Biased random walk on a deterministic fractal

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Anisotropic diffusion on a Sierpínski gasket is studied using renormalization equations for the probability densities of waiting times. The effect of an external field on a random walk is described in terms of the dependence of hopping probabilities, mean waiting times, and standard deviations on the hopping distance and on the intensity of the field. Our procedure can also be applied to other deterministic fractals.

I. INTRODUCTION

A renormalization procedure, originally conceived by Machta[1] for a one-dimensional regular lattice, has recently been applied to fractal diffusion by Van den Broeck.[2] In this scheme the renormalization is established between the probability densities of waiting times of an original lattice and a decimated lattice in which a particular set of the original sites has been removed. For certain regular fractal structures one can construct a decimated lattice identical in structure to the original lattice. As a consequence, the renormalization scheme in such lattices becomes exact. Renormalization methods have been applied extensively to the master equation[3-6] and to discrete wave equations.[7,8] On the other hand, the continuous-time random-walk (CTRW) formalism[9] based on waiting time probability densities is sufficiently general to deal with a wide class of random walks including non-Markovian processes described by generalized master equations, "anomalous diffusion" arising from infinite mean waiting times, and discrete time processes. The extension of the renormalization methods to the CTRW theory further increases its power and range of applicability.

In this paper we calculate the main properties of a random walk on a Sierpínski gasket in the presence of a field. A systematic way to find the renormalization equations (RE's) is developed and the scaling law of the difference between probabilities in different directions is found. The effect of an external field on fractal diffusion has been the subject of both theoretical[6] and numerical[10] studies. In our treatment, the symmetry restrictions of the master equation[6] do not appear. In particular, it is possible for us to choose nonzero probabilities of staying at vertices whose connectivity to other vertices prevents motion along the field, yielding a more realistic description of the process in the presence of the field than has been possible before. The linear response of the system to an external field has also recently been related to superlocalization, i.e., to the spatial decay of the solutions of wave equations on the fractal.[11]

II. RENORMALIZATION EQUATIONS

In the presence of a uniform electric field pointing in a direction that bisects one if its three sides (cf. Fig. 1), the Sierpínski gasket presents two types of vertices, a and b, shown in Fig. 2. We define $p^{(0)}(t), q^{(0)}(t), r^{(0)}(t)$ to be the probability densities that the walker, arriving at a site of type a at time 0, will move at time t to its nearest neighbor in the direction given in Fig. 2. The densities $(p')^{(0)}(t)$ and $(r')^{(0)}(t)$ are defined in the same way for the b-type vertices.

The RE's (Refs. 1 and 2) are the relations between the hopping probability densities of the decimated lattice, $p^{(1)}(t), q^{(1)}(t), \ldots$, and the hopping probability densities of the original lattice, $p^{(0)}(t), q^{(0)}(t), \ldots$. The topology of the Sierpínski gasket and some other deterministic fractals allows the nearest-neighbor connections of the original lattice to be preserved under successive decimations that remove every other site in all directions. In the absence of a field this leads to a single RE;[2] with the field one obtains five RE's for the Sierpínski gasket.

In order to obtain the RE's it is necessary to sum over all the possible paths between the starting site and the arriving site going through the sites that have been re-

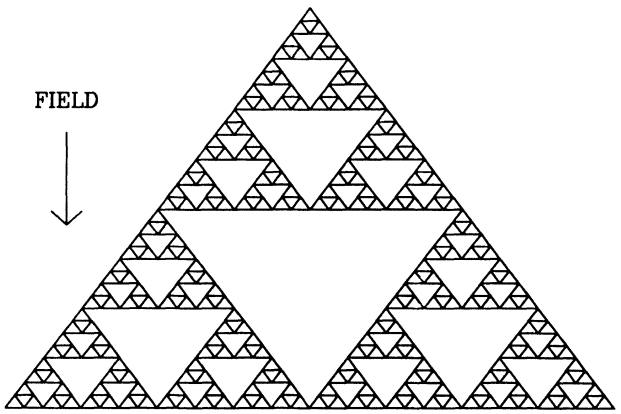

FIG. 1. Sierpínski gasket and direction of the applied field.

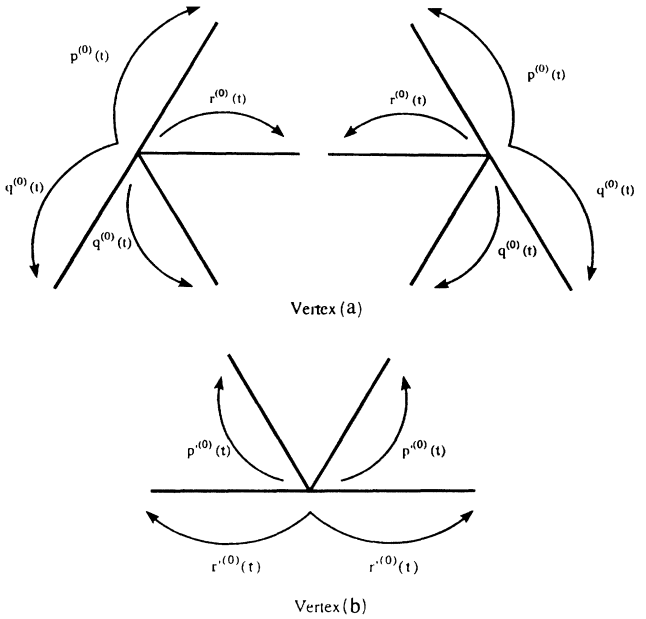

FIG. 2. The two types of vertices on the Sierpínski gasket that must be distinguished in the presence of a field.

moved in the decimation. The two types of vertices have to be considered separately because of the anisotropy produced by the field. The sums over paths can be written compactly in matrix form. In order to do so it is useful to define the following two matrices:

$$\underline{A}^{(0)}(t)=\begin{pmatrix} 0 & r & p & 0 & 0 & q & q \\ r' & 0 & p' & p' & 0 & 0 & 0 \\ q & q & 0 & r & 0 & 0 & 0 \\ 0 & q & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p' & p' \\ p & 0 & 0 & 0 & q & 0 & r \\ p & 0 & 0 & 0 & q & r & 0 \end{pmatrix},$$

$$\underline{B}^{(0)}(t)=\begin{pmatrix} 0 & r' & 0 & p' & r' & p' & 0 \\ r' & 0 & p' & p' & 0 & 0 & 0 \\ 0 & q & 0 & r & 0 & 0 & 0 \\ q & q & r & 0 & 0 & 0 & 0 \\ r' & 0 & 0 & 0 & 0 & p' & p' \\ q & 0 & 0 & 0 & q & 0 & r \\ 0 & 0 & 0 & 0 & q & r & 0 \end{pmatrix}. \tag{1}$$

The rows and columns of $\underline{A}(t)$ and $\underline{B}(t)$ are numbered according to the seven vertex enumerations shown in Figs. 3(a) and 3(b), respectively. The elements A_{ij} and B_{ij} of the matrices are the probability densities of hops from one to another of these sites [t dependences and superscripts (0) of the elements are omitted]. The probability density that the walker jumps from the initial site to a nearest-neighbor site of the decimated lattice starting from an a vertex (b vertex) is obtained in terms of the N-fold time convolution of $\underline{A}^{(0)}(t)$ [of $\underline{B}^{(0)}(t)$]. Taking the Laplace transform, the renormalization equations can be

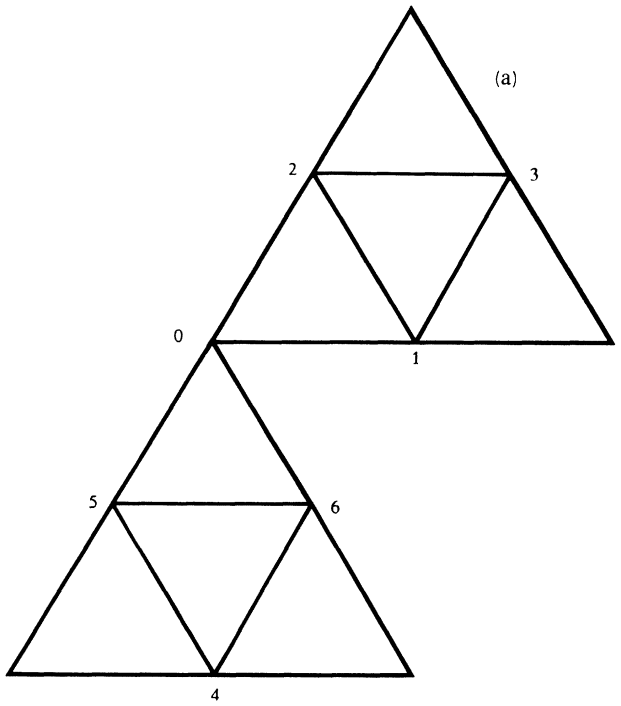

FIG. 3. (a) Enumeration of sites used to construct the renormalization equations. (a): a-type vertex; (b): b-type vertex.

written as follows:

$$\begin{aligned} \tilde{x}^{(1)}(s)&=\sum_{N=0}^{\infty}\langle 0|[\underline{\tilde{A}}^{(0)}(s)]^N|v_x\rangle \\ &=\langle 0|[1-\underline{\tilde{A}}^{(0)}(s)]^{-1}|v_x\rangle , \\ \tilde{y}^{(1)}(s)&=\sum_{N=0}^{\infty}\langle 0|[\underline{\tilde{B}}^{(0)}(s)]^N|v_y\rangle \\ &=\langle 0|[1-\underline{\tilde{B}}^{(0)}(s)]^{-1}|v_y\rangle \end{aligned} \tag{2}$$

for $x=r,p,q$ and $y=p',r'$. Here $\langle 0|$ is the vector with all components zero but the first one, $\langle 0|=(1,0,0,0,0,0,0)$, reflecting the origin of the walker at the site labeled 0. The vectors $|v_x\rangle$ and $|v_y\rangle$ depend on the final arrival site and involve the probabilities for the final step taken by the walker. For example, consider the horizontal jump from the a vertex numbered 0 in Fig. 3(a) to the site on the right of 1 when the sites numbered 1 to 6 are removed. The probability density for this jump is $r^{(1)}(t)$. The vector $|v_r\rangle$ for this calculation contains

the probability densities for jumps from each numbered site (prior to decimation) to this final site:

$$\langle \nu_r| = (0,(\tilde{r}')^{(0)}(s),0,\tilde{q}^{(0)}(s),0,0,0) \ . \tag{3}$$

The RE's obtained from (2) are valid at each stage of decimation and the explicit form, omitting tildes, s dependences, and also the superscripts (n) which indicate the stage of decimation in the rhs of the equation, are

$$\begin{aligned}
r^{(n+1)} &= -\frac{[r^3+(-pq-1)r-pq]r'-p'qr^2+(-p'-p)qr-pp'q^2}{D_a(r,p,q,r',p')} \ , \\
p^{(n+1)} &= \frac{2pp'r^2+(2pp'+p^2)r+p^2}{D_a(r,p,q,r',p')} \ , \\
q^{(n+1)} &= \frac{(2q^2r+2q^2)r'+q^2r+q^2}{D_a(r,p,q,r',p')} \ , \\
(r')^{(n+1)} &= -\frac{(r^2-1)(r')^2+(-2p'qr-2p'q)r'-p'qr-(p')^2q^2}{D_b(r,p,q,r',p')} \ , \\
(p')^{(n+1)} &= \frac{(2pp'r+2pp')r'+pp'r+pp'}{D_b(r,p,q,r',p')} \ ,
\end{aligned} \tag{4}$$

where

$$D_a(r,p,q,r',p') = [r^3+(-pq-1)r-pq]r'+(-p'q-1)r^2+(-3p'-2p)qr+pp'q^2+(-2p'-3p)q+1 \ , \tag{5}$$

and

$$D_b(r,p,q,r',p') = (2r^2-2)(r')^2+(-4p'qr-4p'q)r'-r^2-2p'qr+2(p')^2q^2-4p'q+1 \ . \tag{6}$$

We note that our systematic matrix procedure for writing the renormalization equations yields generic equations whose general form depends only on the connectivity of the lattice. For the isotropic case ($p=q=r=r'=p'\equiv\Psi/4$) all of the above renormalization equations become equal to one another and reproduce the (single) renormalization equation obtained by Van den Broeck:[2]

$$\tilde{\Psi}^{(n+1)}(s) = \frac{[\tilde{\Psi}^{(n)}(s)]^2}{4-3\tilde{\Psi}^{n}(s)} \ . \tag{7}$$

The matrix formulation also allows us to find the RE in the three-dimensional isotropic case, which reads[12]

$$\tilde{\Psi}^{(n+1)}(s) = \frac{[\tilde{\Psi}^{(n)}(s)]^2}{[\tilde{\Psi}^{(n)}(s)]^2-6\tilde{\Psi}^{(n)}(s)+6} \ . \tag{8}$$

The RE's (4) are used to calculate the probability densities of the hops in the different directions and their moments, in particular the mean waiting times and the second moments from which the standard deviations can be calculated for the two types of vertices. The equations for the probability densities are functional relations which cannot be implemented in a numerical calculation. However, the information necessary for the calculation of the moments is obtained from the behavior of the Laplace transform of the probability densities near $s=0$. The total probability (integrated over time) of a given jump, say r, after the nth decimation is given by $\tilde{r}^{(n)}(0)$. The mean value of the waiting time and the second moment for the a vertices after the nth decimation are

$$\begin{aligned}
\langle t_a^{(n)} \rangle &= \int_0^\infty dt\, t[r^{(n)}(t)+p^{(n)}(t)+2q^{(n)}(t)] \\
&= -\left[\frac{d}{ds}\right]_{s=0}[\tilde{r}^{(n)}(s)+\tilde{p}^{(n)}(s)+2\tilde{q}^{(n)}(s)] \ , \\
\langle t_a^{(n)^2} \rangle &= \int_0^\infty dt\, t^2[r^{(n)}(t)+p^{(n)}(t)+2q^{(n)}(t)] \\
&= \left[\frac{d^2}{ds^2}\right]_{s=0}[\tilde{r}^{(n)}(s)+\tilde{p}^{(n)}(s)+2\tilde{q}^{(n)}(s)] \ ,
\end{aligned} \tag{9}$$

and for b vertices

$$\begin{aligned}
\langle t_b^{(n)} \rangle &= \int_0^\infty dt\, t[2(r')^{(n)}(t)+2(p')^{(n)}(t)] \\
&= -\left[\frac{d}{ds}\right]_{s=0}[2(\tilde{r}')^{(n)}(s)+2(\tilde{p}')^{(n)}(s)] \ , \\
\langle t_b^{(n)^2} \rangle &= \int_0^\infty dt\, t^2[2(r')^{(n)}(t)+2(p')^{(n)}(t)] \\
&= \left[\frac{d^2}{ds^2}\right]_{s=0}[2(\tilde{r}')^{(n)}(s)+2(\tilde{p}')^{(n)}(s)] \ .
\end{aligned} \tag{10}$$

The probabilities, mean waiting times, and second moments can easily be obtained from the RE's. The probabilities are obtained by numerical iteration, and the mean waiting times by evaluating the Jacobian matrix of the RE's with the values obtained from the iteration, and applying this matrix to the vector whose components are given by the mean waiting times of the original lattice. The second moments are calculated by applying the same Jacobian matrix to the vector with components given by the second moments of the original lattice, adding this to the application of the gradient of the Jacobian matrix to

the vector of the mean waiting times of the original lattice, and further adding the vector obtained by applying the product of these two matrices to the same vector of mean waiting times.

III. EXPLICIT MODEL

To carry out an explicit calculation we must now make specific assumptions about the transition probability densities in the original lattice prior to any decimation. We assume that the walker jumps at times $t=\Delta t, 2\Delta t, \ldots$. At an a-type vertex a jump to one of its nearest neighbors occurs with probability one at $t=\Delta t$. The probabilities of such jumps in the different directions are r,p,q, with $r+p+2q=1$. At a b-type site the probabilities of the possible jumps are taken to be p and r and here we allow a nonzero probability $1-2(p+r)$ of staying at a site in each time step. Thus, the walker may take his first step away from a b site at any of the time steps with a decreasing probability as time increases. This information is reflected in the probability densities of the hopping times, of which we have five:

$$
\begin{aligned}
r^{(0)}(t)&=r\delta(t-\Delta t)\ ,\\
p^{(0)}(t)&=p\delta(t-\Delta t)\ ,\\
q^{(0)}(t)&=q\delta(t-\Delta t)\ ,\\
(r')^{(0)}(t)&=r\sum_{N=1}^{\infty}[1-2(p+r)]^{N-1}\delta(t-N\Delta t)\ ,\\
(p')^{(0)}(t)&=p\sum_{N=1}^{\infty}[1-2(p+r)]^{N-1}\delta(t-N\Delta t)\ .
\end{aligned}
\tag{11}
$$

The first three indicate a jump at exactly time $t=\Delta t$. The last two express the probability of no jump in the first $N-1$ time intervals followed by a jump at $t=N\Delta t$, with N arbitrary. The total probability of eventually jumping out of a b site (i.e., integrated over all time) is unity. The eventual exit takes place along an r' bond with probability $2r/2(p+r)$ and along a p' bond with probability $2p/2(p+r)$. The Laplace transforms of the hopping probability densities are

$$
\begin{aligned}
\tilde{p}^{(0)}(s)&=pe^{-s\Delta t}\ ,\\
\tilde{q}^{(0)}(s)&=qe^{-s\Delta t}\ ,\\
\tilde{r}^{(0)}(s)&=re^{-s\Delta t}\ ,\\
(\tilde{p}')^{(0)}(s)&=\frac{p}{e^{s\Delta t}-1+2(p+r)}\ ,\\
(\tilde{r}')^{(0)}(s)&=\frac{r}{e^{s\Delta t}-1+2(p+r)}\ .
\end{aligned}
\tag{12}
$$

The presence of the field is taken into account by choosing different values for the (initial) jump probabilities p,q,r. We take

$$
\begin{aligned}
r&=\frac{1}{4}=\tilde{r}^{(0)}(0)\ ,\\
p&=\frac{1-2\alpha}{4}=\tilde{p}^{(0)}(0)\ ,\\
q&=\frac{1+\alpha}{4}=\tilde{q}^{(0)}(0)\ .
\end{aligned}
\tag{13}
$$

The parameter α measures the relative difficulty of jumping against the field and with the field from an a vertex. At a b vertex α measures the probability per step of remaining at the vertex. The first decimation RE's are obtained from (4) and yield to first order in α

$$
\begin{aligned}
&\tilde{r}^{(1)}(0)=\frac{1}{4},\quad \tilde{p}^{(1)}(0)=\frac{1-4\alpha}{4},\quad \tilde{q}^{(1)}(0)=\frac{1+2\alpha}{4}\ ,\\
&(\tilde{r}')^{(1)}(0)=\frac{1+2\alpha}{4},\quad (\tilde{p}')^{(1)}(0)=\frac{1-2\alpha}{4}\ .
\end{aligned}
\tag{14}
$$

Note that the coefficient of the parameter α doubles upon decimation, and this doubling continues generation after generation. To obtain physically meaningful results we choose α so as to yield finite probabilities for macroscopic jumps in the direction against the field *and* toward the field. Thus α must not only be scaled downward but it must be done so in a particular way. The appropriate scaling, as in one-dimensional Brownian motion, is the same as for the distance. In particular, if we let N_G be the total number of generations (i.e., the total number of decimations to be performed so that a single triangle remains in Fig. 1), then α must be taken to be of order 2^{-N_G}. If I is a measure of the intensity of the field, then we can write

$$
\alpha=\frac{I}{2^{N_G}}\ .
\tag{15}
$$

This means that the effect of the field in the early stages of decimation (i.e., on a microscopic scale) is very small but becomes appreciable on a macroscopic scale, i.e., in the later stages of decimation.

Simultaneously it is also convenient to scale the time step Δt so that the mean time to go from a vertex to any other on the decimated lattice is of $O(1)$ after the N_G decimations. The unit of time in this argument is the average time for the walker to go from a vertex to any other in the original (undecimated) lattice. In the isotropic case (i.e., no field) it takes five times as long to reach a next-nearest-neighbor vertex than a nearest-neighbor vertex, so that the time interval Δt should be scaled downward by a factor of 5 for each new generation. The appropriate I-dependent time scaling for our anisotropic case can in principle be (cumbersomely) calculated. Instead, we follow the procedure used for random walks in ordinary (integer-dimensional) lattices and scale the time for the anisotropic problem in the same way as in the isotropic problem (in the ordinary case this scaling leads to the correct diffusion equation with drift in the continuum limit). Thus we take

$$
\Delta t=\frac{1}{5^{N_G}}\ .
\tag{16}
$$

IV. RESULTS

In this section we discuss the results obtained by implementing the renormalization equations and compare them with results obtained from direct simulation of the random walk. Figures 4–6 are obtained from the RE's, while Fig. 7 contains the simulation results.

In Fig. 4 we present numerical calculations of the probabilities and scaled mean waiting times $\langle t_i \rangle / \Delta t 5^{N_D}$, $i=a,b$, i.e., the mean number of steps divided by 5^{N_D}, as functions of the number of decimations N_D, with (obviously) $N_D \leq N_G$. In these graphs we have taken $I=1$ and $N_G=20$. Before discussing the results, we note that ideally one would like to obtain such results for arbitrarily large N_G ($N_G \to \infty$), whereas numerically one is limited to a finite N_G. Fortunately, we can show numerically that $N_G=20$ is sufficiently large to capture the results for arbitrarily large N_G. To see this, consider graphs such as those in Fig. 4 but now calculated for $N_G=21$. The probability difference parameter and the time step are now $\alpha=2^{-21}$ and $\Delta t=5^{-21}$. With these values we obtain *after* the first decimation the *same* values within 2^{-42} for probabilities and mean times as those with which the iteration for $N_G=20$ was started. Therefore the results for the last 20 decimations for $N_G=20$ and $N_G=21$ are indistinguishable. The same is valid for larger N_G, i.e., the last 20 decimations lead to results virtually indistinguishable from those in Fig. 4. Since the interesting variations (dependences on the field) occur only in the last ten or so decimations, this correspondence is all that is necessary to conclude that although obtained for $N_G=20$, Fig. 4 reproduces the behavior of the infinite fractal. As a matter of fact, an even smaller N_G already leads to excellent estimates of the behavior of a very dense fractal: analytic estimation as well as numerical results give deviations between the graphs for $N_G=10$, 15, and 20 of around 0.1%.

In light of these remarks, consider the results of Fig. 4. Figure 4(a) shows the expected small-α result that for low decimation orders (i.e., on a microscopic level) the field has little effect on the hopping probabilities, which are all equal to $\frac{1}{4}$, as in the isotropic system. As the macroscopic level of description $N_D \to N_G$ is approached, the field is felt and deviations from the isotropic behavior set in. The probability of escaping an a vertex increases in the direction of the field (q) and decreases in the other directions, so that the motion away from an a vertex becomes more and more deterministic. At a b vertex eventual escape is less and less likely to occur in a direction opposing the field and thus more likely along a direction perpendicular to the field (r increases). It is easy to check

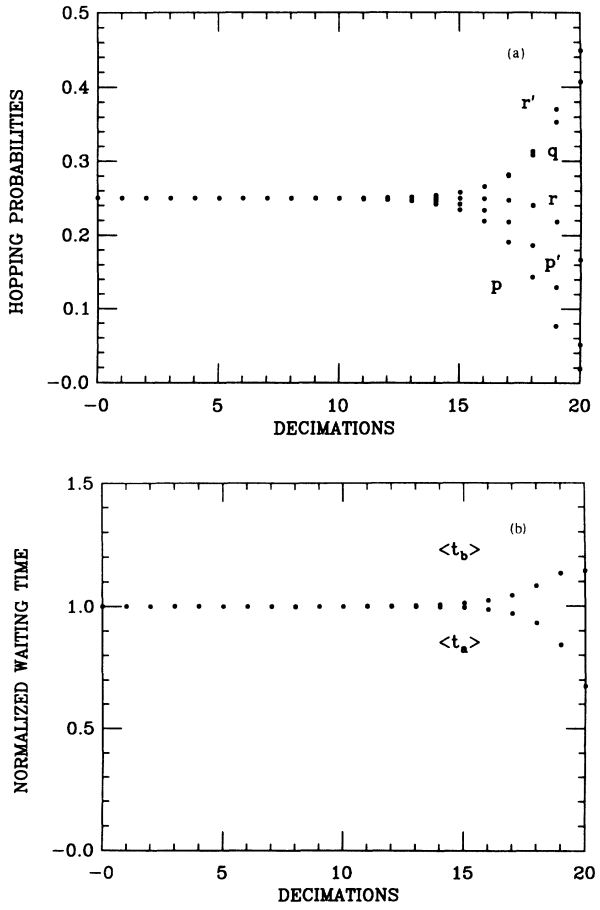

FIG. 4. (a) Hopping probabilities and (b) mean waiting times scaled as in a random walk on an isotropic Sierpínski gasket as a function of the number of decimations for $I=1$.

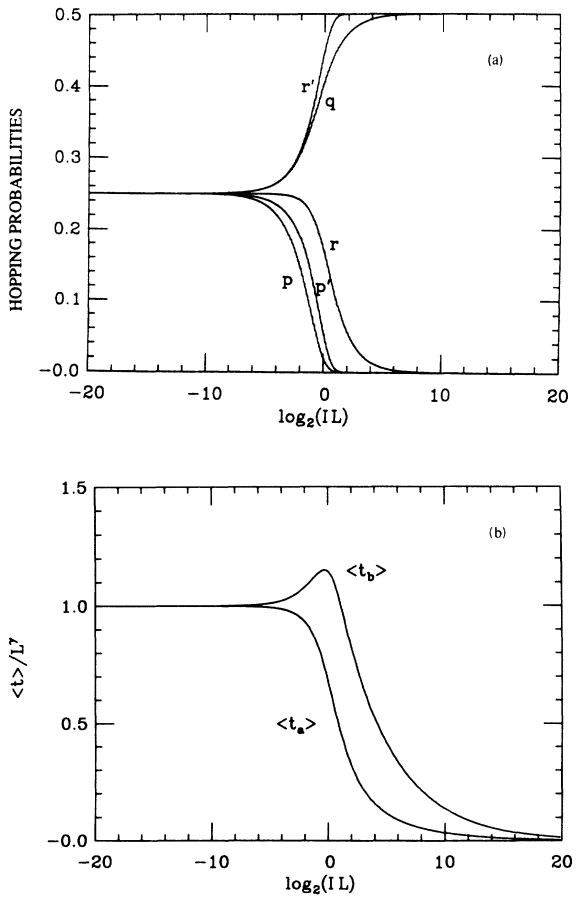

FIG. 5. (a) Hopping probabilities and (b) mean waiting times scaled as in Fig. 4 as a function of $\log_2(IL)$.

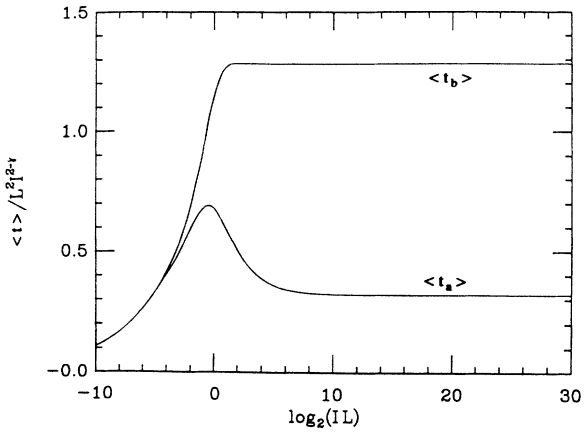

FIG. 6. Rescaled mean waiting times as a function of $\log_2(IL)$.

that the values $q=r'=\frac{1}{2}, r=p=p'=0$ are fixed points of the RE's (4). These values are only approached but not achieved in Fig. 4(a) (cf. subsequent discussion on field intensity).

Figure 4(b) shows that for low orders of decimation the scaled mean time of escape from either an a vertex or a b vertex is unity (i.e., both $\langle t_i \rangle$'s increase as 5^{N_D}, as in the isotropic problem). Upon approaching the macroscopic level of description, the scaled mean time for escape from an a vertex decreases as the motion following the field becomes more deterministic, i.e., the variation of $\langle t_a \rangle$ with N_D becomes slower than 5^{N_D}. At a b vertex the tendency to remain at each step increases as $N_D \to N_G$, and hence the scaled time to escape from a b vertex becomes increasingly longer, i.e., the variation of $\langle t_b \rangle$ with N_D becomes faster than 5^{N_D}. When the lattice is fully decimated $\langle t_a \rangle$ is down to 0.67 and $\langle t_b \rangle$ reaches 1.15.

In calculating the points in Fig. 4 we took the intensity coefficient I introduced in Eq. (15) to be unity. The effect of changing (increasing) I is essentially to shift the abscissa to the left and hence allows for the "continuation" of the graphs. Thus let us reintroduce I and for convenience let us also introduce the length of a hop L, which after the N_Dth decimation is $L=2^{-N_G+N_D}$ (the length unit is equal to the length of the entire lattice). Since the field parameter α depends only on the product IL, the dependence of the hopping probabilities and scaled waiting times on N_D *and* on I can then be represented simultaneously and conveniently in terms of the product of I and L. This abscissa has the added advantage of not

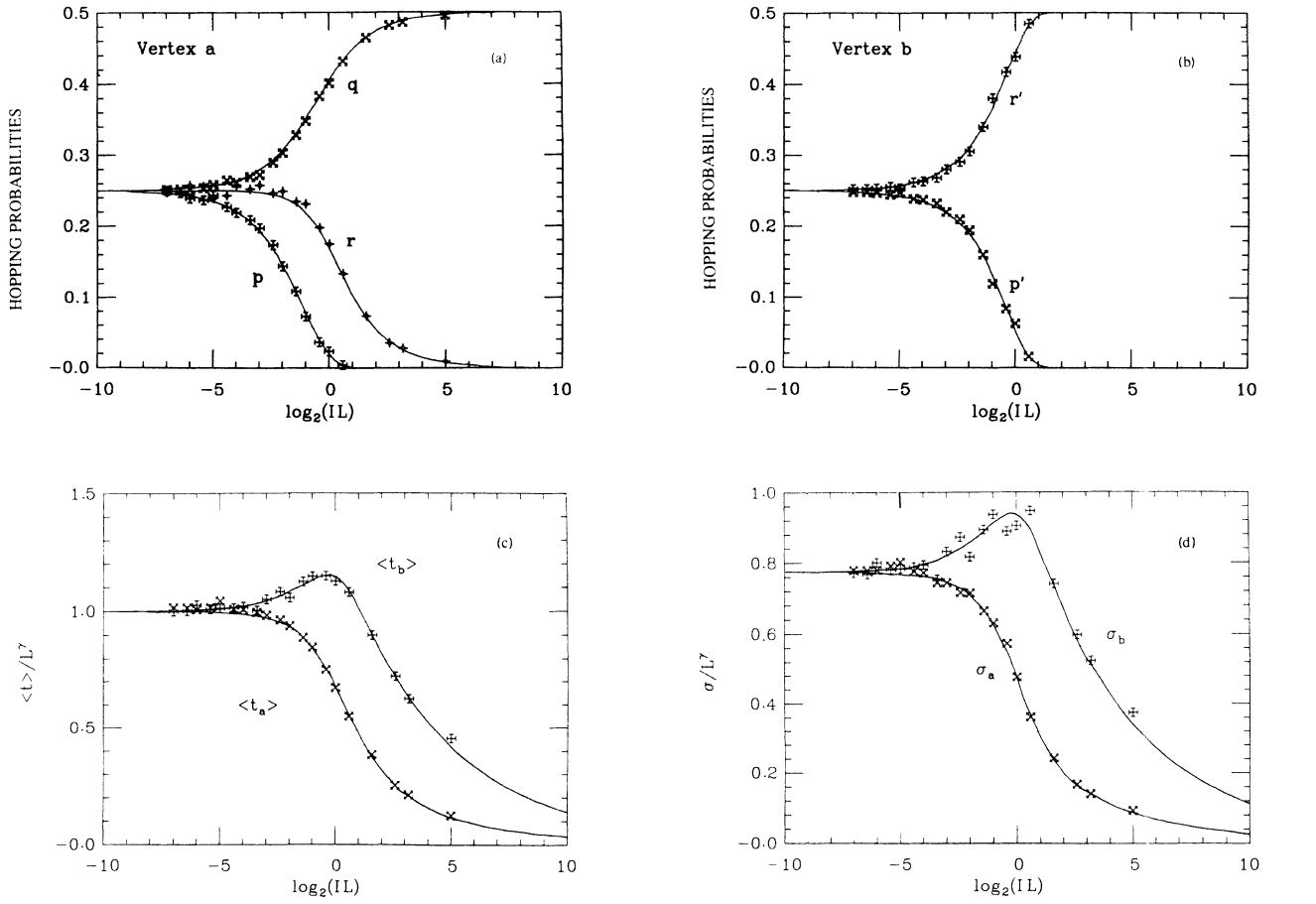

FIG. 7. Comparison of simulations and renormalization equation results for (a) and (b) hopping probabilities, (c) scaled mean waiting times, and (d) scaled waiting time variances.

growing boundlessly as $N_G \to \infty$. Figure 5(a) shows the hopping probabilities as a function of the logarithm (base 2) of IL. It must be understood that for each value of I only values up to $\log_2(IL)=\log_2 I$ can be achieved since the maximum value of L is unity. When $I=1$ the highest physically achievable value of the abscissa is therefore $\log_2 L=0$. Up to this value Fig. 5(a) exactly reproduces Fig. 4(a). As I increases, larger values of the abscissa can be achieved, and the larger field intensities drive the hopping probabilities closer and closer to the fixed points of the RE's.

Figure 5(b) shows the scaled mean waiting times for escape from each type of vertex as a function of $\log_2(LI)$. Note that we have explicitly indicated the appropriate scaling of the mean waiting time in terms of the length L; the appropriate exponent for this scaling is[2]

$$\gamma=\frac{\ln 5}{\ln 2}\ , \tag{17}$$

in terms of which $5^{N_D-N_G}=L^{\gamma}$, so that $\langle t\rangle/5^{N_D}\Delta t=\langle t\rangle/L^{\gamma}$. For $I=1$ only values of the abscissa up to zero are physically meaningful, and these again reproduce the curves of Fig. 4(b). With increasing intensity, the mean waiting time for escape from an a vertex continues to decrease as the motion becomes more deterministic along the q bonds, again reflecting a slower growth with L than L^{γ}. The time for escape from a b vertex reaches a maximum and then decreases with increasing intensity, reflecting the balance between the closure of an escape channel (p' decreases) and the increased efficiency of the other escape channel (r' increases).

That both $\langle t_a\rangle$ and $\langle t_b\rangle$ in Fig. 5(b) eventually vanish is a reflection of the fact that at sufficiently high intensities the time scaling that we have used becomes inappropriate if we wish to deal with scaled times of $O(1)$. In fact, as LI increases the motions acquire more of the characteristics of a one-dimensional random walk as the probabilities of jumps against the field become smaller. To see that this must be the case, consider the way in which the walker can leave each type of vertex when I is large. The motion out of an a-type vertex occurs with overwhelming probability along a "q direction." The other channels of motion out of an a-type vertex, p and r, become increasingly less probable as the intensity increases [see Fig. 5(a)]. The motion out of a b-type vertex occurs with increasing probability along an "r' direction" [Fig. 5(a)] as the p' motion becomes less and less probable. Furthermore, the duration of each step along a q direction is negligible compared to that along an r' direction because the probability α of remaining at a b vertex at each step is very large. In fact, the walk in the horizontal direction that the walker must perform before it gets from a b vertex to an a vertex is essentially a one-dimensional random walk and hence one expects the mean time of escape from a b vertex to go as the square of the length of the path. If a walker could get from the top to the bottom of the fractal taking only q-type steps, then the motion would be ballistic and the mean time of traversal would be proportional to the length of the path. However, even escape from an a vertex from the top to the bottom of the fractal necessarily takes the walker through b-type vertices and hence necessarily involves the type of one-dimensional walk in the horizontal direction mentioned above. Therefore escape from either type of vertex in the large-I limit should behave as a one-dimensional random walk and the mean time of escape should grow as L^2. One would therefore expect that the L^2 scaling of time typical of Brownian motion in one dimension would become more appropriate. In Fig. 6 we show the mean waiting times $\langle t_a\rangle$ and $\langle t_b\rangle$ scaled with L^2. In order to preserve the dependence on only the product LI rather than on L and I separately, we further scale the times $\langle t_i\rangle$ with $I^{2-\gamma}$. Figure 6 then shows $(\langle t_i\rangle/L^2I^{2-\gamma})$ versus $\log_2(IL)$. With increasing intensity these scaled times approach a constant, as expected, indicating that the $\langle t_i\rangle$ now vary as 4^{N_D} (instead of 5^{N_D}). Further information can be extracted from this result: since

$$\frac{\langle t_i\rangle}{L^2I^{2-\gamma}}=\frac{\langle t_i\rangle}{L^{\gamma}}(LI)^{\gamma-2}\ , \tag{18}$$

it follows that with increasing intensity the left-hand side can only approach a constant if $\langle t\rangle/L^{\gamma}$ decays as $(LI)^{2-\gamma}=(LI)^{-0.3213}$. A least-squares analysis of Fig. 4(b) yields precisely this asymptotic result for large I. For $I>2^{15}$

$$\frac{\langle t_a\rangle}{L^{\gamma}}\sim\frac{0.3170}{(LI)^{0.3212}} \tag{19}$$

while for $I>2^{10}$,

$$\frac{\langle t_b\rangle}{L^{\gamma}}\sim\frac{1.2668}{(LI)^{0.3211}}\ . \tag{20}$$

Note the ratio $\langle t_b\rangle/\langle t_a\rangle=4$, a result which can be extracted analytically from the renormalization equations. This ratio can be understood in terms of the arguments preceding Eq. (18). In order for a walker to escape from an a vertex it has to go the bottom of the fractal taking $2^{N_G}/2$ steps of negligible duration and a number of horizontal steps. The horizontal motion associated with this descent is equivalent to the horizontal motion associated with escape from a b vertex in a once-decimated lattice. In other words, the horizontal distance that must be covered to escape from an a vertex is twice as long as the horizontal distance for escape from a b vertex. Since the horizontal motion is essentially a one-dimensional random walk in which it takes four times as long to cover twice the distance, the ratio of escape times from the two vertices is therefore 4.

In Fig. 7 we report the results of direct numerical simulations of the walk on the gasket. The simulations were performed for a Sierpínski gasket with an applied external field such as that described above, and an average over 5000 Monte Carlo trajectories was taken in each case. We find excellent agreement between the analytic and numerical results. In these figures we also exhibit the scaled variance, with $\sigma=(\langle t^2\rangle-\langle t\rangle 2)^{1/2}$. The variances show behavior similar to that of the mean escape times. The scatter of the simulations shows the need to

average over more trajectories to obtain more accurate higher moments.

Finally we note that the results on the mean waiting times can be used to express the distance covered by the walker in a given interval t of time. For an isotropic lattice (no field) the mean displacement of course vanishes. The root-mean-square radial displacement at long times is given by[2]

$$\langle r^2(t)\rangle^{1/2} \sim t^{1/\gamma} = t^{\ln 2/\ln 5} = t^{0.431} . \tag{21}$$

This value grows with time more slowly on a Sierpínski gasket than it does for a random walk on an ordinary lattice of any integer dimension, for which the root-mean-square radial displacement grows as $t^{1/2}$. In the presence of the field the mean displacement no longer vanishes, and for large values of I it is given by

$$\langle r(t,I)\rangle \sim \langle r^2(t,I)\rangle^{1/2} \sim t^{1/2} I^{(\gamma-2)/2} = t^{1/2} I^{0.1606} . \tag{22}$$

The mean displacement grows as the square root of the time rather than linearly with time for the same reasons as laid forth preceding Eq. (18): the motion in the large-I limit is predominantly a one-dimensional random walk in the horizontal directions that the walker must cover before it occasionally gets to a vertex where downward motion along the field becomes possible. The effect of the bias can be described by a scaling expression of the form

$$\langle r^2(t,I)\rangle^{1/2} \sim \langle r^2(t,0)\rangle^{1/2} f(\langle r^2(t,0)\rangle^{1/2}, I) \sim t^{1/\gamma} f(I t^{1/\gamma}) , \tag{23}$$

where $f(0)=1$ and $f(x)=x^{\beta}$ for large x. The power β is determined by the time scaling of the displacement and is in this case $\beta=(\gamma-2)/2$.

V. SUMMARY

We have studied anisotropic diffusion on a Sierpínski gasket by generalizing the renormalization method recently introduced by Van den Broeck.[2] We obtain the dependence of hopping probabilities, waiting times, and variances thereof on the jump size and field intensity. We find that the scaling of the mean time $\langle t\rangle$ to first traverse a given distance L varies as in the isotropic case, $\langle t\rangle \sim L^{\gamma} = L^{\ln 5/\ln 2} = L^{2.3213}$, for small distances but changes to $\langle t\rangle \sim L^2 I^{2-\gamma} = L^2 I^{-0.3213}$ for large distances and strong fields, when the walk becomes more unidirectional in character. Note that even at high field intensities one cannot extract a mobility from these results (i.e., the displacement never becomes proportional to the time) because most of the motion occurs in a direction perpendicular to the field.[11] The variance shows similar scaling behavior. It should be noted that if the field points in any direction other than the one we have chosen, then essentially ballistic motion becomes possible at high-field intensities. The time to traverse the fractal would then become proportional to the length of the path, L, rather than L^2 and the mean distance $\langle r\rangle$ would in turn become proportional to t rather than $t^{1/2}$. The scaling expression (23) in this case holds again, but now with the power $\beta=\gamma-1$.

Finally we note that the procedures developed herein can easily be applied to other deterministic fractals. However, the results that we have presented for the Sierpínski gasket and the dependence of the large-field behavior on the direction of the field even in this relatively simple case indicate the difficulty with attempting to make general statements about the expected behavior of the random walk in other structures.

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