Effect of anisotropy on the self-organized critical state

Imre M. Jánosi
Laboratory of Surface and Interface Physics, Eötvös University, Budapest, Múzeum krt. 6-8, 1088, Hungary
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We have performed computer simulations in order to investigate the effect of anisotropy in a continuous-energy model recently proposed by Y. C. Zhang [Phys. Rev. Lett. **63**, 470 (1989)]. We have found that the most anisotropy-sensitive characteristic function is the peaked energy distribution, and the other quantities (critical exponents, etc.) are affected by the anisotropy in the extreme cases only. We present some quantitative connections between the anisotropy parameter and the characteristic quantities up to five dimensions.

I. INTRODUCTION

Recently Bak, Tang, and Wiesenfeld[1-4] (BTW) have introduced the idea of self-organized criticality to explain the behavior of some far-from-equilibrium dynamical systems. These extended dissipative dynamical systems evolve into structures with long-range fractal spatial correlations[5,6] and/or long-range temporal correlations with a "$1/f$" power spectrum.[7] They have suggested that this behavior may be caused by the self-organization of the systems into a critical state. Systems which exhibit spatial or temporal power-law correlations naturally evolve into this critical state.[2] Tang and Bak[4] have pointed out the analogy with traditional critical phenomena by defining several critical exponents and deriving scaling relations between them. Hwa and Kardar[8] have performed a dynamic renormalization-group calculation to determine various critical exponents in $d \leq 4$ dimensions. They have found slightly different noise exponent values for the energy dissipation function defined by BTW in Ref. 1, and they have pointed out an important principle, namely, that the dynamics should satisfy a conversation law to ensure the self-similarity of the steady-state configurations. Dhar and Ramaswamy[9] have defined a variant of the BTW model and determined the critical exponents exactly in arbitrary dimension. They have found that their model in two dimensions is equivalent to a special case of directed percolation solved earlier.

Several real systems have been suggested as possible candidates for this behavior, for example, the light from quasars, the sand flow in an hourglass, the properties of earthquakes,[10] the flows of rivers such as the Nile, motion of dislocations in a resistor, raindrops running down a windowpane, or even interactive economical systems.[1,2] Jaeger, Liu, and Nagel[11] have performed direct experiments to describe the nature of sand flow along the free surface of sandpiles, and Jánosi and Horváth[12] have made measurements and simulations to gather quantitative information about the dynamics of water droplets on a windowpane. Although they have not found unambiguous evidence of the above-mentioned behavior, it seems that the idea may have widespread applications in the explanation of the ubiquitous power-law correlations in nature.

Zhang[13] has analyzed the scaling theory of self-organized critical phenomena on a continuous-energy model and estimated various exponents. He found that the energy is concentrated around a few discrete values. In dynamical equilibrium states there is a well-defined critical stored energy and it is distributed isotropically and homogeneously in space, even when energy input is not. We have intended to investigate the effect of anisotropy on the overall behavior, therefore we have performed detailed computer simulations. Section II will be concerned with the model and the definitions, and the results of the simulations will be presented in Sec. III. In Sec. IV we give a short summary of our observations.

II. MODEL

For the sake of clarity we describe briefly the Zhang model[13] and our modifications. Let us take a d-dimensional hypercubic lattice, where energy E can be stored on each site. At time t, an ε input energy is added to a randomly selected $\mathbf{r}$ site. E and ε have non-negative continuous values and the distribution of ε is arbitrarily chosen. There is a limit $E_{\max}$ on the allowed energy at any site. If at a given time the energy exceeds the limit value, an activation event occurs:

$$E(\mathbf{r},t+1) \longrightarrow E_{\min} , \tag{1}$$

$$E(\mathbf{n}_i,t+1) \longrightarrow E(\mathbf{n}_i,t) + \frac{E(\mathbf{r},t)-E_{\min}}{2d} , \tag{2}$$

where $\mathbf{r}$ denotes the place of the excited node, $\mathbf{n}_i$ denotes the ith nearest-neighboring site vector ($1 \leq i \leq 2d$). Without loss of generality, we take $E_{\min}=0$ and $E_{\max}=1$ throughout the main part of our investigations. The energy of the active site in the original model is evenly divided for the neighbors, so we call this case *isotropic* penetration. The transferred energy acts an input energy for the neighboring sites, so further activation events may occur. A single input energy may trigger off activation on a set of connected sites known as an activation cluster or avalanche.[1-4] The boundaries are opened, which means that the excess energy can flow out freely from the system. We adopt an adiabatic energy input condition: the system has to be allowed to relax before input energy

is again charged. After a long time the average stored energy reaches a plateau E_c (critical energy) and the system is then said to be in a dynamical equilibrium state.

We intend to examine the effect of anisotropy on the behavior. It is clear that the anisotropy itself would be very complicated, depending on the place and time, therefore we deal with the relatively simple case, the so-called *symmetrical, one-parameter anisotropy*. The symmetry of the anisotropy means that the amount of transported energy is not dependent on the sign of the direction of propagation, i.e., the $\mathbf{n}_+=(n_1,\ldots,n_i+1,\ldots,n_d)$ and the $\mathbf{n}_-=(n_1,\ldots,n_i-1,\ldots,n_d)$ neighbors get the same part of the energy. We would like to characterize the anisotropy with only one parameter. The simplest case is when we divide the d-dimensional lattice into a $(d-a)-$ and an a-dimensional sublattice where a is a positive integer constant and $1\le a\le d-1$. Then the source energy is divided differently for the two sublattices, but inside the sublattices the energy propagation is isotropic. So we can modify the second Zhang rule as follows:

$$E_a(i,t+1)\longrightarrow E_a(i,t)+\left[\frac{E(\mathbf{r},t)}{2d}-\kappa\frac{E(\mathbf{r},t)}{2d}\right], \tag{3a}$$

$$E_{d-a}(j,t+1)\longrightarrow E_{d-a}(j,t)+\left[\frac{E(\mathbf{r},t)}{2d}+\kappa\frac{E(\mathbf{r},t)}{2d}\frac{a}{(d-a)}\right], \tag{3b}$$

where the subscript denotes the transferred energy for one neighboring site in the appropriate sublattice. Here $0\le\kappa\le 1$ is the continuous *anisotropy parameter*, $\kappa=0$ belongs to the isotropic lattice, and $\kappa=1$ characterizes the fully anisotropical situation. In the latter case the original d-dimensional lattice is decoupled to $a-$ or $(d-a)$-dimensional isolated sublattices depending on the sign of the "anisotropy energies." In two dimensions we have only one possibility: we can divide the lattice into two one-dimensional sublattices. In three dimensions we have two different cases; the coordinate reduction may result in one-dimensional or two-dimensional decoupled sublattices. Generally, the number of different classes is $d-1$ under the above-mentioned conditions.

We have numerically measured the critical energy $E_c(\kappa,d)$. We adopted a slightly different method for the measurement, namely, we have computed the average stored energy in the dynamical equilibrium state, but in the *subcritical* regime, i.e., when there is no activation cluster in the system. This resulted in E_c's a little bit smaller than the values given by Zhang,[13] but so we could avoid the problem of the energy charge: in the *supercritical* regime the instantaneous average energy evidently depends on the value of ε. When this added energy is too high, the transient relaxation process may last a long time resulting in, e.g., $E_c>E_{\max}$ average values. For the same reason *the single-site energy distribution* $P(E)$ is measured in the subcritical range also.

We have checked one of the main critical exponents, namely, the τ cluster size distribution exponent obeys the relation $D(s)\sim s^{-\tau}$, where s is the number of activated sites in one avalanche. From the scaling theory[13] and other considerations[8,9] follows

$$\tau=2\left[1-\frac{1}{d}\right]. \tag{4}$$

It is important to note that in the above-mentioned theories there exists an upper critical dimension $d_c=4$ proved in a paper by Obukhov.[14] In $d\ge 4$ dimensions the scaling breaks down.

III. SIMULATION RESULTS

We performed simulations in isotropic and anisotropic systems. The lattices consisted of 8000–10 000 sites in 1–5 dimensions; the typical number of the Monte Carlo steps was 200 000. The parameters were the following: $E_{\min}=0, E_{\max}=1, 0<\varepsilon<1$ (in the main part of the work we used uniform ε distribution).

At the first step we evaluated the *isotropic* single-site energy distribution $P(E)$ in $d=1-5$ dimensions; the results are shown in Figs. 1(a)–(e). In the $d\le 4$ cases the distributions have clearly a peaked shape; there are $2d-1$ peaks for the hypercubic lattice with the isotropic nearest-neighbor interaction. The "backgrounds" (i.e., the occupation probabilities between two peaks) are larger in higher dimensions, and the peaks are wider in the higher energy regimes. We believe that this results in the decay of the peaked distribution in $d\ge 5$ dimensions. Figure 2 shows the peak positions against the lattice dimension. The positions of the middle peaks play a central role (as we see later), and these energy values are very close to the critical stored energy E_c. The fitting of the dimension dependence is the following:

$$E_{\text{middle peak}}(d)=\left(\tfrac{1}{2}+\tfrac{1}{6}d^{-1}\right)E_{\max}\approx E_c(d). \tag{5}$$

The positions of the other peaks correspond to the multiples of some quasiunit or energy quantum E_0:

$$E_0(d)=\frac{E_c(d)}{d}. \tag{6}$$

We plotted the calculated peak position in Fig. 2 as well; the agreement with the measured data is clear in $d\le 4$ dimensions.

The reason of the peaked energy distribution lies in some very robust self-averaging process. The places of the energy peaks depend on the $E_{\max}$ and $E_{\min}$ borders only, and are absolutely independent of the range or distribution of the loaded excess energy ε. For instance, when the input energy is charged in equal parts $\varepsilon_0\le E_{\max}$, the only change in the single-site stored energy distribution are some added δ peaks at the ε_0, $2\varepsilon_0$, $3\varepsilon_0,\ldots\le E_{\max}$ places.

When we "switch on" the anisotropy in the system, the most significant change occurs in the energy distribution. The peaks suffer splitting [Figs. 3(a)–3(d), and 4(a) and 4(b)], and the measure of splitting depends on the value of the anisotropy parameter κ. Figure 3(a) shows that a very low anisotropy in the energy propagation results in the change of the single-site energy distribution. The

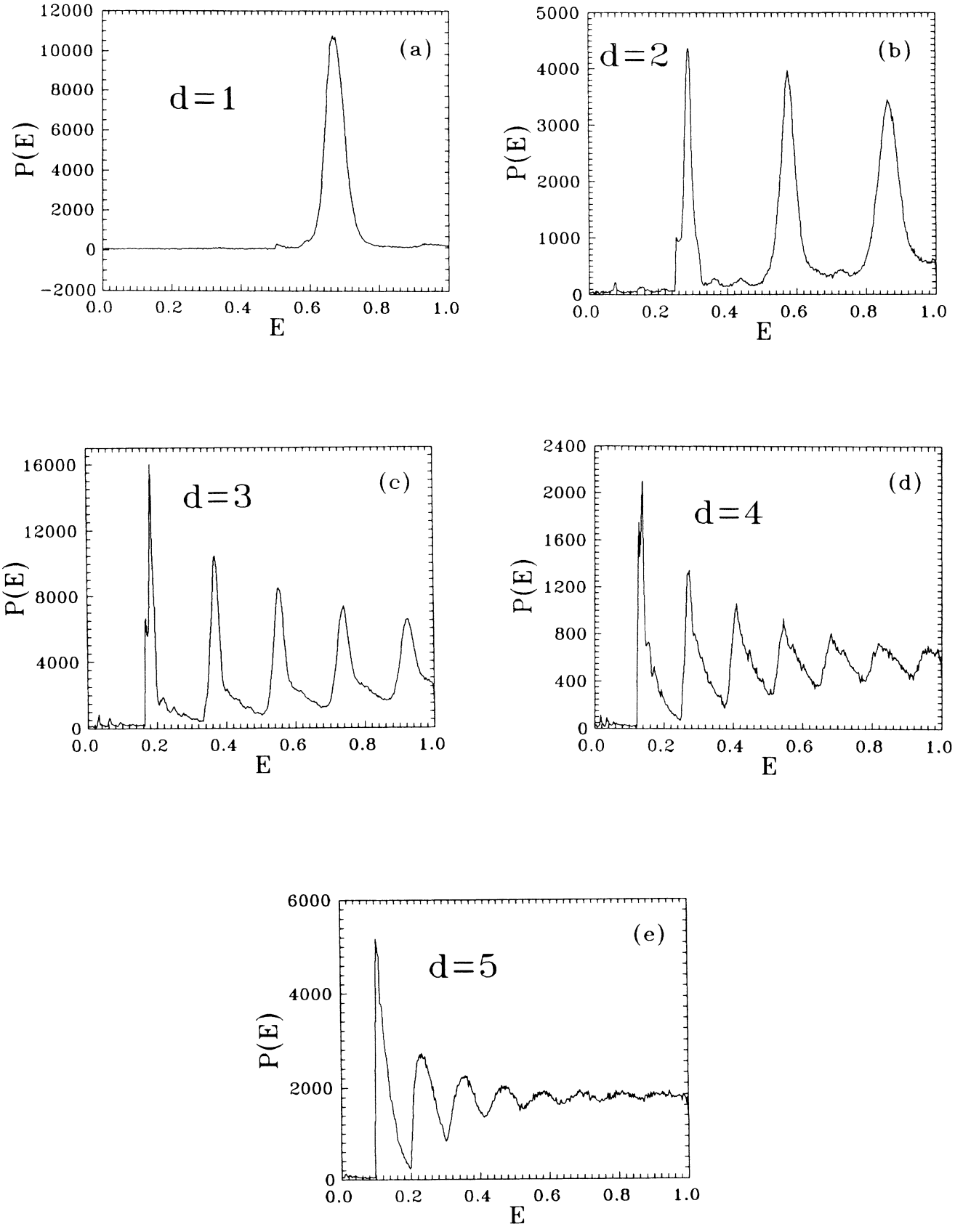

FIG. 1. (a)–(e): Single-site stored energy distribution measured in the subcritical range. $E_{max}=1$, $E_{min}=0, 0 \leq \varepsilon \leq 1$, $P(E)$ in arbitrary units.

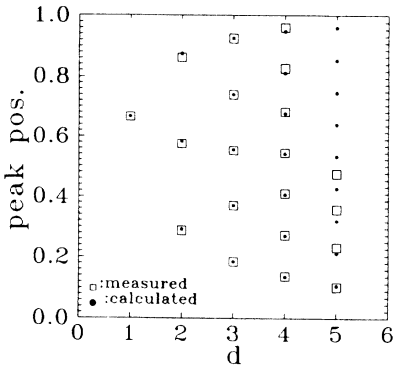

FIG. 2. Peak positions vs the lattice dimension. The calculated values are given by (5) and (6) (see text).

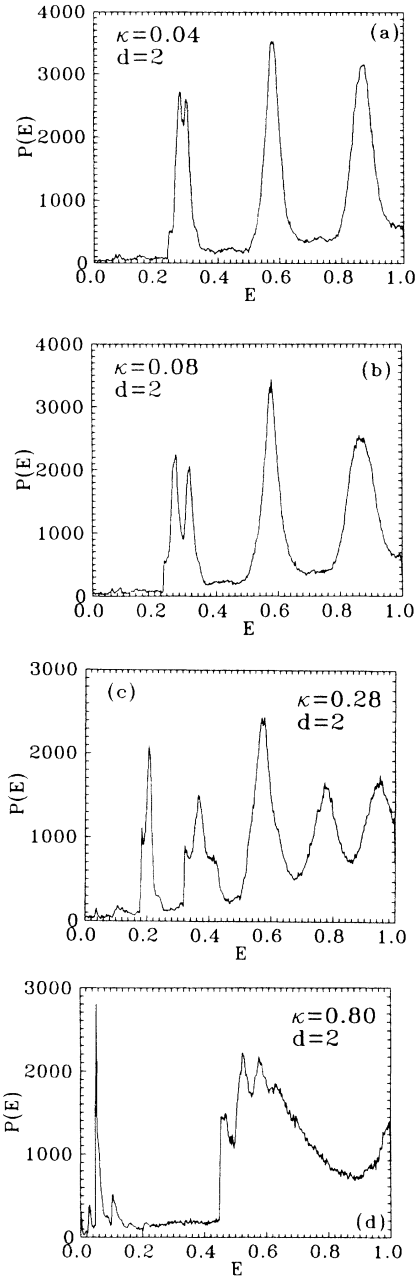

FIG. 3. (a)–(d): Single-site energy distribution at different anisotropy parameters in a two-dimensional lattice.

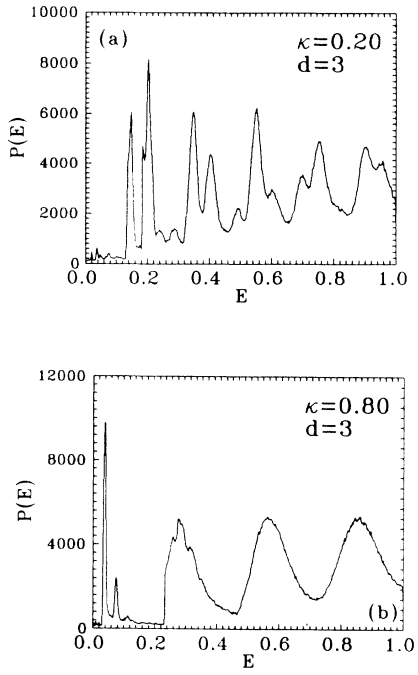

FIG. 4. (a) and (b): Single-site energy distribution at different anisotropy parameters in a three-dimensional lattice.

low-energy peaks split up earlier. When the anisotropy increases, the splitted subpeaks depart from the original position nearly symmetrically until they meet the neighboring subpeak and merge with it, or reach the border of the energy range and disappear from the distribution. It is important to note that the middle peak never splits up, but the position of this peak can slightly change according to the actual value of the average stored energy. The position of the largest peaks against the value of the anisotropy parameter κ [defined in (3)] is plotted in Fig. 5 (two-dimensional case); the solid line shows the value of the average stored energy E_c. In three dimension there are two different cases. We show in Fig. 6 the $a=1$ version, where the "easy directions" of the energy propagation are two-dimensional sublattices. We can conclude from these figures that the peak positions depend almost linearly on the anisotropy parameter. Figure 7 shows the

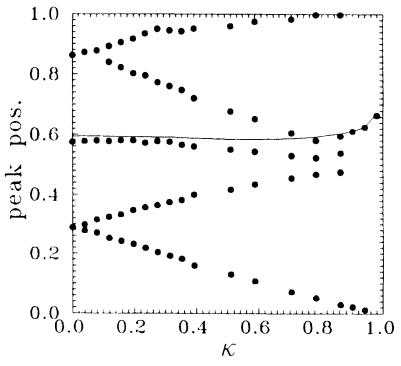

FIG. 5. Peak positions vs the anisotropy parameter in a two-dimensional lattice. The solid line is the average stored energy E_c.

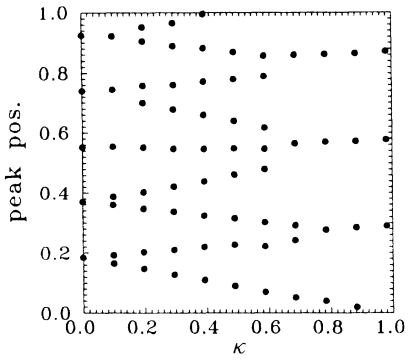

FIG. 6. Peak positions vs the anisotropy parameter in a three-dimensional lattice. Here $a=1$ [see (3)], i.e., the fully anisotropic case ($\kappa=1$) corresponds to totally decoupled two-dimensional sublattices.

height of the peaks against the positions for the two-dimensional case. (The figure consist of the $\kappa \leq 0.72$ points only, for clarity.) On the basis of the height data we can differentiate, e.g., the two-dimensional anisotropic distribution ($\kappa \approx 0.3$) from the three-dimensional isotropic case which have five peaks on the same places but with different height distribution.

We checked the above-mentioned critical exponent and the values for the isotropical cases are the following:

$$\tau \sim \begin{cases} 0 \text{ for 1D} \\ 1.2 \text{ for 2D} \\ 1.55 \text{ for 3D} \\ 1.9 \text{ for 4D} . \end{cases}$$

These values approximately obey the scaling rule

$$\tau = \frac{5}{2}\left[1-\frac{1}{d}\right], \tag{7}$$

which differs from Eq. (4) given by Zhang in Ref. 13. We note that the "cluster-size" here means the total number of elementary excitations or sliding events[1-3] or flip number defined by Kadanoff *et al.*[15] This is the most physically significant quantity in terms of energy dissipation; however, it may differ from the actual size of an avalanche (which we might call an "area" in the case of two dimensions), obviously. Grassberger and Manna performed large-scale simulations[16] for the original discrete BTW sandpile model and they obtained (in agreement with our observation) that the multiple activation event during one avalanche may play an important role in the energy propagation process, at least in the low-dimensional cases. From the other point of view, Zhang's considerations based on the condition of compactness, i.e., the area of the clusters obeys the $s \sim r^d$ relation, where r is some linear size and d is the spatial dimension. It seems that in the higher dimensions the clusters have fractal shapes rather than compact (e.g., Ref. 16 gives $d_f \approx 2.85$ fractal dimension for the three-dimensional clusters). The deviation of our cluster size exponents from the literature might be based on the differences of the models,[15] on the role of the multiple excitations and/or the fractal shape of the clusters,[16] or even on the differences of the cluster size definitions.

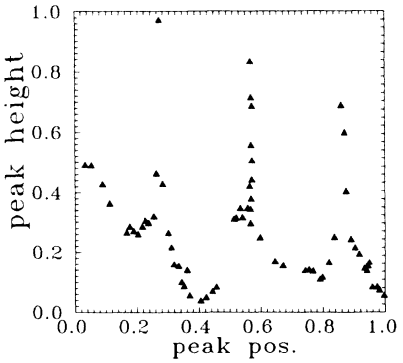

FIG. 7. Peak height vs the peak position in a two-dimensional lattice at different anisotropy parameters. (The three-dimensional case is very similar, but it consists of five peaks in the isotropic starting position.)

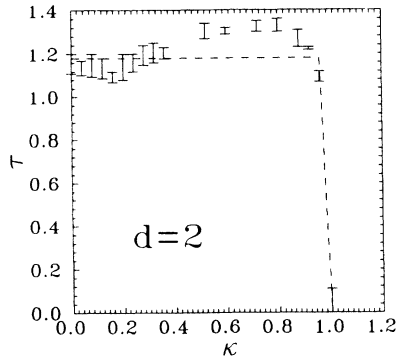

FIG. 8. Critical exponents of the two-dimensional cluster size distributions at different anisotropy parameters. (The dashed lines only guide the eyes.)

The exponents are not sensitive to the anisotropy. As an example we plotted in Fig. 8 the measured τ cluster size exponents against the anisotropy parameter in the two-dimensional case. At very small κ values (small anisotropy) there is no sign of any change in the cluster size distribution, while the single-site energy distribution is remarkably varied, the peaks are split up.

IV. SUMMARY

We performed computer simulations to investigate the self-organized critical state in a continuous-energy model. We introduced a type of anisotropy in the energy propagation process. From the experiments we conclude the following.

(i) The single-site energy distribution has a peaked shape. The positions of the peaks depend on the allowed energy range ($E_{\max}-E_{\min}$) and the lattice dimensionality only, but are absolutely independent of the characteristics of the energy discharge.

(ii) The critical stored energy obeys the scaling relation (5), and the positions of the peaks correspond to the mul-

tiples of the dimension-dependent quasiunits given by (6).

(iii) In the cases of anisotropic energy propagation defined by (3), the peaks suffer splitting, and the measure of the splitting depends almost linearly on the anisotropy parameter κ.

(iv) The critical exponents of the cluster size distribution are affected by the anisotropy on the extreme cases only (Fig. 8.). This means that a very weak coupling between the individual subsystems (large anisotropy) forms an effective higher-dimensional correlation.

(v) All of our investigations seem to support the earlier theoretical result, that in this type of dynamical model there exists an upper critical dimension $d_c=4$, and above this dimension the scaling breaks down.

We believe that the robust self-averaging process resulting in the peaked single-site energy distribution is a very important fingerprint of the self-organized critical state. The lack of unambiguous experimental evidence renders primordial the investigation of relatively simple models, and the theoretical understanding of the quantized energy distribution may help to the deeper insight of the behavior of the complex dynamical systems.

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