Multifractal scaling of velocity derivatives in turbulence

Mark Nelkin*
School of Applied and Engineering Physics, Cornell University, Ithaca, New York 14853
(Received 24 May 1990)

We consider the Reynolds-number dependence of various higher-order quantities evaluated at a single spatial point in an incompressible turbulent flow. In particular, we consider $A_p = \langle (\partial u/\partial x)^p \rangle$, which gives the moments of the velocity derivative distribution, and $B_n = \langle (\partial^n u/\partial x^n)^2 \rangle$, which gives the $2n$th moment of the energy spectrum $E(k)$. We assume that multifractal scaling in the inertial range can be extrapolated down to the locally fluctuating Kolmogorov microscale $\eta(\alpha)$, and we average over the distribution of $\eta(\alpha)$ using the appropriate multifractal spectrum $f(\alpha)$. We show that the scaling of A_p and B_n is the same if $n=p-1$. For $n=1$ and $p=2$, this gives the known result that the average energy dissipation is constant. For $n=2$ and $p=3$, this determines the scaling of the velocity derivative skewness or of the mean-square second derivative. These should scale together according to the Navier-Stokes equations, and we recover this result from the scaling analysis. For $n=3$ and $p=4$, the velocity derivative flatness is related to the sixth moment of the energy spectrum. In the limit of large p, we obtain a new relation between the exponential tail of the probability distribution of $\partial u/\partial x$, and the exponential tail of the energy spectrum $E(k)$. We compare our results for the skewness and flatness of velocity derivatives to an earlier result of Meneveau and Sreenivasan [Nucl. Phys. B Proc. Suppl. **2**, 49 (1987)], and we suggest that a multiscaling analysis of the Reynolds-number dependence of $E(k)$ might lead to an improved degree of universality.

I. SCALING ASSUMPTIONS

The scale-dependent intermittency of the velocity field in high-Reynolds-number fluid turbulence has been the subject of intensive study for many years. Recent experimental work[1] has established that the statistical properties of the turbulent energy dissipation, $\varepsilon(x) = \nu(\partial u/\partial x)^2$, averaged over an inertial range distance r, are universal and are well described by the currently fashionable multifractal formalism. In this paper we assume that the multifractal spectrum $f(\alpha)$ of ε is known. We refer the reader to Ref. 1 for historical and experimental details.

The underlying motivation for phenomenological scaling arguments is to suggest testable relations among experimental quantities. Twelve years ago[2] it seemed plausible that turbulence might be fractally homogeneous and describable by a single scaling exponent. We now know[1] that this is not true, but that does not mean that all measurable quantities are independent. Our starting point is a simple prediction by Meneveau and Sreenivasan[3] for the skewness and flatness of the velocity derivative as a function of Reynolds number. In critically examining this prediction, we find an analytical result with a somewhat different structure. The numerical changes for skewness and flatness are quite small, but the behavior for higher-order moments is different. More important is our suggestion that high moments of the probability distribution of velocity derivatives are simply related to high moments of the energy spectrum $E(k)$. This suggests a multiscaling analysis of $E(k)$ similar to that recently applied by Wu *et al.*[4] to the power spectrum of temperature fluctuations in free convection.

We start from the 1962 Obukhov assumption[5] that the 1941 Kolmogorov scaling[6] is locally valid, but that the dissipation fluctuates from one location to another. The dissipation fluctuations are described by a multifractal distribution. We consider the pth-order velocity structure function:

$$D_p(r) = \langle [u(x+r) - u(x)]^p \rangle = \langle [\Delta u(r)]^p \rangle \ . \qquad (1)$$

We assume that locally $[\Delta u(r)]^p$ scales as $r^{p\alpha/3}$ when r is an inertial range distance satisfying $\eta \ll r \ll L$, where η is the (fluctuating) dissipation length scale, L is the integral length scale of the turbulence, and α is the appropriate scaling variable for a one-dimensional cut through the three-dimensional dissipation field. The inertial range scaling of $D_p(r)$ is obtained by averaging over α with a probability distribution r^z, where $z = 1 - f(\alpha)$, and $f(\alpha)$ is again taken for one-dimensional cuts. When this average is carried out with use of the usual steepest-descent methods, the behavior of $D_p(r)$, for $1 \le p \le 15$, is in good agreement with the directly measured behavior. See Ref. 1 for details. This strongly suggests but does not prove that the multifractal analysis, combined with the Obukhov assumption,[5] describes a genuine property of high-Reynolds-number turbulence.

In this paper we are concerned with single-point quantities, such as

$$A_p = \left\langle \left| \frac{\partial u}{\partial x} \right|^p \right\rangle = \lim_{r \to 0} r^{-p} D_p(r) \ , \qquad (2)$$

and

$$B_n = \left\langle \left| \frac{\partial^n u}{\partial x^n} \right|^2 \right\rangle = \lim_{r \to 0} \frac{\partial^{2n} D_2(r)}{\partial r^{2n}} = \int_0^\infty k^{2n} E(k) dk \ . \qquad (3)$$

The A_p are the moments of the probability distribution of the velocity derivative, and the B_n are the moments of the (one-dimensional) energy spectrum $E(k)$. In all of our equations, we are interested only in the scaling exponents, and we neglect numerical prefactors. To calculate these exponents, we take local Kolmogorov scaling seriously and assume that we can extrapolate the inertial range scaling down to the local dissipation scale $\eta(\alpha)$. This can, in turn, be expressed in terms of the average dissipation scale η_k through[7]

$$\eta(\alpha)\sim\eta_K^{4/(\alpha+3)} . \tag{4}$$

To evaluate A_p, we combine Eqs. (1), (2), and (4), and integrate over the appropriate probability distribution to obtain

$$A_p\sim\int d\alpha\, \eta_K^{-y(p)} , \tag{5}$$

where

$$y(p)=\frac{4}{3+\alpha}\left[f(\alpha)-1-\frac{p\alpha}{3}+p\right] . \tag{6}$$

To do the integral in Eq. (5), we use steepest descent to find the value of α, which maximizes the exponent $y(p)$, and we assume that A_p scales as η_K raised to this maximum value of $y(p)$. The exponent $y(p)$ is maximum when

$$(3+\alpha)\left[\frac{df}{d\alpha}-(p/3)\right]=f(\alpha)-\frac{p\alpha}{3}+p-1 . \tag{7}$$

As is usual with this formalism,[1] we make the Legendre transformation from the $[\alpha,f(\alpha)]$ representation to the $[q,D(q)]$ representation through

$$q=\frac{df}{d\alpha},\quad f(\alpha)=q\alpha-(q-1)D(q) . \tag{8}$$

Consider first the special case of $p=2$, where we are evaluating the mean-square velocity derivative. Combining Eqs. (7) and (8) in this case, we find that

$$q=q_1=\frac{df}{d\alpha}=1,\quad \alpha=\alpha_1=f(\alpha_1) . \tag{9}$$

The maximum value of $y(2)$ from Eq. (6) is just $\frac{4}{3}$, so that the mean-square velocity derivative scales as $\eta_K^{-4/3}$. This is the expected result since η_K scales as $\nu^{3/4}$. Thus the mean-square velocity derivative scales as ν^{-1}, which is equivalent to the mean dissipation being independent of viscosity, as expected. The value of α for which this occurs is approximately 0.87, and has been discussed elsewhere.[8]

II. FORMAL RESULTS

For an arbitrary value of p, the extremum condition of Eq. (7) can be simplified using the Legendre transform relations of Eq. (8). Making the appropriate substitutions, α and $f(\alpha)$ can be eliminated to obtain

$$(Q-1)D(Q)=2p-1-3Q , \tag{10}$$

where Q is the value of q which solves the transcendental equation (10) for a given value of p. We want to study the scaling of A_p with the Taylor microscale Reynolds number R_λ. The quantity of direct physical interest is the velocity derivative skewness, flatness, or generalized hyperflatness defined by

$$C_p=\frac{A_p}{(A_2)^{p/2}}\sim R_\lambda^{x(p)} . \tag{11}$$

Since R_λ is proportional to $\nu^{-1/2}$, and η_K is proportional to $\nu^{3/4}$, the scaling exponent $x(p)$ is given by

$$x(p)=\tfrac{3}{2}y_{\max}(p)-p . \tag{12}$$

Substituting Eqs. (6), (8), and (10) into Eq. (12), we obtain

$$x(p)=3(2Q-p)=\tfrac{3}{2}(Q-1)[1-D(Q)] , \tag{13}$$

where Q and $D(Q)$ for a given p must be found from the experimental $D(q)$ curve and the solution of the transcendental equation (10). The same procedure can be used to calculate the mean-square values of higher derivatives, B_n. The analogs of Eqs. (5) and (6) become

$$B_n\sim\int d\alpha\, \eta_K^{-r(n)} , \tag{14}$$

where

$$r(n)=\frac{4}{3+\alpha}\left[f(\alpha)-1-\frac{2\alpha}{3}+2n\right] . \tag{15}$$

If we maximize $r(n)$ with respect to α, and transform from the $[\alpha,f(\alpha)]$ representation to the $[q,D(q)]$ representation using Eq. (8), we find that the value of q which maximizes $r(n)$ is the solution of the same transcendental equation (10) which we obtained previously, provided that we make the substitution of $n=p-1$ in Eq. (10).

In the 1941 Kolmogorov theory,[6] B_n diverges as the $(\frac{3}{2})n-1$ power of R_λ. We therefore define a new scaling exponent $t(n)$ by

$$t(n)=\tfrac{3}{2}[r_{\max}(n)-n]+1 . \tag{16}$$

The scaling of B_n with R_λ is then given by

$$R_\lambda^{(3/2)n-1}B_n\sim R_\lambda^{t(n)} . \tag{17}$$

With some algebraic manipulations, we find that the exponent $t(n)$ is related to the exponent $x(p)$ through

$$t(n)=x(n+1) , \tag{18}$$

where $x(p)$ is the solution of Eq. (13). This completes our formal results, which relate the moments of the velocity derivative distribution to the moments of the energy spectrum.

III. VELOCITY DERIVATIVE SKEWNESS AND FLATNESS

The exponent $x(3)$ describes the divergence of velocity derivative skewness with R_λ. The exponent $r(2)$ describes the mean-square second derivative, which is the fourth moment of the energy spectrum $E(k)$. From the Navier-Stokes equations plus the assumption of local isotropy, we have[9] the well-known relationship that

$$D_3(r) = -\tfrac{4}{5}\langle \varepsilon \rangle r + 6\nu \frac{dD_2(r)}{dr} , \tag{19}$$

where the structure functions $D_n(r)$ are defined by Eq. (1), and $\langle \varepsilon \rangle$ is the average rate of energy dissipation per unit mass. Expanding Eq. (19) in a power series in r, the term linear in r cancels identically, and the term proportional to r^3 expresses the balance between the production and dissipation of the mean-square vorticity. In particular, the relation

$$t(2) = x(3) , \tag{20}$$

which we obtained in Sec. II from phenomenological scaling arguments, is consistent with this dynamical result. The mean-square second derivative and mean-cube first derivative must scale in the same way as a consequence of the Navier-Stokes equations.

The exponent $x(4)$ describes the divergence of velocity derivative flatness with R_λ. The scaling relation that we have obtained, $t(3)=x(4)$, states that the sixth moment of the energy spectrum contains the same scaling information as the velocity derivative flatness. This relation has no dynamical basis, but it is an intuitively appealing conclusion which has been suggested before.[10] It is, in fact, simply a consequence of dimensional analysis. The 1941 Kolmogorov dimensional arguments are assumed correct but must be applied to the locally fluctuating dissipation.

To evaluate the skewness and flatness numerically, we note that the 1962 Kolmogorov lognormal approximation[11] is a good approximation when q is not too large. This approximation is most conveniently expressed as

$$D(q) \approx 1 - (\mu/2)q , \tag{21}$$

and the universal exponent μ is experimentally determined[1] to be approximately 0.25. Because $D(p/2) \approx 1$ for small values of p, the solution of Eq. (10) for small values of p is well approximated by $Q \approx p/2$. If this approximation were substituted into the left-hand side of Eq. (13), we would obtain a null result; but if it is substituted into the right-hand side, we obtain an excellent first approximation:

$$x(p) \approx \tfrac{3}{4}(p-2)[1-D(p/2)] . \tag{22}$$

Equation (22) is the same result presented earlier by Meneveau and Sreenivasan,[3] and is in reasonably good agreement with experiment for the dependence of velocity derivative skewness and flatness on Reynolds number.

IV. HIGHER-ORDER MOMENTS

In the limit of large p, Eqs. (21) and (22) are no longer good approximations. We assume that the limit of $D(q)$ as $q \to \infty$ exists and has the nonzero value D_∞. In that limit, the solution of Eq. (10) is

$$Q = (2p - 1 + D_\infty)/(3 + D_\infty) \approx 2p/(3+D_\infty) . \tag{23}$$

Substituting into Eq. (13), we obtain

$$\lim_{p\to\infty} [x(p)/p] = 3(1-D_\infty)/(3+D_\infty) . \tag{24}$$

It is very difficult to measure D_∞ reliably or even to establish its existence; but a nonzero value suggests that the probability distribution of $\partial u/\partial x$ is asymptotically exponential for large values of the velocity derivative. This result has some approximate experimental support,[12] but it remains controversial as to whether or not it is accurately true. Within our scaling picture, the exponents $x(p)$ are simply related to the exponents $t(n)$, which describe the scaling of the $2n$th moment of the energy spectrum $E(k)$. To obtain $t(n)$ proportional to n for large n, it is natural to assume an asymptotically exponential form for $E(k)$ at large k, a result for which there is considerable theoretical support.

V. CONCLUSIONS

Our principal conclusion is that the scaling of powers of the velocity derivative $\partial u/\partial x$ is simply related to the scaling of higher derivatives. This is a natural consequence of Kolmogorov dimensional analysis, but depends in an essential way on the fluctuations of the local dissipation length scale $\eta(\alpha)$. At the level of the velocity derivative skewness, this conclusion agrees with a well-known dynamical consequence of the Navier-Stokes equations. For higher-order quantities, the result has no direct dynamical support. With use of the available experimental data for the multifractal spectrum of the dissipation, the skewness and flatness are predicted to increase with R_λ as the 0.15 and 0.38 power, respectively, in reasonable agreement with experiment. See Ref. 3 for details.

It should be noted, however, that this conclusion is controversial, and that the experiments are not universally accepted. In a recent paper, Kraichnan[13] has proposed a model in which nearly exponential tails to the velocity derivative distribution occur without any multifractal properties, and with a skewness and flatness that are independent of Reynolds number. We have no theoretical reason to disagree with the Kraichnan model, but in our opinion the overall experimental support for a multifractal picture is sufficiently strong and sufficiently universal that it should not be ignored. This does not imply, of course, that our brute-force extension of the multifractal scaling picture to very small scales need be correct. In a subject where there is very little firmly based theory, however, it is likely to be useful to make explicitly testable predictions from whatever phenomenological theories are available, especially when these theories already correlate a considerable body of experimental data.

Unfortunately, high moments of the distribution of $\partial u/\partial x$ and of the energy spectrum $E(k)$ are notoriously difficult to measure accurately, and these are the basic ingredients of our scaling analysis. At a more general level, however, our analysis suggests that the simple 1941 Kolmogorov scaling for the energy spectrum $E(k)$ as a function of k and R_λ should not quite work. A formalism, in which various moments scale independently, is essentially the same as the recent multiscaling formalism introduced by Wu *et al.*[4] in the analysis of the frequency spectrum of temperature fluctuations in free convection. This suggests that a similar analysis be attempted for the energy spectrum $E(k)$ in incompressible turbulence. The hope is that data from moderate Reynolds numbers in the laboratory and in computer simulations could be collapsed onto a single curve along with the high-Reynolds-number data from atmospheric experiments. If this could done, it would have the important implication that the detailed shape of the spectrum in the dissipation range would contain the same physical information usually available only from the inertial range. This would allow the more accurate low-Reynolds-number data, for which no inertial range exists, to be used in a more physically significant way.

ACKNOWLEDGMENTS

I would like to thank K. R. Sreenivasan and Charles Meneveau for many stimulating conversations on this subject. I would also like to thank the New York University Physics Department for their hospitality.

---

*Present address: Physics Department, New York University, 4 Washington Place, New York, NY 10003.

[1]C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. (to be published).

[2]M. Nelkin and T. L. Bell, Phys. Rev. A **17**, 363 (1978).

[3]C Meneveau and K. R. Sreenivasan, Nucl. Phys. B (Proc. Suppl.) **2**, 49 (1987).

[4]X. Z. Wu, L. Kadanoff, A. Libchaber, and M. Sano, Phys. Rev. Lett. **64**, 2140 (1990).

[5]A. M. Obukhov, J. Fluid Mech. **13**, 77 (1962).

[6]A. N. Kolmogorov, as described in L. D. Landau and E. M. Lifshitz, *Fluid Mechanics,* 2nd ed. (Pergamon, New York, 1987), Chap. III.

[7]C. Meneveau and M. Nelkin, Phys. Rev. A **39**, 3732 (1989).

[8]K. R. Sreenivasan and C. Meneveau, Phys. Rev. A **38**, 6287 (1988).

[9]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics,* 2nd Ed. (Pergamon, New York, 1987), p. 140, Eq. (34.21).

[10]J. C. Wyngaard and Y. H. Pao, in *Statistical Models and Turbulence,* Vol. 12 of *Lecture Notes in Physics,* edited by M. Rosenblatt and C. Van Atta (Springer-Verlag, Berlin, 1972), p. 384.

[11]A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).

[12]B. Castaing, Y. Gagne, and E. J. Hopfinger, Physica D (to be published).

[13]R. H. Kraichnan, Phys. Rev. Lett. **65**, 575 (1990).