Markov processes in Hilbert space and continuous spontaneous localization of systems of identical particles

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Stochastic differential equations describing the Markovian evolution of state vectors in the quantum Hilbert space are studied as possible expressions of a universal dynamical principle. The general features of the considered class of equations as well as their dynamical consequences are investigated in detail. The stochastic evolution is proved to induce continuous dynamical reduction of the state vector onto mutually orthogonal subspaces. A specific choice, expressed in terms of creation and annihilation operators, of the operators defining the Markov process is then proved to be appropriate to describe continuous spontaneous localization of systems of identical particles. The dynamics obtained in such a way leaves practically unaffected the standard quantum evolution of microscopic systems and induces a very rapid suppression of coherence among macroscopically distinguishable states. The classical behavior of macroscopic objects as well as the reduction of the wave packet in a quantum measurement process can be consistently derived from the postulated universal dynamical principle.

I. INTRODUCTION

A. General considerations

The quantum description of physical phenomena meets some conceptual difficulties that motivate the uneasiness that many people feel with this theoretical scheme. The root of these difficulties can be traced back to the fact that the theory incorporates, at its axiomatic level, two different principles of evolution. The first principle, expressed by the Schrödinger equation, has to be applied in ordinary situations in which *a truly quantum system* evolves undisturbed. Such an evolution is deterministic and linear. The second principle has to be applied when a measurement takes place. The theory introduces a specific assumption to describe this situation, i.e., the postulate of wave-packet reduction (WPR), according to which the wave function undergoes a sudden stochastic change. As all of us have learned from the classic treatise[1] of Dirac, "a measurement always causes the system to jump into an eigenstate of the dynamical variable that is being measured." Note that, at the level of the normalized wave function, WPR is a nonlinear process.

The fundamental difficulty of the theory can then be simply summarized. If one takes into account that, after all, a measuring apparatus is a physical system made up of quantum constituents, one is led to pretend that the measuring process should be describable as the evolution of the closed physical system $S+A$, S and A denoting the measured system and the apparatus, respectively. The problem is that, in such a way, one gets, as the result of the measuring process, a superposition of macroscopically distinguishable states that appears to contradict both WPR, which provides a statistical mixture, and common sense. Removing such contradictions is the task of the theory of quantum measurement, but, in our opinion, as well as in other people's opinions,[2-5] a solution of the problem not implying some kind of breakdown of quantum mechanics in certain conditions does not seem to be possible.

The above features render the theory not fully internally consistent and partially ambiguous. On the one hand, it requires acceptance of the idea that there are systems that are not *truly* quantum mechanical, leading therefore to a dualistic attitude in the description of physical phenomena; on the other hand, due to the impossibility of a neat definition of the borderline between the two classes of physical systems, it compels the physicist to disregard from time to time the exact equations of the theory and to supplement them with vague verbal assertions.[6]

At a more philosophical level one can remark that the adoption of two different principles of evolution reflects the embarassing position of the orthodox quantum interpretation with respect to the problem of the subject-object distinction. The theory accepts in its postulates that some distinction must be made, but it cannot prescribe where or when to put it.[4]

An important step towards greater physical exactness and logical consistency would be made if one could devise a manner to account for the stochastic jumps mentioned by Dirac in terms of precise dynamical processes governed by definite mathematical equations. Needless to say, since these jumps violate both the deterministic and the linear nature of the Schrödinger equation, such a program requires one to accept a modification of it. Various attempts of introducing nonlinear and stochastic elements in the dynamical equations have been made. The crucial difficulty one has to face when trying to follow this line derives from the miraculous accuracy of the predictions of the theory for all microscopic systems. How can one devise modifications of the standard theory such that they have a negligible impact for small systems and at the same time are able to yield a fast dynamical suppression of the unwanted superpositions of macroscopically distinguishable states? Nevertheless, dynamical reduction models whereby a superposition of states continuously evolves into one of its terms have been considered by various authors.[7-12] These works have concentrated on achieving such an evolution, but they have left unsolved two basic problems.

(i) *The preferred-basis problem.* Which are the states to which the dynamical reduction process leads?

(ii) *The system-dependence problem.* How can the coherence suppressing process become more and more effective when going from microscopic to macroscopic systems?

A new approach to the problem of modifying the standard quantum dynamics has recently been proposed[13-15,5] We shall refer to it as quantum mechanics with spontaneous localization (QMSL). The model is based on the assumption that each constituent of any system suffers, at randomly distributed times with an appropriate mean frequency, a sudden collapse consisting in a localization of the wave function within an appropriate range. An important difference between dynamical reduction models and QMSL is that in QMSL finite changes of the state vector occur instantaneously –quantum jumps really take place, even though they are spontaneous and their occurrence does not require the intervention of any observer or the interaction with a macroscopic apparatus. However, the model gives a precise answer to the questions raised under (i) and (ii) above—the preferred basis is that of localized states for the constituents and the correct system dependence follows rigorously.

B. Features of QMSL

QMSL assumes that each particle of a system of n distinguishable particles labeled by index i experiences, with mean frequency λ^i, a sudden spontaneous localization described by[16]

$$|\psi\rangle \rightarrow |\psi_{\mathbf{x}}^i\rangle = L_{\mathbf{x}}^i|\psi\rangle \ . \tag{1.1}$$

In Eq. (1.1), where state vectors refer to the n-particle system, $L_{\mathbf{x}}^i$ is a norm reducing, positive, self-adjoint, linear operator representing the localization of particle i around point $\mathbf{x}$. It contains a parameter, having dimensions of length, that measures the size of the localization volume. In Refs. 13 and 14, $L_{\mathbf{x}}^i$ was chosen to be a Gaussian function, centered in $\mathbf{x}$, of the position operator of particle i; the length parameter was denoted by $1/\sqrt{\alpha}$. Since the process (1.1) does not conserve the norm, we rewrite it in the nonlinear norm-conserving form

$$\begin{aligned} &|\phi\rangle \rightarrow |\phi_{\mathbf{x}}^i\rangle = |\psi_{\mathbf{x}}^i\rangle / \||\psi_{\mathbf{x}}^i\|| \ , \\ &|\psi_{\mathbf{x}}^i\rangle = L_{\mathbf{x}}^i|\phi\rangle \ , \end{aligned} \tag{1.2}$$

and contextually assume that the probability density for the occurrence of $\mathbf{x}$ is

$$\mathcal{P}_i(\mathbf{x}) = \|\psi_{\mathbf{x}}^i\|^2 \ . \tag{1.3}$$

Assumption (1.3) requires that

$$\int d^3\mathbf{x} (L_{\mathbf{x}}^i)^2 = 1 \ . \tag{1.4}$$

The process we have described is a Markov process in Hilbert space. Due to the occurrence of sudden finite changes of the state vector, we call it a hitting process. It is to be noted that assumption (1.3), which makes the occurrence of the collapses more likely where the wave function is larger, is strictly analogous to the postulate about the probabilities of the outcomes of a measurement in standard quantum mechanics. As already stated, however, the QMSL collapses occur universally and spontaneously and do not require any actual measurement to be performed.

We draw attention to the following features of QMSL: The stochastic modification of the standard dynamics introduced by QMSL induces, according to the collapses that have occurred, the decomposition of a statistical ensemble described by a state vector into subensembles each described by a state vector. So, any member of the ensemble is associated at all times with a definite state vector. The possibility of associating with each individual closed system (in particular, with the system $S + A$ considered by measurement theory) a definite wave function at any time opens the way to the interpretation of the wave function itself as a real property of the system, laying the foundations of an objective description of physical phenomena.[6,12,15,5]

The collapses correspond to approximate localizations of the constituents. This corresponds to a definite choice of the preferred basis. It is remarkable that this choice by itself solves the problem of system dependence. In fact, each localization of a single constituent, when applied to a system for which the center-of-mass position is a collective variable (i.e., the wave function of the system is the product of a center-of-mass wave function times an internal wave function), is sufficient to localize the whole system. It follows that the frequency of the process for the whole system is the sum of the frequencies for the single constituents. This fact allows one to choose the parameters in such a way that the process is completely ineffective for microscopic systems; nevertheless, it is sufficient to quickly suppress coherence between distant states of macroscopic systems. It has also been shown[17] that different choices for collapse mechanisms (in particular, the assumption that they involve, besides position, also the momentum variable) do not share the cumulative properties of frequencies.

The QMSL model, as presented above, is consistent only in the case of systems of distinguishable particles. In fact, the QMSL collapses (1.1) or (1.2) do not preserve the symmetry properties of the state vector. A natural generalization of QMSL to the case of identical particles can be obtained[18] by using the whole set of positions of all particles to distinguish among different configurations of the system. Such a version of QMSL, however, does not appear to have nice features when translated into the language of second quantization. Another kind of QMSL model uses the set of densities around all points of space to discriminate among different configurations. These models will be described in Sec. IV C below, but the approach of Sec. III, also based on densities, will turn out to be much preferable.

C. CSL and other recent developments

It has been proved recently[19] that it is possible to devise a dynamical reduction model that exhibits all the appealing features of QMSL. It will be referred to as the continuous spontaneous localization (CSL) model. In CSL one assumes a stochastic evolution equation of the Ito form

$$d|\psi\rangle=[-iH\,dt+dh-\tfrac{1}{2}\overline{(dh)^2}]|\psi\rangle\ , \qquad (1.5)$$

where dh is a random, self-adjoint, linear operator. It is built up with the operators representing the density of particles around all points of space and contains in its definition (which will be given in Sec. III below) a length parameter $1/\sqrt{\alpha}$ and a strength parameter γ. The process defined by Eq. (1.5) does not conserve the norm. This fact needs an interpretation, which leads to the introduction of another process,

$$d|\phi\rangle=[-iH\,dt+dh_\phi-\tfrac{1}{2}\overline{(dh_\phi)^2}]|\phi\rangle\ , \qquad (1.6)$$

which is norm conserving and nonlinear, due to the dependence of dh_ϕ on $|\phi\rangle$. Equation (1.6) embodies an assumption concerning the probabilities to be assigned to the state vectors $|\phi\rangle$ that is the counterpart of the assumption (1.3) for a hitting process. The procedure leading from the process (1.5) to (1.6) will be explained in detail in Sec. II.

Stochastic equations having a formal structure of the type (1.6) have been considered in previous works,[9] but there the random terms appearing at the right-hand side (rhs) had a specific form devised to describe a specific measurement that was supposed to be performed. The considered equations, therefore, did not have the universal character of the processes envisaged in Refs. 13–15, 17–19, and here. Other recent investigations[20–22] deal with dynamical reduction models similar to the one considered in Ref. 19 and here. In Ref. 21 an equation very close to Eq. (1.6) is introduced (without deriving it from a linear process), but it is not specialized to the use of densities around space points to discriminate among different configurations. The idea of using densities is considered in Ref. 22, where, however, the dynamical equation has a more complicated structure than here.

D. Aims and contents of the present paper

This paper deals first with a new presentation and discussion of the class of Markov processes in Hilbert space to which the CSL theory belongs. Then, the general formalism is specialized to CSL in the framework of second quantization and the physical consequences of the theory are thoroughly investigated. Finally, the relationship between the classes of stochastic processes used by QMSL and CSL is discussed.

In Sec. II A, linear Markov processes in Hilbert space are introduced and the basic physical assumption concerning probabilities is stated and embodied into the evolution equation. Section II B is devoted to the general proof that the considered Markov processes lead to dynamical reduction of the state vector on the common eigenspaces of the operators that define the process. In Sec. II C, the equation for the statistical operator is derived.

Section III A defines the CSL theory in second-quantized form. Section III B investigates the physical consequences of CSL. First, it is shown that, under appropriate assumptions, the center-of-mass and the internal motions of a system decouple, the stochastic terms in the dynamical equation do not affect the internal structure and the center-of-mass wave function obeys a stochastic differential equation of the CSL type. Secondly, the rates of reduction on the approximate position eigenstates of a macroscopic object are evaluated and it is shown that it is possible to choose the two parameters defining the stochastic process in such a way that reduction is both effective for macroscopic objects and completely negligible for microscopic particles. Finally, considering together the localization process and the Schrödinger evolution, it is shown that the stochasticity introduced in the behavior of macroscopic bodies is also negligible.

The class of hitting processes (the discontinuous stochastic processes used in QMSL) is reconsidered in Sec. IV A. An appropriate infinite frequency limit, under which a hitting process reduces to a continuous process of the type introduced in Sec. II A, is discussed in Sec. IV B. Finally, two examples of hitting processes going into the CSL process defined in Sec. III A in the infinite frequency limit are presented in Sec. IV C. Some concluding remarks are contained in Sec. V.

II. MARKOV PROCESSES IN HILBERT SPACE

A. Raw and physical processes

In the Hilbert space, we consider the Markov process $|\psi_{\mathbf{B}}(t)\rangle$ satisfying the Ito stochastic differential equation

$$d|\psi\rangle=(C\,dt+\mathbf{A}\cdot d\mathbf{B})|\psi\rangle\ , \qquad (2.1)$$

where C is an operator, $\mathbf{A}\equiv\{A_i\}$ is a set of opertors, and $\mathbf{B}\equiv\{B_i\}$ is a real Wiener process such that

$$\overline{dB_i}=0\ ,\qquad \overline{dB_i dB_j}=\delta_{ij}\gamma\,dt\ , \qquad (2.2)$$

γ being a real constant; the dot product has the obvious meaning

$$\mathbf{A}\cdot d\mathbf{B}=\sum_i A_i dB_i \ . \tag{2.3}$$

The index i can be continuous, in which case the sum becomes an integral and the Kronecker δ becomes a Dirac δ. Given an initial state $|\psi(0)\rangle$, Eq. (2.1) generates at time t an ensemble of state vectors $|\psi_{\mathbf{B}}(t)\rangle$, where $\mathbf{B}$ denotes a particular realization $\mathbf{B}(t)$ of the Wiener process. To simplify notation, the dependence of $|\psi\rangle$ on t and $\mathbf{B}$ will be often dropped, as in Eq. (2.1). The process (2.1) and the ensemble generated by it will be called the raw process and ensemble. In the raw ensemble, each state vector $|\psi_{\mathbf{B}}(t)\rangle$ has the same probability as the particular realization $\mathbf{B}(t)$ that originates it through Eq. (2.1).

The raw process (2.1) does not conserve the norm, in general. In fact, using Ito calculus, one finds

$$\begin{aligned} d\|\psi\|^2 &= \langle \psi|d\psi\rangle+\langle d\psi|\psi\rangle+\overline{\langle d\psi|d\psi\rangle} \\ &= \langle \psi|(\mathbf{A}+\mathbf{A}^\dagger)|\psi\rangle\cdot d\mathbf{B}+\langle\psi|(C+C^\dagger)|\psi\rangle dt \\ &\quad +\langle\psi|\mathbf{A}^\dagger\cdot\mathbf{A}|\psi\rangle\gamma\, dt \ , \end{aligned} \tag{2.4}$$

where we have used the notation $d|\psi\rangle=|d\psi\rangle$. If the state vectors $|\psi_{\mathbf{B}}(t)\rangle$ were of norm 1, their probabilities, given by the raw ensemble, could naturally be interpreted as the physical probabilities. The vectors $|\psi_{\mathbf{B}}(t)\rangle$ being not of norm 1, let us consider the ensemble of the normalized vectors

$$|\chi_{\mathbf{B}}(t)\rangle=|\psi_{\mathbf{B}}(t)\rangle/\|\psi_{\mathbf{B}}(t)\| \ , \tag{2.5}$$

having the same probabilities as the corresponding vectors $|\psi_{\mathbf{B}}(t)\rangle$ [i.e., as the realizations $\mathbf{B}(t)$ of the Wiener process] and the ensemble of the normalized vectors

$$|\phi_{\mathbf{B}}(t)\rangle=|\psi_{\mathbf{B}}(t)\rangle/\|\psi_{\mathbf{B}}(t)\| \ , \tag{2.6}$$

whose probabilities are those of the vectors $|\psi_{\mathbf{B}}(t)\rangle$ times their squared norms $\|\psi_{\mathbf{B}}(t)\|^2$. We use different symbols for the vector functions $|\chi_{\mathbf{B}}(t)\rangle$ and $|\phi_{\mathbf{B}}(t)\rangle$, in spite of the fact that the right-hand sides of Eqs. (2.5) and (2.6) coincide, because their probabilities are different, so that as random vector functions they are different. In fact, as we shall see and as is obvious, they obey different stochastic differential equations. We choose as the physical probabilities those of the vectors (2.6) rather than those of the vectors (2.5). The ensemble of the vectors $|\phi_{\mathbf{B}}(t)\rangle$ and the stochastic process in the Hilbert space that generates it will be called the physical ensemble and process. As mentioned in the Introduction, the prescription leading to the physical ensemble is the counterpart of the assumption (1.3) of QMSL and of the postulate of standard quantum mechanics on the probabilities of the outcomes of a measurement.

Let us now investigate the relation between the raw and the physical processes. Indicating by $p(\mathbf{B}(t,t_0))$ the probability of the realization $\mathbf{B}(t,t_0)$ of the Wiener process or equivalently of the state vector $|\psi_{\mathbf{B}}(t)\rangle$ and by $q(\mathbf{B}(t,t_0))$ the probability of the state vector $|\phi_{\mathbf{B}}(t)\rangle$, one has by definition

$$q(\mathbf{B}(t,t_0))=\|\psi_{\mathbf{B}}(t,t_0)\|^2 p(\mathbf{B}(t,t_0)) \ . \tag{2.7}$$

It is easily shown that, because of linearity of Eq. (2.1) together with the Markov nature of the Wiener process $\mathbf{B}$, the procedure leading from the raw to the physical ensemble can be performed just at the considered final time or, in addition, any number of times between the initial and the final times. It follows that Eq. (2.7) can be substituted by its specialization to the infinitesimal time interval (t_0,t_0+dt), i.e.,

$$q(d\mathbf{B})=(1+d\|\psi\|^2)p(d\mathbf{B}) \ . \tag{2.8}$$

The possibility of considering the physical ensemble depends on fulfillment of the condition that the total probability associated with the distribution q is 1. This amounts to requiring that, for any $|\psi\rangle$, the average relative to the distribution p of the weighting factor $\|\psi\|^2$ is 1, i.e., $\overline{d\|\psi\|^2}=d\|\psi\|^2=0$. From Eq. (2.4), one finds

$$C+C^\dagger=-\gamma\,\mathbf{A}^\dagger\cdot\mathbf{A} \ . \tag{2.9}$$

When this condition is taken into account, denoting by $-iH$ the anti-Hermitian part of C, Eq. (2.1) becomes

$$d|\psi\rangle=(-iH\,dt+\mathbf{A}\cdot d\mathbf{B}-\tfrac{1}{2}\gamma\,\mathbf{A}^\dagger\cdot\mathbf{A}\,dt)|\psi\rangle \ . \tag{2.10}$$

This equation is of the form (1.5) for self-adjoint $\mathbf{A}$. Equation (2.4) simplifies to

$$d\|\psi\|^2=\langle\psi|(\mathbf{A}+\mathbf{A}^\dagger)|\psi\rangle\cdot d\mathbf{B} \ . \tag{2.11}$$

Then Eq. (2.8) becomes

$$q(d\mathbf{B})=(1+2\mathbf{R}\cdot d\mathbf{B})p(d\mathbf{B}) \ , \tag{2.12}$$

where

$$\mathbf{R}=\tfrac{1}{2}\langle\psi|(\mathbf{A}+\mathbf{A}^\dagger)|\psi\rangle \tag{2.13}$$

and the probability distribution q is normalized. Indicating by $d\mathbf{B}'$ the random variable whose distribution is q, one has

$$\begin{aligned} &\overline{dB_i'}=2R_i\gamma\,dt \ , \\ &\overline{dB_i'dB_j'}=\delta_{ij}\gamma\,dt \ , \end{aligned} \tag{2.14}$$

so that

$$d\mathbf{B}'=d\mathbf{B}+2\mathbf{R}\gamma\,dt \tag{2.15}$$

and $\mathbf{B}'$ is a diffusion process having the same diffusion as $\mathbf{B}$ and drift $2\mathbf{R}\gamma$. The meaning of the process $\mathbf{B}'$ and of its differential $d\mathbf{B}'$ follows from that of the probability distribution q that defines them. The set of all realizations $\mathbf{B}'(t)$ coincides with that of all realizations $\mathbf{B}(t)$ (in fact, both sets coincide with the set of all functions satisfying a given initial condition), but their probabilities, according to the definition (2.7) of q, are those of the physical ensemble instead of those of the raw ensemble. The stochastic differential equation for the physical process can now easily be written. We first write the equation for the process generating the normalized vectors $|\chi\rangle$. From Eqs. (2.10) and (2.11), by direct evaluation, one gets

$$d|\chi\rangle=[(-iH-\tfrac{1}{2}\gamma\,\mathbf{A}^{\dagger}\cdot\mathbf{A}-\gamma\,\mathbf{A}\cdot\mathbf{R}+\tfrac{3}{2}\gamma\mathbf{R}\cdot\mathbf{R})dt+(\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}]|\chi\rangle\,,\quad \mathbf{R}=\tfrac{1}{2}\langle\chi|(\mathbf{A}+\mathbf{A}^{\dagger})|\chi\rangle\,. \tag{2.16}$$

It is easily checked that Eq. (2.16) conserves the norm and that this feature does not depend on $\mathbf{B}$ having drift zero. The physical process is obtained by replacing to each realization $\mathbf{B}(t)$ of the random function $\mathbf{B}(t)$ an equal realization having the appropriate different probability, i.e., and equal realization $\mathbf{B}'(t)$ of the random function $\mathbf{B}'(t)$. This amounts to replacing $d\mathbf{B}$ by $d\mathbf{B}'$ in Eq. (2.16) so that we get

$$d|\phi\rangle=[(-iH-\tfrac{1}{2}\gamma\,\mathbf{A}^{\dagger}\cdot\mathbf{A}-\gamma\,\mathbf{A}\cdot\mathbf{R}+\tfrac{3}{2}\gamma\mathbf{R}\cdot\mathbf{R})dt+(\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}']|\phi\rangle\,,\quad \mathbf{R}=\tfrac{1}{2}\langle\phi|(\mathbf{A}+\mathbf{A}^{\dagger})|\phi\rangle\,. \tag{2.17}$$

It is convenient to rewrite Eq. (2.17) in terms of the original Wiener process $\mathbf{B}$. One gets

$$d|\phi\rangle=\{[-iH-\tfrac{1}{2}\gamma(\mathbf{A}^{\dagger}-\mathbf{R})\cdot\mathbf{A}+\tfrac{1}{2}\gamma(\mathbf{A}-\mathbf{R})\cdot\mathbf{R}]dt+(\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}\}|\phi\rangle\,,\quad \mathbf{R}=\tfrac{1}{2}\langle\phi|(\mathbf{A}+\mathbf{A}^{\dagger})|\phi\rangle\,. \tag{2.18}$$

We note that the equations for the norm-conserving processes (2.16) and (2.17) or (2.18), contrary to Eqs. (2.1) or (2.10), are nonliner.

The case in which $\mathbf{A}$ is a set of self-adjoint operators is of particular interest. In this case Eq. (2.18) becomes

$$d|\phi\rangle=\{[-iH-\tfrac{1}{2}\gamma(\mathbf{A}-\mathbf{R})^2]dt+(\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}\}|\phi\rangle\,,\quad \mathbf{R}=\langle\phi|\,\mathbf{A}|\phi\rangle\,. \tag{2.19}$$

This equation is of the form (1.6); it has been considered independently by Gisin.[20]

B. Reduction of the state vector

We shall now show that, in the case in which $\mathbf{A}$ is a set of commuting self-adjoint operators, the non-Schrödinger terms in Eq. (2.19) induce for large times the reduction of the state vector on the common eigenspaces of the operators $\mathbf{A}$. The inputs and the outcomes of the present section are essentially the same as those of the proofs of reduction given by one of the authors in a series of papers.[7,8,19] The proof given here is somewhat more direct in that it works on the squared amplitudes without making use of the Fokker-Planck equation for their probability density.

Since we are interested here in discussing the physical effects of the new terms, we disregard for the moment the Schrödinger part of the dynamical equation. Then Eq. (2.19) becomes simply

$$d|\phi\rangle=[-\tfrac{1}{2}\gamma(\mathbf{A}-\mathbf{R})^2dt+(\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}]|\phi\rangle\,,\quad \mathbf{R}=\langle\phi|\,\mathbf{A}|\phi\rangle\,. \tag{2.20}$$

Let us write

$$\mathbf{A}=\sum_{\sigma}\mathbf{a}_{\sigma}P_{\sigma}\,, \tag{2.21}$$

where the orthogonal projections P_σ sum up to the identity and it is understood that $\mathbf{a}_\sigma\neq\mathbf{a}_\tau$ (i.e., $a_{\sigma i}\neq a_{\tau i}$ for at least one value of i) for $\sigma\neq\tau$. We consider the real nonnegative variables

$$\langle\phi|P_\sigma|\phi\rangle=z_\sigma\,, \tag{2.22}$$

having the property

$$\sum_\sigma z_\sigma=1\,. \tag{2.23}$$

In terms of such variables, one finds

$$\mathbf{R}=\sum_\sigma\mathbf{a}_\sigma z_\sigma\,, \tag{2.24}$$

$$(\mathbf{A}-\mathbf{R})|\phi\rangle=\sum_\sigma\sum_\tau z_\tau(\mathbf{a}_\sigma-a_\tau)P_\sigma|\phi\rangle\,, \tag{2.25}$$

$$(\mathbf{A}-\mathbf{R})^2|\phi\rangle=\sum_\sigma\left[\sum_\tau z_\tau(\mathbf{a}_\sigma-\mathbf{a}_\tau)\right]^2P_\sigma|\phi\rangle\,. \tag{2.26}$$

It follows that the stochastic differential equation (2.20) can be written

$$dP_\sigma|\phi\rangle=\left[-\tfrac{1}{2}\gamma\left[\sum_\tau z_\tau(\mathbf{a}_\sigma-\mathbf{a}_\tau)\right]^2dt+\sum_\tau z_\tau(\mathbf{a}_\sigma-\mathbf{a}_\tau)\cdot d\mathbf{B}\right]P_\sigma|\phi\rangle\,. \tag{2.27}$$

Use of this equation in the relation

$$d\langle\phi|P_\sigma|\phi\rangle=\langle\phi|P_\sigma(dP_\sigma|\phi\rangle)+(d\langle\phi|P_\sigma)P_\sigma|\phi\rangle+\overline{(d\langle\phi|P_\sigma)(dP_\sigma|\phi\rangle)} \tag{2.28}$$

gives for the variables z_σ the set of stochastic differential equations

$$dz_\sigma=2z_\sigma\sum_\tau z_\tau(\mathbf{a}_\sigma-\mathbf{a}_\tau)\cdot d\mathbf{B}\,. \tag{2.29}$$

Qualitatively, Eq. (2.29) shows that the diffusion of $\{z_\sigma\}$ vanishes when $\{z_\sigma\}$ approaches the solutions of the set of equations

$$z_\sigma\sum_\tau z_\tau(\mathbf{a}_\sigma-\mathbf{a}_\tau)=0\,, \tag{2.30}$$

so that the values of $\{z_\sigma\}$ eventually accumulate towards such solutions. A formal proof of the fact that $\{z_\sigma\}$ asymptotically reduces to one of the solutions of Eq. (2.30) is easily obtained. From Eq. (2.29) one finds

$$dz_\sigma^2=2z_\sigma dz_\sigma+\left[2z_\sigma\sum_\tau z_\tau(\mathbf{a}_\sigma-\mathbf{a}_\tau)\right]^2\gamma\,dt \tag{2.31}$$

and in turn

$$d\overline{z_\sigma^2}=\overline{dz_\sigma^2}=\overline{\left[2z_\sigma\sum_\tau z_\tau(\mathbf{a}_\sigma-\mathbf{a}_\tau)\right]^2}\gamma\,dt\,. \tag{2.32}$$

It follows that

$$\frac{d}{dt}\overline{z_\sigma^2}\geq0\,. \tag{2.33}$$

This result, together with the boundedness property

$$\overline{z_\sigma^2} \leq 1 , \tag{2.34}$$

entails that, for $t \to \infty$,

$$\frac{d}{dt}\overline{z_\sigma^2} \to 0 . \tag{2.35}$$

Using again Eq. (2.32), we get

$$z_\sigma \sum_\tau z_\tau(\mathbf{a}_\sigma - \mathbf{a}_\tau) \to 0 . \tag{2.36}$$

It is shown in Appendix A that the only solutions of the set of equations (2.30) are of the form $z_1 = 0, \ldots, z_\sigma = 1, \ldots,$ corresponding to $|\phi\rangle$ lying in one of the common eigenspaces of the operators $\mathbf{A}$. Since Eqs. (2.27) do not change the Hilbert space ray to which each component $P_\sigma|\phi\rangle$ belongs, we conclude that $|\phi\rangle$ asymptotically reduces to one of its initial components $P_\sigma|\phi(0)\rangle$ times a real normalizing factor.

The probabilities of the various possible issues are also easily calculated. In fact, since $\overline{dz_\sigma} = \overline{dz_\sigma} = 0$, one has

$$\overline{z_\sigma} = z_\sigma(0) . \tag{2.37}$$

On the other hand,

$$\overline{z_\sigma} \to \mathrm{Prob}[z_\sigma(\infty) = 1] \tag{2.38}$$

so that one finds

$$\mathrm{Prob}[z_\sigma(\infty) = 1] = z_\sigma(0) , \tag{2.39}$$

i.e.,

$$\mathrm{Prob}[|\phi(\infty)\rangle \propto P_\sigma|\phi(0)\rangle] = \langle \phi(0)|P_\sigma|\phi(0)\rangle . \tag{2.40}$$

As one can see, this result is a direct consequence[7,8,19] of the martingale property $\overline{dz_\sigma} = 0$.

We have shown in this section that the non-Schrödinger terms in Eq. (2.19) produce in the long-time limit the reduction of the state vector on the common eigenspaces of the operators $\mathbf{A}$. The time rate of such a process and its competition with Schrödinger evolution are most easily studied in the framework of the statistical operator.

C. Statistical operator

The statistical operator corresponding to the physical ensemble and its evolution equation are easily obtained from the definition

$$\rho = \overline{\|\psi\|^2 \frac{|\psi\rangle}{\|\psi\|} \frac{\langle\psi|}{\|\psi\|}} = \overline{|\psi\rangle\langle\psi|} \tag{2.41}$$

and Eq. (2.10), or from

$$\rho = \overline{|\phi\rangle\langle\phi|} \tag{2.42}$$

and Eq. (2.18). Using once more the Ito calculus in evaluating $d\rho$, one gets

$$\frac{d\rho}{dt} = -i[H,\rho] + \gamma\, \mathbf{A}\rho\cdot\mathbf{A}^\dagger - \tfrac{1}{2}\gamma\{\mathbf{A}^\dagger\cdot\mathbf{A},\rho\} , \tag{2.43}$$

where the symbols [,] and { , } denote the commutator and the anticommutator, respectively. This is the Lindblad[23] form for the generator of a quantum dynamical semigroup. It is remarkable that the general Lindblad generator can be obtained from a stochastic process in Hilbert space. As we shall see in Sec. IV A hitting processes give rise to particular forms of the generator. The derivation of Eq. (2.43) we have presented provides a description of the ensemble to which $\rho(t)$ corresponds such that each member of it has a definite state vector at any time.

III. CONTINUOUS LOCALIZATION OF A SYSTEM OF IDENTICAL PARTICLES

A. Definition of the process

We now apply the general formalism introduced in Sec. II to the continuous spontaneous localization of a system of identical particles. Let us consider the creation and annihilation operators $a^\dagger(\mathbf{x},s)$, $a(\mathbf{x},s)$ of a particle at point $\mathbf{x}$ with spin component s satisfying canonical commutation or anticommutation relations. We define a locally averaged density operator,

$$N(\mathbf{x}) = \sum_s \int d^3\mathbf{y}\, g(\mathbf{y}-\mathbf{x}) a^\dagger(\mathbf{y},s) a(\mathbf{y},s) , \tag{3.1}$$

where $g(\mathbf{x})$ is a spherically symmetric, positive real function peaked around $\mathbf{x}=0$, normalized in such a way that

$$\int d^3\mathbf{x}\, g(\mathbf{x}) = 1 , \tag{3.2}$$

so that

$$\int d^3\mathbf{x}\, N(\mathbf{x}) = N , \tag{3.3}$$

N being the total number operator. The operators $N(\mathbf{x})$ are self-adjoint and commute with each other. In the following we choose

$$g(\mathbf{x}) = (\alpha/2\pi)^{3/2} \exp(-\tfrac{1}{2}\alpha \mathbf{x}^2) , \tag{3.4}$$

where α is a parameter such that $\alpha^{-3/2}$ represents essentially the volume over which the average is taken in the definition of $N(\mathbf{x})$. The improper vectors

$$|q,s\rangle = \mathcal{N} a^\dagger(\mathbf{q}_1,s_1) \ldots a^\dagger(\mathbf{q}_n,s_n)|0\rangle \tag{3.5}$$

are the normalized common eigenvectors of the operators $N(\mathbf{x})$ belonging to the eigenvalues

$$n(\mathbf{x}) = \sum_{i=1}^{n} g(\mathbf{q}_i - \mathbf{x}) . \tag{3.6}$$

We identify now, with reference to Sec. II A, the index i which labels the operators A_i with the space point $\mathbf{x}$ and the operators A_i with the density operators $N(\mathbf{x})$. Then Eq. (2.10) becomes

$$d|\psi\rangle = \Big[-iH\,dt + \int d^3\mathbf{x}\, N(\mathbf{x}) dB(\mathbf{x}) - \tfrac{1}{2}\gamma \int d^3\mathbf{x}\, N^2(\mathbf{x}) dt \Big] |\psi\rangle , \tag{3.7}$$

where

$$\overline{dB(\mathbf{x})}=0\ ,\qquad \overline{dB(\mathbf{x})dB(\mathbf{y})}=\gamma\delta^3(\mathbf{x}-\mathbf{y})dt\ . \tag{3.8}$$

This is, in a different notation, the process already considered in Ref. 19 for identical particles. The generalization to several kinds of particles is immediate. The process (3.7) can also be preented in terms of nonsingular, correlated random variables, as shown in Appendix B.

The equation for the statistical operator (2.43) reads in the present case

$$\frac{d\rho}{dt}=-i[H,\rho]+\gamma\int d^3\mathbf{x}\,N(\mathbf{x})\rho N(\mathbf{x}) -\tfrac{1}{2}\gamma\left\{\int d^3\mathbf{x}\,N^2(\mathbf{x}),\rho\right\}. \tag{3.9}$$

In the representation of vectors (3.5), Eq. (3.9) becomes

$$\frac{d}{dt}\langle q',s'|\rho|q'',s''\rangle=-i\langle q',s'|[H,\rho]|q'',s''\rangle +\gamma\sum_{i,j}[G(\mathbf{q}_i'-\mathbf{q}_j'')-\tfrac{1}{2}G(\mathbf{q}_i'-\mathbf{q}_j') -\tfrac{1}{2}G(\mathbf{q}_i''-\mathbf{q}_j'')] \times\langle q',s'|\rho|q'',s''\rangle\ . \tag{3.10}$$

where

$$G(\mathbf{y}'-\mathbf{y}'')=\int d^3\mathbf{x}\,g(\mathbf{y}'-\mathbf{x})g(\mathbf{y}''-\mathbf{x}) =(\alpha/4\pi)^{3/2}\exp[-\tfrac{1}{4}\alpha(\mathbf{y}'-\mathbf{y}'')^2]\ . \tag{3.11}$$

For a single particle, Eq. (3.10) reduces to

$$\frac{d}{dt}\langle \mathbf{q}'|\rho|\mathbf{q}''\rangle =-i\langle \mathbf{q}'|[H,\rho]|\mathbf{q}''\rangle +\gamma(\alpha/4\pi)^{3/2} \times\{\exp[-\tfrac{1}{4}\alpha(\mathbf{q}'-\mathbf{q}'')^2]-1\}\langle \mathbf{q}'|\rho|\mathbf{q}''\rangle\ . \tag{3.12}$$

We note that, taking

$$\lambda=\gamma(\alpha/4\pi)^{3/2}\ , \tag{3.13}$$

this equation coincides with the equation for a single particle considered in Ref. 14.

B. Physical consequences

1. Separation of the center-of-mass motion

We discuss the physical implications of the modified dynamical equation (3.7) under the assumption that the order of magnitude of the length parameter $\alpha^{-1/2}$ is such that it can reasonably be admitted that the internal wave function of a macroscopic body is sharply localized with respect to $\alpha^{-1/2}$.

Let $\mathbf{Q}$ be the center-of-mass (c.m.) coordinate of the system of identical particles that constitutes the considered macroscopic body,

$$\mathbf{Q}=\frac{1}{n}\sum_{i=1}^{n}\mathbf{q}_i\ , \tag{3.14}$$

and write

$$\mathbf{q}_i=\mathbf{Q}+\tilde{\mathbf{q}}_i\ . \tag{3.15}$$

The coordinates $\tilde{\mathbf{q}}_i$ with respect to the c.m. sum up to zero, so that they are functions of $3N-3$ independent internal variables, which we indicate by r. The internal variables r, together with the c.m. coordinate $\mathbf{Q}$, are functions of the coordinates $\mathbf{q}_i$. The internal variables, as defined here, describe also rotations of the n-particle system. We arbitrarily assume tha the orientation of the system is sharply defined in the wave function similarly to the truly internal structure. Instead, one should consider three orientation variables, to be treated along the same lines as the c.m. coordinate, and $3N-6$ truly internal variables, to be assumed sharply localized in the wave function. The problem would then be considerably more complicated without gaining very much as regards physical insight. So, we consider the wave function

$$\psi(q,s)=\Psi(\mathbf{Q})\chi(r,s)\ ,\qquad \chi(r,s)=\begin{bmatrix}S\\A\end{bmatrix}\Delta(r,s)\ , \tag{3.16}$$

where the symbol $\binom{S}{A}$ means symmetrization or antisymmetrization with respect to interchanges of the arguments $(\mathbf{q}_i,s_i)$. The wave functions Ψ and χ are understood to be separately normalized. The function $\Delta(r,s)$ is assumed to be sharply (with respect to $\alpha^{-1/2}$) peaked around the value r_0 of r.

The action of the operator $N(\mathbf{x})$ on the wave function (3.16) is easily worked out. One finds

$$N(\mathbf{x})\Psi(\mathbf{Q})\chi(r,s) =\Psi(\mathbf{Q})\begin{bmatrix}S\\A\end{bmatrix} \times\sum_i(\alpha/2\pi)^{3/2} \times\exp\{-\tfrac{1}{2}\alpha[\mathbf{Q}+\tilde{\mathbf{q}}_i(r)-\mathbf{x}]^2\}\Delta(r,s)\ . \tag{3.17}$$

According to our assumption, the factor in front of the function Δ varies much more slowly than Δ itself, so that we can take $r=r_0$ in the factor. In other words, we treat the factor as if $\Delta(r,s)$ were of the form $\delta^{3N-3}(r-r_0)\xi(s)$. Then

$$N(\mathbf{x})\Psi(\mathbf{Q})\chi(r,s)=F(\mathbf{Q}-\mathbf{x})\Psi(\mathbf{Q})\chi(r,s)\ , \tag{3.18}$$

where

$$F(\mathbf{Q}-\mathbf{x})=\sum_i(\alpha/2\pi)^{3/2}\exp\{-\tfrac{1}{2}\alpha[\mathbf{Q}+\tilde{\mathbf{q}}_i(r_0)-\mathbf{x}]^2\}\ . \tag{3.19}$$

According to Eq. (3.18), the operator $N(\mathbf{x})$ acts only on the factor Ψ of ψ. As a consequence, in the assumption that

$$H=H_{\mathbf{Q}}+H_r\ , \tag{3.20}$$

if Ψ and χ satisfy the equations

$$d\Psi = \left[-iH_Q dt + \int d^3\mathbf{x}\, F(\mathbf{Q}-\mathbf{x}) dB(\mathbf{x}) - \tfrac{1}{2}\gamma \int d^3\mathbf{x}\, F^2(\mathbf{Q}-\mathbf{x}) dt \right] \Psi \,, \tag{3.21}$$

$$d\chi = -iH_r dt\, \chi \,, \tag{3.22}$$

respectively, the wave function (3.16) satisfies Eq. (3.7). It is seen that, under our assumptions, the c.m. and the internal motions decouple as in the absence of the stochastic terms in Eq. (3.7). Furthermore, the stochastic terms do not affect the internal structure, while the c.m. wave function obeys a stochastic differential equation, again of the type (2.10), whose consequences will be discussed below.

2. Reduction rates

The operators $F(\mathbf{Q}-\mathbf{x})$ appearing in Eq. (3.21), which correspond to the operators A_i of Eq. (2.10), are real functions of the c.m. position operator $\mathbf{Q}$. They are a set of commuting self-adjoint operators, so that, as we know from the results of Sec. II B, the non-Schrödinger terms in Eq. (3.21) induce the reduction of the state vector on the eigenvectors of the position $\mathbf{Q}$. Of course, such a process requires an infinitely long time, while, in finite times, only the reduction on approximate eigenstates of $\mathbf{Q}$ takes place. We discuss here the time rate of the localization process by studying the time dependence of the off-diagonal elements of the statistical matrix $\langle \mathbf{Q}'|\rho|\mathbf{Q}''\rangle$. Again, we disregard the effect of the Schrödinger term, this approximation being justified by the fact that, for the values of $|\mathbf{Q}'-\mathbf{Q}''|$ in which we are interested, the reduction process will turn out to be very fast.

Equation (2.43) becomes in the present case

$$\frac{\partial}{\partial t}\langle \mathbf{Q}'|\rho|\mathbf{Q}''\rangle = -\Gamma(\mathbf{Q}',\mathbf{Q}'')\langle \mathbf{Q}'|\rho|\mathbf{Q}''\rangle \,, \tag{3.23}$$

where

$$\Gamma(\mathbf{Q}',\mathbf{Q}'') = \gamma \int d^3\mathbf{x}\left[\tfrac{1}{2}F^2(\mathbf{Q}'-\mathbf{x}) + \tfrac{1}{2}F^2(\mathbf{Q}''-\mathbf{x}) - F(Q'-\mathbf{x})F(\mathbf{Q}''-\mathbf{x})\right] . \tag{3.24}$$

Equation (3.23) gives

$$\langle \mathbf{Q}'|\rho(t)|\mathbf{Q}''\rangle = \exp(-\Gamma t)\langle \mathbf{Q}'|\rho(0)|\mathbf{Q}''\rangle \,. \tag{3.25}$$

It is easily found that Γ is an even function of $\mathbf{Q}'-\mathbf{Q}''$. Since it is assumed that very many constituents of the considered body are contained in a volume $\alpha^{-3/2}$, we can use the macroscopic density approximation, consisting in replacing the sum by an integral in Eq. (3.19). Then one writes

$$F(\mathbf{Q}-\mathbf{x}) = \int d^3\tilde{\mathbf{y}}\, D(\tilde{\mathbf{y}})(\alpha/2\pi)^{3/2} \times \exp\left[-\tfrac{1}{2}\alpha(\mathbf{Q}+\tilde{\mathbf{y}}-\mathbf{x})^2\right] , \tag{3.26}$$

where $D(\tilde{\mathbf{y}})$ is the number of particles per unit volume in the neighborhood of the point $\mathbf{y}=\mathbf{Q}+\tilde{\mathbf{y}}$.

A further approximation, which we call the sharp scanning approximation, can be used, since we are not interested here in the details of the function Γ for $\mathbf{Q}'-\mathbf{Q}''\to 0$. The sharp scanning approximation consists in replacing the normalized Gaussian function appearing in Eq. (3.26) by the corresponding δ function. Then one has

$$F(\mathbf{Q}-\mathbf{x}) = D(\mathbf{x}-\mathbf{Q}) \,, \tag{3.27}$$

so that it results

$$\Gamma(\mathbf{Q}'-\mathbf{Q}'') = \gamma \int d^3\mathbf{x}\left[D^2(\mathbf{x}) - D(\mathbf{x})D(\mathbf{x}+Q'-\mathbf{Q}'')\right] , \tag{3.28}$$

where suitable changes of the integration variable have also been made. The physical meaning of Γ is easily understood by making reference to a homogeneous macroscopic body of density D_0. Then

$$\Gamma = \gamma D_0 n_{\text{out}} \,, \tag{3.29}$$

n_{out} being the number of particles of the body in the c.m. position $\mathbf{Q}'$ that do not lie in the volume occupied by the body in the c.m. position $\mathbf{Q}''$. The ratio between the macroscopic frequency (3.29) and the microscopic frequency (3.13) is $n_{\text{out}} D_0 (4\pi/\alpha)^{3/2}$.

The results (3.25) and (3.29) have to be compared with the result

$$\langle \mathbf{Q}'|\rho(t)|\mathbf{Q}''\rangle = \exp(-\lambda_{\text{macro}} t)\langle \mathbf{Q}'|\rho(0)|\mathbf{Q}''\rangle \,, \tag{3.30}$$

$$\lambda_{\text{macro}} = n\lambda \,, \tag{3.31}$$

valid for $|\mathbf{Q}'-\mathbf{Q}''| \gg \alpha^{-1/2}$, obtained in Ref. 14 for the case of distinguishable particles. We note that in the present case an additional factor $D_0(4\pi/\alpha)^{3/2}$ appears in the macro-to-micro ratio, but such a factor is multiplied by the number of uncovered particles n_{out} rather than by the total number n. Clearly, this is a consequence of indistinguishability of particles and of the choice of density as the dynamical variable governing the process. In Ref. 14, the length parameter $\alpha^{-1/2}$ was chosen to be of the order of 10^{-5} cm and the microscopic frequency λ was suggested to be of the order of $10^{-16}\ \text{s}^{-1}$ with the aim of obtaining $\lambda_{\text{macro}} \approx 10^{7}\ \text{s}^{-1}$ for a typical macroscopic number $n \approx 10^{23}$. We repeat here the same choice,

$$\alpha^{-1/2} \approx 10^{-5}\ \text{cm} \tag{3.32}$$

and look for a value of γ such that the macroscopic frequency Γ is again of the order of $10^{7}\ \text{s}^{-1}$ for $n_{\text{out}} \approx 10^{13}$. Since $D_0 \approx 10^{24}\ \text{cm}^{-3}$, we get

$$\gamma \approx 10^{-30}\ \text{cm}^3 s^{-1} \,, \tag{3.33}$$

corresponding, according to Eq. (3.13), to $\lambda \approx 10^{-17}\ \text{s}^{-1}$. This value is such that nothing changes in the dynamics of a microscopic particle even in the case in which it has an extended wave function.

3. Position and momentum spreads

According to Eq. (3.28) or to the original expression (3.24), the diagonal elements $\langle \mathbf{Q}|\rho|\mathbf{Q}\rangle$ of the statistical operator in the position representation are not affected by the reduction process, as a consequence of the process being a localization. Of course, this does not mean that the time evolution of $\langle \mathbf{Q}|\rho|\mathbf{Q}\rangle$ is the same as given by the pure Schrödinger dynamics, because this dynamics is different, for mixtures of localized states, from that for nonlocalized states. So, some changes are expected in the time dependence of both position and momentum spreads, as a consequence of the presence of the localization process. An explicit evaluation of these effects is necessary in order to check that no unacceptable behavior arises.

The equation for the statistical operator, in operator form, is written

$$\frac{d}{dt}\rho = -i[H,\rho] + \gamma \int d^3\mathbf{x}[F(\mathbf{Q}-\mathbf{x})\rho F(\mathbf{Q}-\mathbf{x}) - \tfrac{1}{2}\{F^2(\mathbf{Q}-\mathbf{x}),\rho\}] , \quad (3.34)$$

where we retain now the Schrödinger term. We consider the case of a free macroscopic body, so that, in our notation, $H = P^2/2(M\hbar)$, M being the total mass. For a dynamical variable S, we define the mean value

$$\langle S \rangle = \mathrm{tr}(S\rho) . \quad (3.35)$$

The time derivative of $\langle S\rangle$, according to Eq. (3.34), is given by

$$\frac{d}{dt}\langle S \rangle = -i\,\mathrm{tr}([S,H]\rho) + \gamma \int d^3\mathbf{x}\,\mathrm{tr}\{[F(\mathbf{Q}-\mathbf{x})SF(\mathbf{Q}-\mathbf{x}) - \tfrac{1}{2}\{S,F^2(\mathbf{Q}-\mathbf{x})\}]\rho\} . \quad (3.36)$$

From Eq. (3.36), through tedious but elementary calculations, one gets

$$\frac{d}{dt}\langle Q_i \rangle = \frac{1}{M}\langle P_i \rangle , \quad (3.37a)$$

$$\frac{d}{dt}\langle P_i \rangle = 0 , \quad (3.37b)$$

and

$$\frac{d}{dt}\langle Q_i^2 \rangle = \langle Q_i P_i + P_i Q_i \rangle , \quad (3.38a)$$

$$\frac{d}{dt}\langle Q_i P_i + PQ_i \rangle = 2\frac{1}{M}\langle P_i^2 \rangle , \quad (3.38b)$$

$$\frac{d}{dt}\langle P_i^2 \rangle = \tfrac{1}{2}\gamma\delta_i\hbar^2 , \quad (3.38c)$$

where

$$\delta_i = \int d^3\mathbf{y} \left[\frac{\partial F(\mathbf{y})}{\partial y_i}\right]^2 . \quad (3.39)$$

System (3.37) is the same as in the case of the pure Schrödinger (PS) evolution, so that

$$\langle Q_i \rangle = \langle Q_i \rangle_0, \quad \langle P_i \rangle = \langle P_i \rangle_0 , \quad (3.40)$$

where the suffix 0 indicates the PS solution satisfying the same initial conditions. System (3.38) differs from the corresponding PS system for the nonzero r.h.s. in Eq. (3.38c). One easily finds

$$\langle Q_i^2 \rangle = \langle Q_i^2 \rangle_0 + \gamma\delta_i \frac{\hbar^2}{6M^2} t^3 , \quad (3.41a)$$

$$\langle Q_i P_i + P_i Q_i \rangle = \langle Q_i P_i + P_i Q_i \rangle_0 + \gamma\delta_i \frac{\hbar^2}{2M} t^2 , \quad (3.41b)$$

$$\langle P_i^2 \rangle = \langle P_i^2 \rangle_0 + \gamma\delta_i \frac{\hbar^2}{2} t . \quad (3.41c)$$

The new feature of Eqs. (3.41) with respect to the standard case is the momentum diffusion coefficient $\frac{1}{2}\gamma\delta_i\hbar^2$ appearing in the third equation. The extra terms in the two first equations simply reflect the consequences of momentum diffusion on the other dynamical variables through the standard evolution. The same set of equations was obtained in Ref. 14 in the case of distinguishable particles, the factor $\gamma\delta_i$ being replaced there by $n\lambda\alpha$.

To evaluate the quantities δ_i, the sharp scanning approximation is no longer sufficient, because here the derivative of the function F is required. We then use the macroscopic density approximation (3.26). For definiteness and simplicity, we make reference to a homogeneous macroscopic parallelepiped of density D_0 having edges of lengths L_i parallel to the coordinate axes. Then, as shown in Appendix C, one has with high accuracy

$$\delta_i = (\alpha/\pi)^{1/2} D_0^2 S_i , \quad (3.42)$$

where $S_i = L_1 L_2 L_3 / L_i$ is the transverse section of the macroscopic parallelepiped.

If the choice (3.32) and (3.33) is used for α and γ together with $D_0 \approx 10^{24}$ cm^{-3}, one gets from Eq. (3.42) for the momentum diffusion coefficient,

$$\tfrac{1}{2}\gamma\delta_i\hbar^2 \approx 10^{-32}\ (\mathrm{g\,cm\,s^{-1}})^2\,\mathrm{s}^{-1} S_i\,\mathrm{cm}^{-2} . \quad (3.43)$$

For an ordinary macroscopic body, this value appears too small to give detectable effects. For a very small macroscopic particle, due to the $1/M^2$ factor in the extra term of Eq. (3.41a), a non-negligible stochasticity could appear. For $L_j \approx 10^{-4}$ cm [this is the smallest order of magnitude for which the approximations leading to Eq. (3.42) remain valid], a time of the order of 10^2 s is required to make the extra term of the order of 10^{-10} cm^2. We do not know whether this kind of effect could be used to provide a significant experimental bound on the product $\gamma\sqrt{\alpha}$ contained in the momentum diffusion coefficient. We note, however, that the value (3.42) could overestimate δ_i, because of the assumption of a rectangular profile for density.

IV. CONNECTION BETWEEN CONTINUOUS AND HITTING PROCESSES

A. Hitting processes

We come now back to the type of stochastic process in Hilbert space that was the basis of QMSL. Let us consider the collapse process

$$|\psi\rangle \rightarrow |\psi_{\mathbf{b}}^l\rangle = (\beta^l/\pi)^{K/4} \exp[-\tfrac{1}{2}\beta^l(\mathbf{A}^l - \mathbf{b})^2]|\psi\rangle , \quad (4.1)$$

where $\mathbf{A}^l=\{A_i^l;\ i=1,\ldots,K\}$ are commuting self-adjoint operators and $\mathbf{b}=\{b_i;\ i=1,\ldots,K\}$ are real random variables. Index l identifies the process (4.1) within a family of similar processes. The process (4.1) does not conserve the norm, so that we consider the process

$$\begin{aligned}&|\phi\rangle\to|\phi_{\mathbf{b}}^l\rangle=|\psi_{\mathbf{b}}^l\rangle/\||\psi_{\mathbf{b}}^l\|\ ,\\&|\psi_{\mathbf{b}}^l\rangle=(\beta^l/\pi)^{K/4}\exp[-\tfrac{1}{2}\beta^l(\mathbf{A}^l-\mathbf{b})^2]|\phi\rangle\ ,\end{aligned} \tag{4.2}$$

and assume that the probability density for the occurrence of $\mathbf{b}$ is

$$\mathcal{P}_l(\mathbf{b})=\||\psi_{\mathbf{b}}^l\|^2\ . \tag{4.3}$$

The total probability is 1 since

$$(\beta^l/\pi)^{K/2}\int d^K\mathbf{b}\exp(-\beta^l\mathbf{b}^2)=1\ . \tag{4.4}$$

The process (4.2), in conjunction with the assumption (4.3), can be called the physical collapse process. We next assume that the lth physical collapse process takes place at random times according to a Poisson process with mean frequency μ^l, being understood that in the interval between two collapses the system evolves according to the Schrödinger equation. The stochastic process in Hilbert space constructed in such a way enjoys the Markov property but, contrary to the process considered in Sec. II A, is such that the state vector associated to a member of the statistical ensemble does not evolve continuously in time. We call it a hitting process.

Due to the random distribution in time of collapses, the evolution of the statistical ensemble is at any rate smooth, in the sense that the weight of any state vector is a differentiable function of time. In particular, a differential equation for the statistical operator can easily be written[13,14] and turns out to be

$$\frac{d\rho}{dt}=-i[H,\rho]+\sum_l\mu^l[T^l(\rho)-\rho]\ , \tag{4.5}$$

where

$$\begin{aligned}T^l(\rho)=(\beta^l/\pi)^{K/2}\int d^K\mathbf{b}\exp[-\tfrac{1}{2}\beta^l(\mathbf{A}^l-\mathbf{b})^2]\rho\\ \times\exp[-\tfrac{1}{2}\beta^l(\mathbf{A}^l-\mathbf{b})^2]\ .\end{aligned} \tag{4.6}$$

The generator appearing in Eq. (4.5) is a particular Lindblad generator. We note that the second of Eqs. (4.2) could be written more generally in the form $|\psi_{\mathbf{b}}^l\rangle=L_{\mathbf{b}}^l|\phi\rangle$, but, in order that the process have a physical interpretation as a hitting process, the operators $L_{\mathbf{b}}^l$ must satisfy the sum rule

$$\int d^K\mathbf{b}(L_{\mathbf{b}}^l)^\dagger L_{\mathbf{b}}^l=1\ .$$

This fact entails the simple structure number times ρ of the last term in Eq. (4.5). Therefore, contrary to the case of continuous processes considered in Sec. II, the most general Lindblad generator cannot be obtained from a hitting process.

B. Infinite frequency limit

It is natural to ask whether there is a relation between the hitting process described in the foregoing section and the continuous process considered in Sec. II A. The answer is affirmative. It is possible to show[24] that if one takes the infinite frequency limit

$$\mu^l\to\infty,\quad \beta^l\to 0,\quad \tfrac{1}{2}\mu^l\beta^l=\gamma\ , \tag{4.7}$$

the hitting process (4.2),(4.3) goes into the continuous process (2.19), with $\mathbf{A}$ having the structure

$$\mathbf{A}=\{A_i^l;\ l=1,2,\ldots;\ i=1,\ldots,K\}\ , \tag{4.8}$$

and similarly for $\mathbf{B}$.

Here we confine ourselves to notice that the statistical operator equation (4.5) goes, in the infinite frequency limit, into Eq. (2.43) with $\mathbf{A}^\dagger=\mathbf{A}$, $\mathbf{A}$ having the structure (4.8). In fact, as it is shown in Appendix D,

$$T^l(\rho)=\rho+\tfrac{1}{2}\beta^l\sum_i[A_i^l\rho A_i^l-\tfrac{1}{2}\{(A_i^l)^2,\rho\}]+O[(\beta^l)^2]\ , \tag{4.9}$$

so that, in the limit (4.7),

$$\frac{d\rho}{dt}=-i[H,\rho]+\gamma\sum_{l,i}[A_i^l\rho A_i^l-\tfrac{1}{2}\{(A_i^l)^2,\rho\}]\ . \tag{4.10}$$

This equation coincides with Eq. (2.43) with $\mathbf{A}^\dagger=\mathbf{A}$, $\mathbf{A}$ having the structure (4.8).

C. systems of identical particles.

We consider here two different instances of hitting processes inducing the localization of a system of identical particles, which go into the continuous process considered in Sec. III in the infinite frequency limit.

In the first instance,[25] a collapse process is considered at each space point $\mathbf{x}$,

$$\begin{aligned}&|\phi\rangle\to|\phi_z^{\mathbf{x}}\rangle=|\psi_z^{\mathbf{x}}\rangle/\||\psi_z^{\mathbf{x}}\|\ ,\\&|\psi_z^{\mathbf{x}}\rangle=[\beta(\mathbf{x})/\pi]^{1/4}\exp\{-\tfrac{1}{2}\beta(\mathbf{x})[N(\mathbf{x})-z]^2\}|\phi\rangle\ ,\end{aligned} \tag{4.11}$$

where $N(\mathbf{x})$ is the density operator defined by Eq. (3.1) and z is a real random variable. We then assume that the probability density for the occurrence of z given $\mathbf{x}$ is

$$\mathcal{P}_{\mathbf{x}}(z)=\||\psi_z^{\mathbf{x}}\|^2 \tag{4.12}$$

and that the process at $\mathbf{x}$ occurs with frequency density $\mu(\mathbf{x})$. The infinite frequency limit is defined by

$$\mu(\mathbf{x})\to\infty,\quad \beta(\mathbf{x})\to 0,\quad \tfrac{1}{2}\mu(\mathbf{x})\beta(\mathbf{x})=\gamma\ . \tag{4.13}$$

In the second instance, we consider the collapse process

$$\begin{aligned}&|\phi\rangle\to|\phi_{[n]}\rangle=|\psi_{[n]}\rangle/\||\psi_{[n]}\|\ ,\\&|\psi_{[n]}\rangle=W^{1/2}\exp\left[-\tfrac{1}{2}\beta\int d^3\mathbf{x}[N(\mathbf{x})-n(\mathbf{x})]^2\right]|\phi\rangle\ ,\end{aligned} \tag{4.14}$$

where $n(\mathbf{x})$ is a real random function. The collapse is assumed to occur with mean frequency μ and the functional probability density for the occurrence of $n(\mathbf{x})$ is assumed to be

$$\mathcal{P}([n])=\|\psi_{[n]}\|^2 \tag{4.15}$$

with respect to a suitably defined functional measure such that

$$W\int d[n]\exp\left[-\beta\int d^3\mathbf{x}[n(\mathbf{x})]^2\right]=1 \ . \tag{4.16}$$

The infinite frequency limit is defined by

$$\mu\to\infty, \quad \beta\to 0, \quad \tfrac{1}{2}\mu\beta=\gamma \ . \tag{4.17}$$

In the limits (4.13) or (4.17), both hitting processes just considered become the continuous physical process

$$\begin{aligned} d|\phi\rangle &= \left[\left[-iH-\tfrac{1}{2}\gamma\int d^3\mathbf{x}[N(\mathbf{x})-R(\mathbf{x})]^2\right]dt + \int d^3\mathbf{x}[N(\mathbf{x})-R(\mathbf{x})]dB(\mathbf{x})\right]|\phi\rangle \ , \\ R(\mathbf{x}) &= \langle\phi|N(\mathbf{x})|\phi\rangle \ , \end{aligned} \tag{4.18}$$

which corresponds to the raw process (3.7). The statistical operator obeys Eq. (3.9).

V. CONCLUSIONS

We have presented a mathematically precise and logically consistent modification of nonrelativistic quantum mechanics aiming at solving the problems raised by quantum measurement. The modification consists in superimposing upon the ordinary Schrödinger evolution a Markov process in Hilbert space giving rise to a continuous spontaneous localization of the wave function. The necessary requirements that are to be satisfied by any modified version of quantum mechanics are met by the theory. In fact, as shown in Sec. III, the proposed modification does not affect in any appreciable way the predictions of standard quantum mechanics for microscopic objects. On the other hand, when macroscopic systems are involved, the modification of the dynamical equation has, as its practically unique effect, that of inducing a very fast suppression of coherence among macroscopically distinguishable states.

The theory discussed here allows one to describe naturally quantum measurement processes by dynamical equations valid for all physical systems. It is worthwhile repeating, that, in this theoretical scheme, any member of the statistical ensemble has at all times a definite wave function. As a consequence, the wave function itself can be interpreted as a real property of a single closed physical system.

The continuous localization process considered in the present work constitutes progress with respect to the localization by a hitting process, both because it avoids the consideration of sudden collapses of the wave function and because it allows one to express synthetically the law of evolution in the form of a stochastic differential equation for the wave function. We note however that, as shown in Sec. IV, there are hitting processes having a physical content as close as one wants to a given continuous process of the type considered here. The challenging problem remains open of getting a satisfactory relativistic generalization of the present formalism.

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APPENDIX A

We are interested in solving the set of equations

$$z_\sigma\sum_\tau z_\tau(\mathbf{a}_\sigma-\mathbf{a}_\tau)=0 \tag{A1}$$

for real, non-negative values of the unknown variables z_σ, under the condition

$$\sum_\sigma z_\sigma=1 \ . \tag{A2}$$

Note that Eqs. (A1) are not independent, each being a consequence of the remaining ones.

It is immediately verified that, for any σ, the set of values $z_1=0,\ldots,z_\sigma=1,\ldots$ is an acceptable solution. It is also easily shown that such solutions are the only ones, provided $\mathbf{a}_\sigma\neq\mathbf{a}_\tau$ for $\sigma\neq\tau$. In fact, suppose that, say, $z_1\neq 0$ and $z_2\neq 0$. The two first equations of the set (A1) can then be written

$$\sum_\tau z_\tau(\mathbf{a}_1-\mathbf{a}_\tau)=0, \quad \sum_\tau z_\tau(\mathbf{a}_2-\mathbf{a}_\tau)=0 \ . \tag{A3}$$

By subtraction one gets

$$\sum_\tau z_\tau(\mathbf{a}_1-\mathbf{a}_2)=0 \ . \tag{A4}$$

Owing to the condition (A2), Eq. (A4) gives $\mathbf{a}_1=\mathbf{a}_2$, contrary to the hypothesis.

APPENDIX B

The process (3.7) can also be presented in terms of nonsingular, correlated, random variables $dw(\mathbf{y})$. In fact, the second term at the r.h.s. of Eq. (3.7) can be written

$$dh=\sum_s\int d^3\mathbf{y}\,a^\dagger(\mathbf{y},s)a(\mathbf{y},s)dw(\mathbf{y}) \ , \tag{B1}$$

where

$$dw(\mathbf{y})=\int d^3\mathbf{x}\,g(\mathbf{y}-\mathbf{x})dB(\mathbf{x}) \ . \tag{B2}$$

The variables $dw(\mathbf{y})$ have the properties

$$\begin{aligned} &\overline{dw(\mathbf{y})}=0 \ , \\ &\overline{dw(\mathbf{y})dw(\mathbf{y}')}=\gamma\int d^3\mathbf{x}\,g(\mathbf{y}-\mathbf{x})g(\mathbf{y}'-\mathbf{x})dt \\ &\qquad =\gamma G(\mathbf{y}-\mathbf{y}')dt \ , \end{aligned} \tag{B3}$$

where the function G is given by Eq. (3.11). The last term at the r.h.s. of Eq. (3.7) is $-\frac{1}{2}$ times

$$\overline{(dh)^2}=\gamma \sum_{s,s'} \int d^3\mathbf{y}\, d^3\mathbf{y}'\, a^\dagger(\mathbf{y},s)a(\mathbf{y},s)a^\dagger(\mathbf{y}',s') \times a(\mathbf{y}',s')G(\mathbf{y}-\mathbf{y}')dt \ . \tag{B4}$$

APPENDIX C

We evaluate here the factor (3.39) for the case of a homogeneous macroscopic parallelepiped. In the considered case, the function F in the macroscopic density approximation (3.26) is given by

$$\begin{aligned} F(\mathbf{y})&=D_0(\alpha/2\pi)^{3/2}\int_V d^3\tilde{\mathbf{y}} \exp[-\tfrac{1}{2}\alpha(\tilde{\mathbf{y}}+\mathbf{y})^2] \\ &=\tfrac{1}{8}D_0 \prod_{i=1}^{3} \{\mathrm{erf}[(\alpha/2)^{1/2}(y_i+L_i)] \\ &\qquad -\mathrm{erf}[(\alpha/2)^{1/2}y_i]\} \ . \end{aligned} \tag{C1}$$

Using this expression in Eq. (3.39), one finds

$$\delta_1=\tfrac{1}{8}D_0^2(1/\pi\alpha)^{1/2}[1-\exp(-\tfrac{1}{4}\alpha L_1^2)] \times E[(\alpha/2)^{1/2}L_2]E[(\alpha/2)^{1/2}L_3] \ , \tag{C2}$$

where

$$E(x)=\int_{-\infty}^{+\infty} dz[\mathrm{erf}(z+x)-\mathrm{erf}(z)]^2 \ . \tag{C3}$$

Since

$$x=(\alpha/2)^{1/2}L_i \gg 1 \ ,$$

we can take $E(x)=4x$, so that

$$\delta_1=(\alpha/\pi)^{1/2}D_0^2L_2L_3 \ . \tag{C4}$$

APPENDIX D

The operation $T^l(\rho)$ defined by Eq. (4.6) can be written

$$\begin{aligned} T(\rho)=&\exp(-\tfrac{1}{2}\beta\mathbf{A}^2) \\ &\times(\beta/\pi)^{K/2}\int d^K\mathbf{b}\exp(-\beta\mathbf{b}^2)\exp(\beta\mathbf{A}\cdot\mathbf{b})\rho \\ &\times\exp(-\beta\mathbf{b}^2)\exp(-\tfrac{1}{2}\beta\mathbf{A}^2) \ , \end{aligned} \tag{D1}$$

having suppressed the index l. Inserting the expansion

$$\exp(\beta\mathbf{A}\cdot\mathbf{b})=1+\beta(\mathbf{A}\cdot\mathbf{b})+\tfrac{1}{2}\beta^2(\mathbf{A}\cdot\mathbf{b})^2+\cdots \tag{D2}$$

into Eq. (D1) and taking into account that

$$(\beta/\pi)^{K/2}\int d^K\mathbf{b}\exp(-\beta\mathbf{b}^2)b_i=0 \ , \tag{D3a}$$

$$(\beta/\pi)^{K/2}\int d^K\mathbf{b}\exp(-\beta\mathbf{b}^2)b_ib_j=\frac{1}{2\beta}\delta_{ij} \tag{D3b}$$

$$(\beta/\pi)^{K/2}\int d^K\mathbf{b}\exp(-\beta\mathbf{b}^2)b_{i_1}b_{i_2}\cdots b_{i_n} = \begin{cases} 0 \ \ (\text{odd } n)\ , \\ O((\beta)^{-n/2}) \ \ (\text{even } n)\ , \end{cases} \tag{D3c}$$

one finds

$$\begin{aligned} T(\rho)=&\exp(-\tfrac{1}{2}\beta\mathbf{A}^2) \\ &\times\left[\rho+\tfrac{1}{2}\beta\sum_i(A_i\rho A_i+\tfrac{1}{2}\{A_i^2,\rho\})+O(\beta^2)\right] \\ &\times\exp(-\tfrac{1}{2}\beta\mathbf{A}^2) \ . \end{aligned} \tag{D4}$$

Finally, using the expansion

$$\exp(-\tfrac{1}{2}\beta\mathbf{A}^2)=1-\tfrac{1}{2}\beta\sum_i A_i^2+O(\beta^2) \tag{D5}$$

and restoring the index l, one gets Eq. (4.9).

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