} S_0 + \left[\frac{m}{qE_0} \right]^3 \frac{v_z^5}{4\times 2} \partial_z^2 g_1 S_0 + \frac{q}{mE_0} v_z \sum_i \partial_z g_i \partial_{g_i} Q_0 = I''(Q_0) \ , \tag{27b}$$

where

$$I''(Q_0) = \exp\left[-\left[\frac{m}{qE_0} \right]^2 \frac{v_z^4}{4\times 2} \partial_z g_1 \right] I'(Q_0) \ .$$

Continuing in this fashion, and combining Eqs. (26a), (27a), etc., the distribution can be formally written as

$$f_M = R_0(\mathbf{v}, \{g_i\}) \exp\left[-\left[\frac{m}{qE_0} \right] \frac{v_z^2}{2} g_1 + \left[\frac{m}{qE_0} \right]^2 \frac{v_z^4}{4\times 2} \partial_z g_1 - \left[\frac{m}{qE_0} \right]^3 \frac{v_z^6}{6\times 4\times 2} \times \partial_z^2 g_1 + \cdots \right] \tag{28}$$

where R_0 obeys an equation of the form similar to Eqs. (26b) and (27b). Following the procedure that led to the DGE from Eq. (28), this expansion can also be obtained from Eq. (28) by restricting the g_i's to the range,

$$|g_i| < \left[qE_0 \Big/ \frac{mv_z^2}{2} \right]^i \ . \tag{29}$$

Alternatively, the g_i's define the velocity range (in the z direction) for which the DGE constitutes a good approximation to the distribution function.

If g_i, $i = 1, \ldots, N$ (or equivalent, $\partial_z^i g_1$, $i = 1, \ldots, N-1$) do not satisfy Eq. (29) over a significant region of the v_z range (with all others doing so), the exponentials not corresponding to these g_i's can be expanded in a power series, resulting in a distribution that belongs to the family whose lowest-order term is space dependent. The lowest-order member of the family (i.e., where g_1 is significant) is given by Eq. (20). The next order member of the family, valid in the regime where both g_1 and $\partial_z g_1$ do not satisfy Eq. (29) over a large region of v_z space, can be obtained from Eq. (22a).

IV. APPLICATION TO QUASI-LORENTZ GAS MODEL

In this section, the theory developed in the preceding section is used to obtain the electron distribution of the hydrodynamic state in the presence of density gradients for a quasi-Lorentz model.[9] This model is a Lorentz gas model ($m/M \ll 1$, where M is the mass of an atom/molecule that constitutes the background gas) modified to account for energy losses to the background gas. To obtain an analytic solution, the two-term spherical harmonic approximation for f will be used (i.e., $f = f_0^s + f_1^s \cos\theta$). In this case, Eq. (1) becomes [assuming one-dimensional variation in space]

$$\partial_t f_0^s + \frac{v}{3} \partial_z f_1^s + \frac{1}{3v^2} \partial_v (v^2 a f_1^s) = I(f_0^s) \ , \tag{30a}$$

$$f_1^s = -\frac{v}{\nu_m} \partial_z f_0^s - \frac{a}{\nu_m} \partial_v f_0^s \ , \tag{30b}$$

where

$$I(f_0^s) = \frac{1}{v^2} \partial_v [\alpha(v) f_0^s + \beta(v) \partial_v f_0^s] \ , \tag{30c}$$

$$\nu_m = N\sigma(v) v \ ,$$

with

$$\alpha(v) = \frac{m}{M} v^3 \nu_m, \quad \beta(v) = \frac{kT}{M} v^2 \nu_m \ .$$

$\sigma(v)$ is the collision cross section for momentum transfer; N and T are the density and temperature, respectively, of the background gas. In Eq. (30b), it has been assumed that the relative rate of change of f_1 is small compared with the collision frequency. This is consistent with the conditions defining the hydrodynamic regime.

Using the spherical harmonic approximation and Eq. (30b) in Eq. (4a), the current density for the quasi-Lorentz gas model is

$$J_z = -\int \frac{4\pi}{3} \frac{v^2}{\nu_m} \partial_z f_0^s v^2 dv - \int \frac{4\pi}{3} \frac{v}{\nu_m} a \partial_v f_0^s v^2 dv \ . \tag{31a}$$

In the notation of Eq. (4b), Eq. (31a) becomes

$$J_z = Wn - D_0 \partial_z n + D_1 \partial_z^2 n + J_R' \ , \tag{31b}$$

where the term in $\mathbf{J}_R$, Eq. (4b), containing the second derivative of n has been explicitly identified. The factor multiplying this derivative is the kurtosis coefficient, D_1. Explicit expressions for W, D_0, and D_1 are given later in this section. From the discussion in Sec. II, in the hydrodynamic regime, the MKD for a quasi-Lorentz gas model is defined as

$$f_0^s(v,z) = f_M^1(v, n(z,t)) = n f_M(v,n) \ . \tag{32}$$

Using Eqs. (5), (30), and (32), the equivalent of Eq. (9a) for this case is found to be

$$\begin{aligned} -\frac{1}{3v^2}\partial_v \left[\frac{v^2 a^2}{\nu_m}\partial_v f_M + 2\frac{v^3 a}{\nu_m} g_1 f_M \right] + \left[\frac{d_v(v^3 a/\nu_m)}{3v^2} g_1 - \frac{v^2}{3\nu_m} g_2 \right] f_M - \frac{1}{3v^2}\partial_v \left[2\frac{v^3 a}{\nu_m} \sum_i \partial_{g_i} f_M \partial_z g_i \right] \\ + \frac{d_v(v^3 a/\nu_m)}{3v^2} \sum_i \partial_{g_i} f_M \partial_z g_i - \frac{v^2}{3\nu_m} \left[\sum \partial_{g_i} f_M \left[\frac{\partial_z g_i}{g_1} \{ g_2 + g_1^2 \} + n \partial_z^2 g_i - n \partial_z g_i \partial_z \ln g_1 \right] \right. \\ \left. + \sum \partial_{g_i}^2 f_M (\partial_z g_i)^2 \right] + N(f_M) = \frac{1}{v^2}\partial_v(\alpha f_M + \beta \partial_v f_M) \ , \end{aligned} \tag{33}$$

where $N(f_M)$ is obtained from Eq. (8b) and $d_v S(v) \equiv dS(v)/dv$. Equation (33) is the complete working equation from which various approximations to f_M can be obtained, which differ in their range of validity. Some of these solutions are discussed in the subsequent subsections.

An alternate procedure for obtaining an equation for $f_M(v,n)$ [Eq. (33)] is to expand $f_M(\mathbf{v},n)$ in spherical harmonics [instead of $f(\mathbf{v},\mathbf{r},t)$], and proceed from Eq. (9a) [instead of Eq. (1)]. This has not been done in order to start the example from the usual equations for f_0^s and f_1^s, Eqs. (30).

A. Density gradient expansion: small spatial derivatives

As mentioned in Sec. III, this range has been discussed extensively in the literature.[9,11–19] It is instructive to obtain the density gradient expansion as a solution to Eq. (33) in the range $\partial_z \sim \delta \partial_z'$, where δ is a small parameter (see Sec. II A). Again, it is not necessary to specify what δ is at this time; it is strictly used to assist in the ordering of the various terms. Ambiguous ordering of terms has led to a significant amount of misunderstanding in the literature regarding the range of validity of the various solutions obtained and the interpretation of the W and D_0, that appear in Eq. (31b).[18]

Thus proceeding as in Sec. III A, Eq. (33) is rewritten in this range as follows:

$$\begin{aligned} \left[\frac{a^2}{3\nu_m} + \frac{kT}{M}\nu_m \right] \partial_v \ln f_M = -\frac{m}{M}\nu_m v - \frac{2va}{3\nu_m} g_1' \delta + \frac{1}{v^2 f_M} \left[\int_0^v \left[\frac{d_v(v^3 a/\nu_m)}{3} g_1' \delta - \frac{v^4}{3\nu_m} g_2' \delta^2 \right] f_M dv \right] \\ - \frac{2va}{3\nu_m} \sum_i \partial_{g_i'} \ln f_M \partial_z' g_i' \delta + \frac{1}{v^2 f_M} \left[\int_0^v \frac{d_v(v^3 a/\nu_m)}{3} \sum_i \partial_{g_i'} f_M dv \right] \partial_z' g_i' \delta + O(\delta^2) \ . \end{aligned} \tag{34}$$

Expanding $\ln f_M$ in a power series in δ (essentially a Rytov expansion of the solution[22]),

$$\ln f_M = \sum_i \delta^i S_i = S_0 + \delta S_1 + \delta^2 S_2 + \cdots \ , \tag{35a}$$

$$f_M = f_0 [1 + \delta S_1 + \delta^2 (S_2 + S_1^2/2) + \cdots] \tag{35b}$$

and

$$f_M^{-1} = f_0^{-1} [-\delta S_1 - \delta^2 (S_2 - S_1/2) \cdots] \ , \tag{35c}$$

where

$$f_0 = e^{S_0} \ .$$

The distribution function is normalized such that

$$\int_0^\infty f_0 v^2 dv = 1 \ ,$$

whereas all other contributions vanish. Using Eqs. (35) in Eq. (34), and equating coefficients of each power of δ to zero,

$$\left[\frac{a^2}{3\nu_m} + \frac{kT}{M}\nu_m \right] \partial_v S_0 = -\frac{m}{M}\nu_m v \quad \text{for } \delta^0$$

with solution

$$S_0 = -\int_0^v \left[\frac{a^2}{3\nu_m} + \frac{kT}{M}\nu_m \right]^{-1} \frac{m}{M}\nu_m u \, du \ . \tag{36a}$$

Since S_0 is not a function of g_i', $\partial_{g_i'} S_0 = \partial_{g_i'} f_0 = 0$. In the next order

$$\left[\frac{a^2}{3\nu_m} + \frac{kT}{M}\nu_m \right] \partial_v S_1 = -\frac{2va}{\nu_m} g_1' + \frac{1}{v^2 f_0} \left[\int_0^v \left[\frac{d_u(u^3 a/\nu_m)}{3} f_0 \right] du \right] g_1' \quad \text{for } \delta'$$

with solution

$$S_1=-\int_0^v\left[\left[\frac{a^2}{3\nu_m}+\frac{kT}{M}\nu_m\right]^{-1}\left[\frac{2wa}{\nu_m}-\frac{1}{w^2f_0}\int_0^w\frac{d_u(u^3a/\nu_m)}{3}f_0du\right]\right]dw\,g_1' \; . \tag{36b}$$

From this equation, $\partial_{g_1'}S_1=0$, $i\neq 1$, $\partial_{g_1'}S_1=S_1/g_1'$, and in next order,

$$\left[\frac{a^2}{3\nu_m}+\frac{kT}{M}\nu_m\right]\partial_v S_2=-\left[\frac{a^2}{3\nu_m}+\frac{kT}{M}\nu_m\right]\partial_v S_1^2/2$$
$$-\left[\frac{2va}{3\nu_m}\frac{S_1}{g_1'}+\frac{1}{w^2f_0}\int_0^v\left[\frac{u^4}{3\nu_m}+\frac{d_u(u^3s/\nu_m)}{3}\frac{S_1}{g_1'}\right]f_0du\right]g_2' \quad \text{for } \delta^2$$

with solution

$$S_2=-S_1^2/2-\int_0^v\Phi^{(2)}(w)dw\,g_2' \; , \tag{36c}$$

where

$$\Phi^{(2)}(w)=\left[\frac{a^2}{3\nu_m}+\frac{kT}{M}\nu_m\right]^{-1}\left[\frac{2wa}{3\nu_m}\frac{S_1}{g_1'}+\frac{1}{w^2f_0}\int_0^w\left[\frac{u^4}{3\nu_m}+\frac{d_u(u^3a/\nu_m)}{3}\frac{S_1}{g_1'}\right]f_0du\right]$$

Using Eq. (36) and (35b) in Eq. (32) (and returning to the unprimed variables,), the hydrodynamic MKD in this range is obtained; namely,

$$f_M^{(1)}=nf_0\left[1-\frac{S^1}{g_1}g_1-\left[\int_0^v\Phi^{(2)}(w)dw\right]g_2+O(\delta^3)\right] . \tag{37}$$

This result [obtained from Eq. (35b)] is valid in the regime where each exponential factor (δ^iS_i) in the expansion given in Eq. (35a) is less than one. This requirement imposes on the magnitude of the g_i's conditions that are equivalent to those given by Eqs. (21) and (26). As mentioned in the discussion following Eq. (21), an approximate condition may be substituted for Eqs. (21) and (26); namely, $g_i<(qE/\bar{\varepsilon})^i$. The first term in Eq. (36c) cancels out the contribution to the $O(\delta^2)$ term in the expansion of f_M [Eq. (35b)] coming from the square of the $O(\delta)$ term. The resulting solution, given by Eq. (37), is linear in the g_i's and is the density gradient expansion for $f_M^{(1)}$ (see Sec. III A). Equation (37) is accurate to $O(\delta^3)$.

From Eq. (31b), the coefficients of the n, $\partial_z n$, and $\partial_z^2 n$ terms in Eq. (37) can be identified as the drift velocity W, diffusion coefficients D_0, and kurtosis coefficient D_1, respectively. They are explicitly given by

$$W=-\frac{4\pi}{3}\int\frac{u}{\nu_m}a\partial_u f_o u^2du \; , \tag{38a}$$

$$D_0=-\frac{4\pi}{3}\left[\int\frac{v^2}{\nu_m}f_0v^2dv-\int\frac{v}{\nu_m}a(\partial_v f_0S_1+f_0\partial_v S_1)v^2dv\right] , \tag{38b}$$

$$D_1=\frac{4\pi}{3}\left[\int\frac{v^2}{\nu_m}f_0S_1v^2dv+\int\frac{v}{\nu_m}a\left[\partial_v f_0\int_0^v\Phi^{(2)}(u)du+f_0\Phi^{(2)}(v)\right]v^2dv\right] . \tag{38c}$$

These results can readily be obtained by *a priori* assuming an expansion for f in the form of Eq. (37). The derivation given above indicates that such expansion is valid in the range where $\partial_z^i n/n$ is of $O(\delta^i)$, i.e., there is a definite ordering in the expansion (see discussion in Sec. III C). In the range where this ordering is no longer valid, it is necessary to obtain other solutions to Eq. (33) for f_M^1.

Consider next the regime where g_1 and g_2 are comparable to each order and are both of $O(\delta)$. Following the same procedure as used to obtain Eq. (37) from Eq. (34), the MKD is found to be in this regime,

$$f_M^1=nf_0\left[1-S_1^{\mathrm{DGE}}+\int_0^v\Psi^{(2)}(u)du\,g_2+O(\delta^2)\right] , \tag{39}$$

where

$$\Psi^{(2)}(u)=\left[\frac{a^2}{3\nu_m}+\frac{kT}{M}\nu_m\right]^{-1}\left[\frac{1}{u^2f_0}\int_0^u\frac{w^4}{3\nu_m}f_0dw\right]$$

and S_1^{DGE} given by Eq. (36b). Among the higher-order terms, there are some that are nonlinear in the g_i's. To $O(\delta^2)$ Eq. (39) is linear in the g_i's and differs from Eq. (37), to this order, by the term proportional to g_2. Using Eq. (39) in Eq. (31b), expressions for the transport parameters [to $O(\delta^2)$] can also be obtained:

$$W=v_d^{\mathrm{DGE}} \; ,$$

$$D_0=D_0^{\mathrm{DGE}} \; ,$$

$$D_1=\frac{4\pi}{3}\left[\int\frac{v^2}{\nu_m}f_0\frac{S^1}{g_1}v^2dv-\int\frac{v}{\nu_m}a\left[\partial_v f_0\int_0^v\Psi^{(2)}(u)du+f_0\Psi^{(2)}(v)\right]v^2dv\right] ,$$

where v_d^{DGE} and D_0^{DGE} are given by Eqs. (38a) and (38b), respectively. Equation (39) and the expressions for W

and D_0 have been obtained by Penetrante and Bardsley using a different procedure.[18]

It is important to note that the expressions for the current density [Eq. (31)] obtained using Eqs. (39) and (37) are not equal. To $O(\delta^2)$, Eq. (39) leads to a term proportional to $\partial_z^2 n$, whereas Eq. (37) does not. In order to obtain terms proportional to $\partial_z^2 n$ using Eq. (37), it is necessary to go to $O(\delta^3)$; in which case, additional terms also proportional to $\partial_z^2 n$ than those found in Eq. (39) come into play. Moreover, going to $O(\delta^3)$ with Eq. (39) brings in terms that are nonlinear in the g_i's. Such terms are not found in Eq. (37). As far as the transport parameters, Eqs. (39) and (37) yield the same drift velocity and diffusion coefficient, but different kurtosis, and higher-order coefficients.

B. Large density gradients

When the density gradient is sufficiently large that terms proportional to g_1 approach $O(1)$, while terms proportional to g_i, $i \neq 1$, are of $O(\delta)$, the equation for f_0 is found from Eq. (34) to be

$$\partial_v \left[v^2 \left[\frac{a^2}{3\nu_m} + \frac{kT}{M}\nu_m \right] \partial_v f_0 + v^3 \left[\frac{2g_1 a}{3\nu_m} + \frac{m}{M}\nu_m \right] f_0 \right] = -g_1 \frac{d_v(v^3 a/\nu_m)}{3} f_0 \; . \quad (40)$$

A solution to this equation for a given velocity dependence of ν_m may be very difficult. It is however possible to solve this equation numerically and used to tabulate J as a function of a and g_1 for use with Eq. (2a). An approximate analytic solution can be obtained by treating the rhs of Eq. (40) as a perturbation, in which case, the solution becomes

$$f_0(v,g_1) = C_0(1-Q_0)\exp\left[-\int_0^v bv\,dv\right] , \quad (41a)$$

where

$$b = \frac{2g_1(a/3\nu_m)+(m/M)\nu_m}{(a^2/3\nu_m)+(kT/M)\nu_m} \quad (41b)$$

$$Q_0 = -g_1 \int_0^v \left[\exp\left[\int_0^v bu\,du\right]\right] \frac{\int_0^v \left[d_u(u^3/a\nu_m)\exp\left[-\int_0^u bw\,dw\right]\right] du}{3v^2[(a^2/3\nu_m)+(kT/M)\nu_m]} dv \quad (41c)$$

with the normalization condition $4\pi\int_0^\infty f_0 v^2 dv = 1$. Higher-order terms can similarly be included. The resulting expression for the distribution constitutes an expansion about a nonuniform state, that given by Eq. (41).

These results may be modified to account, in lowest order, for the effects of terms proportional to g_2 and $\partial_z g_1$. Following the procedure presented in Sec. II, with an alternate velocity transformation to Eq. (27a); namely,

$$\xi = v - (a^2/v_s^3) g_1$$

the lowest-order solution f_0 is found to be given by Eq. (41a) with a replaced by $(a+\eta)$, where η is defined by Eq. (28b) with v_z replaced by v. Letting $f_M^1 = n f_0(v,g_1)$ and in Eq. (4b), the current density and the transport parameters can be determined. This is explicitly carried out in the next section for the case of a constant collision frequency.

C. Explicit evaluation of distribution function and transport parameters: constant collision frequency ν_0.

In this case (and neglecting the contribution from Q_0), $\nu_m = \nu_0$ and Eq. (41a) becomes

$$f_0(v,g_1) = C_0 e^{-bv^2/2} , \quad (42a)$$

where

$$C_0 = (b/\pi)^{3/2} \quad (42b)$$

with b given by Eq. (41b). From Eqs. (42), (31), and (4b), the drift velocity and diffusion coefficient are found to be

$$W = a/\nu_{m_0} + O(\partial_z g_1) \quad (43a)$$

and

$$D_0 = D_{00}/(1+dg_1) , \quad (43b)$$

where

$$D_{00} = \frac{\frac{2}{3}(a/\nu_{m_0})^2 + (2kT/m)}{(m/M)\nu_{m_0}} \quad (43c)$$

and

$$d = a/(3\nu_{m_0}^2 m/M) \; . \quad (43d)$$

Note that, in lowest order, the drift velocity is not affected by the density gradient. The effect of the gradient is to enhance (decrease) the value of the diffusion coefficient in regions with negative (positive) density gradient. Moreover, the diffusion process is found to be nonlinear in g_1. The parameter d has the units of distance, and, from Eq. (43d), it corresponds to the distance covered in an energy exchange time $(m\nu/M)$ by a particle traveling at the drift velocity (a/ν). For $g_1 \ll d^{-1}$, Eq. (43b) can be expanded to yield $D_0 \simeq D_{00}(1-dg_1)$, which when used in the continuity equation [Eq. (2a)] results in a linear equation that contains terms proportional to the third space derivative. For $g_1 d \lesssim 1$, the full expression needs to be retained. This expression for D_0 renders the continuity equation [Eq. (2a)] nonlinear. Equation (2a) may be written in this case as

$$\partial_t n + W\partial_z n - \partial_z(D_0 \partial_z n) = 0 , \quad (44)$$

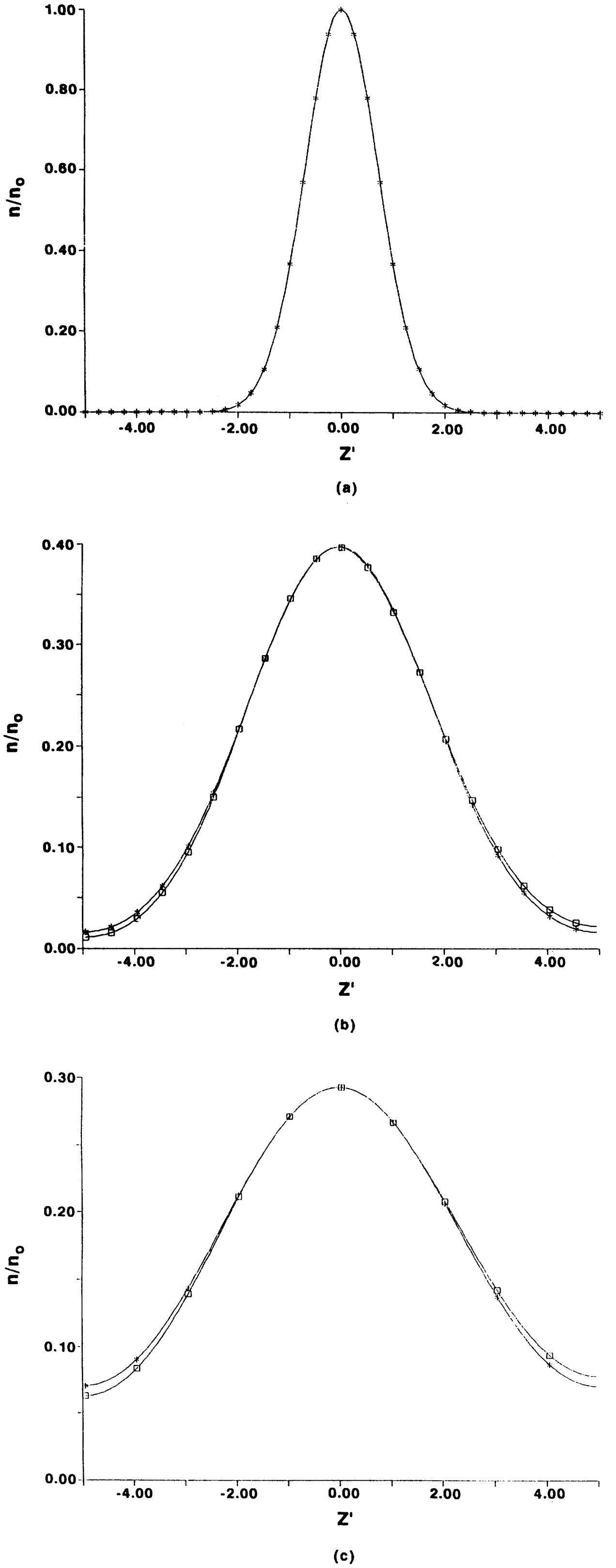

FIG. 1. Effect of nonlinear diffusion on the evolution of an initial Gaussian profile, $n_0\exp[-z/w)^2]$. The normalized time t' and position z' correspond to $D_{00}t/\omega^2$ and z/ω, respectively. The results for $d/\omega=0.25$ (indicated by *) and $d/\omega=0$ (indicated by □) are shown for t' equal to (a) 0, (b) 1.25, and (c) 2.5.

where W and D_0 are given by Eqs. (43a) and (43b) with D_{00} [Eq. (43c) being the linear diffusion coefficient], and d [Eq. (43d) playing the role of a nonlinearity parameter]. Equation (44) has a singularity at $d(\partial_z n/n)=-1$ and is valid in the regime $|d(\partial_z n/n)|<1$. To elucidate the effect of the nonlinearity, the time evolution of an initial Gaussian density profile, $n_0\exp[-(z/w)^2]$, has been calculated by a numerical solution of Eq. (44). The results are shown in Fig. 1 for $d/\omega=0.25$. For comparison, results with $d/\omega=0$ are also shown. A reference frame moving with the drift velocity has been chosen so that only diffusion effects are evident. The normalized time t' and position z' are given by $D_{00}t/\omega^2$ and z/ω, respectively. The effect of the nonlinearity is evident. As expected from the density dependence of D_0 [Eq. (43b)], the leading (trailing) edge of the pulse shows greater (lesser) spreading than for the case $d/\omega=0$. For a Gaussian profile, the effect of the nonlinearity is greater towards the tails of the profile where $\partial_z n/n$ is larger. Physically, the nonlinear diffusion process is due to the fact that the relative exchange of particles between adjacent cells (in space) progressively increases with velocity. This gives rise to a "shear" in particle flow in phase space. The next effect of the density transport (as discussed in Sec. II) is a relative increase (decrease) in the tail of the velocity distribution in regions with negative (positive) density gradients (in the presence of an electric field) and a diffusion coefficient that depends on the gradient [Eq. (43b)].

V. CONCLUDING REMARKS

The effect of density gradients on the transport of electron density has been analyzed using the concept of a MKD. With this approach, it has been possible to (1) derive the DGE in the regime of "small" density gradients without *a priori* assumptions about the form of the expansion, (2) determine the range of validity of the DGE (i.e., quantification of the term small), and (3) investigate the consequences of "large" density gradients. It has been found (Sec. III A) that in the regime where the normalized spatial derivatives of the density are small, the MKD takes the form of a DGE [Eq. (14)]. The regime of validity of the DGE has been determined and presented in Sec. III E. From these results, the term small has been explicitly defined. In particular, the DGE has been shown [Eqs. (24) and (29)] to be valid over most of the velocity range when the mean electron energy is less than the potential energy in a distance corresponding to the scale of density variation.

When the magnitude of the normalized derivatives of the density become significant over parts of the spatial domain of interest, (Sec. III C), the DGE is not uniformly valid over the domain. Procedures for obtaining MKD functions that are uniformly valid over larger spatial domains have been introduced in Sec. III C and III D. In particular, when the contribution from the density gradient becomes of the same order as that of the electric field an equation for the distribution has been obtained and formally solved [Eq. (20)]. The solution has the form of a DGE in terms of the spatial gradient. Unlike the DGE, its lowest-order term is space-time dependent. The specific form for these expressions has been obtained for the case of a quasi-Lorentz gas model (Sec. IV). It has been shown that, in lowest order, the effect of the nonlinearity is to enhance (decrease) the diffusion in regions with negative (positive) slopes. These effects are more likely to be experimentally observed for conditions that result in large values of the nonlinearity parameter d [Eq. (43d)], [it is evident from Eq. (44) that the magnitude of the nonlinear effect depends on both the magnitude of the density gradient and d]. The values for the nonlinearity parameter and the profile chosen in this paper are such that Eq. (25) is weakly nonlinear. To extend the results to higher values for d and/or the density gradients, higher-order terms [such as Q in Eq. (41a)] need to be included in Eqs. (43).

ACKNOWLEDGMENTS

The author would like to thank G. Vradis for assistance with the numerical calculations. This work is supported by the Office of Naval Research under Contract No. N00014-84-K-0537.

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