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Integrability and nonintegrability of quantum systems. II. Dynamics in quantum phase space

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Based on the concepts of integrability and nonintegrability of a quantum system presented in a previous paper [Zhang, Feng, Yuan, and Wang, Phys. Rev. A **40**, 438 (1989)], a realization of the dynamics in the quantum phase space is now presented. For a quantum system with dynamical group $\mathcal{G}$ and in one of its unitary irreducible-representation carrier spaces $\mathfrak{h}_\Lambda$, the quantum phase space is a $2M_\Lambda$-dimensional topological space, where M_Λ is the quantum-dynamical degrees of freedom. This quantum phase space is isomorphic to a coset space $\mathcal{G}/\mathcal{H}$ via the unitary exponential mapping of the elementary excitation operator subspace of $\mathfrak{g}$ (algebra of $\mathcal{G}$), where $\mathcal{H}$ ($\subset \mathcal{G}$) is the maximal stability subgroup of a fixed state in $\mathfrak{h}_\Lambda$. The phase-space representation of the system is realized on $\mathcal{G}/\mathcal{H}$, and its classical analogy can be obtained naturally. It is also shown that there is consistency between quantum and classical integrability. Finally, a general algorithm for seeking the manifestation of "quantum chaos" via the classical analogy is provided. Illustrations of this formulation in several important quantum systems are presented.

I. INTRODUCTION

As the understanding of classical nonlinear phenomena deepens, a natural question is what are the generic behaviors (if any) of a quantum system when its classical counterpart (if it exists) is chaotic. Indeed, in the past decade, the search for the answer to this question has given rise to one of the most exciting research fields (which has loosely been termed *quantum Chaos*[1]) in theoretical physics.

The framework of quantum mechanics is the Hilbert space, in which physical states are represented by state vectors. To explore the quantum manifestation of classical chaos, it is natural to choose as starting points the spectrum and wave functions[2–4] and explore the classical pattern manifested in these basic quantum building blocks.[5,6] However, quantum chaos is not well defined because many concepts which are essential for the analysis of classical chaos are meaningless in quantum mechanics, although in the nearly classical limits, quantum phenomena cannot and should not differ much from those of classical mechanics.[7] Therefore addressing the problem of "chaos" in quantum mechanics certainly demands a fresh look at the fundamental structure of quantum theory.[8]

In a previous paper[9] (hereafter referred to as I), starting from the axiomatic structure of quantum mechanics we presented definitions of quantum dynamical degrees of freedom and the quantum phase space (QPS) from which emerges a concept of quantum integrability. Also, a theorem and its proof are given in I about the relationship between quantum integrability and dynamical symmetry. In the present paper, our goals are (1) to construct the explicit structures of QPS; (2) to explore the dynamical formulation of the quantum system defined on the QPS, and (3) to study the correspondence of quantum-classical mechanics.[10]

A starting point for the search of the manifestation of chaos in quantum systems should be the quantum-classical correspondence. To date, most attempts to study this problem have concentrated in quantizing classical nonintegrable systems[11] while not nearly enough attention has been given to studying the classical limits of quantum integrable and nonintegrable systems even though this approach may be more general, especially since many quantum systems do not have classical counterparts.

It is worth noting that even for quantum systems in which the classical counterparts do not exist, the "*classical limits*" had been derived via quantum-statistical mechanics by Lieb[12] (for spin systems) and Simon[13] (for general compact groups). In their derivations, the Lie group coherent states (from now on referred to as coherent states) theory developed by Klauder,[14] Perelomov,[15] and Gilmore[16] about two decades ago was utilized to construct the partition functions. Detailed properties of the coherent states have recently been reviewed[17] and will not be repeated here. In the work of Lieb and Simon, explicit conditions are provided under which the quantum and classical partition functions are identical, i.e., by using the coherent states, the quantum partition function will reach its classical limit if the Q and P representations become identical. Hence it was recognized that the coherent states are indeed the best theoretical tool to obtain classical limits.

There are three approaches to studying the classical limits of quantum mechanics.

The first is due to Schrödinger:[18] to construct a quantum wave packet whose time evolution follows the classical trajectory (i.e., the coordinate and momentum expectation values of the wave packet are solutions of Hamilton's equations) and satisfies the Schrödinger equation. This approach is only successful in constructing the wave packets of a harmonic oscillator and has proven to be extremely useful in the development of quantum optics.[19]

The second approach is due to Dirac:[20] to construct a quantum Poisson bracket such that the basic structures of quantum and classical mechanics can be put in one-to-one correspondence. This approach has been partially realized by Konstant, Souriau, and Kirillov independently in the 1970s and is now known in the literature as geometric quantization.[21] The geometric quantization begins with a symplectic geometry (M,ω) from which a quantum system can be constructed via the concept of coadjoint representations of an algebra g. Still, it is more important to have an approach which starts from a quantum structure and "terminates" at the classical structure. For certain quantum systems, Yaffe derived their classical limits as large N limits[22] by retracing the path of geometric quantization via the coherent state theory.[14–17] However, generically speaking, a retracing procedure may inherently possess the difficulty of nonuniqueness. The reason is that for a given irreducible representation (irrep) of $\mathcal{g}$, there are in general several inequivalent symplectic manifolds (although only one for rank-1 Lie algebras) due to the existence of different coadjoint orbits. In other words, more than one symplectic manifold can in principle be constructed for a given quantum system. Hence, to find a one-to-one correspondence between quantum and classical structures via the retracing approach is still an interesting yet open question.

The third is from the Feynman path integral: in the Feynman path integral, the quantum propagator is an integration over all possible paths between the initial and final points in the configuration space. This approach is based on the underlying classical mechanics and therefore concerns naturally the problem of quantum-classical correspondence. However, it cannot in principle be directly applied to quantum systems which do not have classical counterparts. It was Klauder[23] who developed a new formulation of the path integral on the general phase space via his *continuous states*[14] (i.e., a general family of coherent states). In such a formalism, the connection of the quantum and classical mechanics is generically provided. Indeed, if the phase-space structures (i.e., the symplectic and complex structures) can be explicitly developed even for systems without classical counterparts, such a formalism can be very useful not only in understanding formal properties but in practice as well.

The above three approaches are based on the three pictures of quantum mechanics: the Schrödinger picture, the Heisenberg picture, and the Feynman path-integral representation. In all these approaches, coherent states have played an important role. However, coherent states are only one representation of the Hilbert space while the phase space of a quantum system which is the basic structure determining the classical limit must logically exist independent of representations of the Hilbert space. To emphasize this point, we will now discuss briefly a procedure to construct a phase space of a quantum system by using the coherent state theory. For a given quantum system (denoted by a dynamical group $\mathcal{G}$ and its irrep space $\mathfrak{h}_\Lambda$),[9] one can obtain a set of coherent states[14,15] by choosing an arbitrary "initial" state $|x\rangle \in \mathfrak{h}_\Lambda$. These coherent states are labeled by the coset (or phase) space $\mathcal{G}/\mathcal{G}_s$ where $\mathcal{G}_s$ is the maximal invariant (with respect to $|x\rangle$) subgroup of $\mathcal{G}$. In principle, different choices of $|x\rangle$ may result in different inequivalent $\mathcal{G}/\mathcal{G}_s$ for the same quantum system. In each $\mathcal{G}/\mathcal{G}_s$ one can find a classical system. Since these classical systems must physically be equivalent, different $\mathcal{G}/\mathcal{G}_s$ should be related to each other via additional constraints. If one does not choose a suitable $|x\rangle$, the resulting classical limit is a constrained classical system[24] and therefore a complicated one although the physics may be equivalent.

It is worth noting that there are two exceptional cases for which $\mathcal{G}/\mathcal{G}_s$ is unique. One is when $\mathcal{G}$ is a rank-1 Lie group [e.g., the Heisenberg-Weyl group, SU(2) and SU(1,1)] and the other is when $\mathfrak{h}_\Lambda$ carries a nondegenerate irrep (see Appendix) of $\mathcal{G}$. The first case includes a limiting class of physical systems. The second is usually unphysical. This is because a physical system is generally restricted by some symmetries and therefore the physical Hilbert space cannot be a nondegenerate irrep space of $\mathcal{G}$.

A typical example to demonstrate the above argument is the finite n-body quantum system. For such a system, the dynamical group is SU(r), where r is the total number of single particle levels. For a nondegenerate irrep space of SU(r), with *any* weight state as $|x\rangle$, the coherent states possess the phase space $SU(r)/C$, where $C = U(1)^{\otimes(r-1)}$ is the Cartan Abelian subgroup of SU(r). However, most realistic finite many-body system must have bosons and/or fermions as building blocks. If the system is bosonic, the physical Hilbert space must be a *fully symmetric* irrep space of SU(r), a degenerate irrep space. In this case, if one chooses the lowest (or highest) weight state as $|x\rangle$, the phase space is the desired $SU(r)/U(r-1)$. If the system is fermionic, then the Hilbert space is a fully antisymmetric irrep space of SU(r), again a degenerate irrep space. When we also take the lowest (or highest) weight state as $|x\rangle$, the phase space is the desired $SU(r)/S(U(r-n)\otimes U(n))$. However, if one does not take the lowest (or highest) weight state as $|x\rangle$, then the phase space is $SU(r)/C$ or some other coset spaces but not $SU(r)/U(r-1)$ or $SU(r)/S(U(r-n)\otimes U(n))$ irrespective of whether the system is bosonic or fermionic. In fact, all other coset spaces are equivalent to $SU(r)/U(r-1)$ or $SU(r)/S(U(r-n)\otimes U(n))$ by some additional constraints which may be given by those coherent state expectation values of the fully degenerate Casimir operators[9] in a subgroup chain of SU(r). It is worth noting that when $r=2$ ($n=1$), the dynamical group is reduced to a rank-1 group and all possible phase spaces are reduced to SU(2)/U(1) and therefore become unique, as we have pointed out earlier.

The above discussion implies that there exists one fundamental inherent phase space for a given quantum system from which the classical limit can be obtained without additional constraint(s). Since the coherent state theory[14,15] does not provide, although sometimes imposes,[16] a *prior* and *physical* condition to restrict the choice of $|x\rangle$, there must exist an intrinsic quantum property to determine the inherent phase space. Thus it is imperative to know what this intrinsic property is and how it can be used to explicitly construct such an inherent phase space. In this paper, the answer to this question will be given. The system's intrinsic property is *the quantum-dynamical degrees of freedom* (QDDF) defined in paper I (Ref. 9) and the construction of the inherent phase space (QPS) for arbitrary quantum systems is restricted by the QDDF. The correspondence of quantum-classical mechanics is then explicitly obtained. The results, as we have expected, reveal that the QPS is an inherent geometry (with natural symplectic and complex structures) of the quantum system, and its existence does not depend on the underlying classical basis. The primary role played by the coherent states is to provide a link between quantum phase space and the Hilbert space.[14] Furthermore, we will show how to construct systematically the canonical form of the quantum phase-space coordinates for all semisimple Lie groups with Cartan decomposition. From such an explicit realization of canonical coordinates, one can realize and compute the "classical limit" and thus provide a natural route to investigation of the quantum manifestation of chaos.

This paper is organized as follows. In Sec. II, the topological and algebraic structures of QPS, i.e., its dimension, symplectic structure, and coordinate realization, are explicitly constructed; from this a unified way to connect quantum and classical theories can be given. A phase-space representation of quantum mechanics is then realized via the coherent states of QPS within the Hilbert space. In Sec. III, based on the structure of QPS and the phase-space representation, a classical analogy of the quantum system is obtained. This carries an explicit realization of quantum-classical correspondence. Thus all the conclusions obtained in paper I are shown to be consistent with the classical theory. Furthermore, in accordance with the *working definition* of "quantum chaos,"[25] i.e., the dynamical behavior of a quantum system whose classical analogy is chaotic, a general algorthim containing a prescription of manifesting quantum chaos via its classical analogy can be constructed. This is discussed in Sec. IV. These discussions can be summarized in the program shown in Fig. 1. In Sec. V several basic quantum systems often used in the study of "quantum chaos" are studied to illustrate the present procedure. Finally, summary and conclusions are given in Sec. VI.

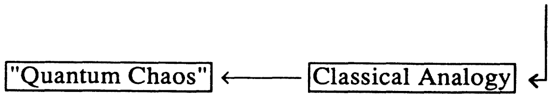

FIG. 1. $\mathcal{G}$ and $\mathfrak{h}_\Lambda$ are the dynamical group and the Hilbert space, respectively. For details, see Sec. II.

II. QUANTUM PHASE SPACE

A. The construction of quantum-dynamical degrees of freedom

In this subsection, we shall briefly discuss the QDDF concept of I. For a given quantum system, there is always a dynamical group $\mathcal{G}$.[9] The Hamiltonian H of the system and its various transition operators $\{A\}$ was expressed in Eq. (3) of paper I [hereafter such equations will be listed as Eq. (I-3), e.g.] as functions of the generators $\{T_i, i=1,\ldots,n\}$ of $\mathcal{G}$:

$$H=H(T_i), \quad A=A(T_i) . \tag{I-3}$$

By definition,

$$[T_i,T_j]=\sum_{k=1}^{n} C_{ij}^k T_k . \tag{I-4}$$

In Eq. (I-4), C_{ij}^k are the structure constants of the algebra $\mathfrak{g}$ of $\mathcal{G}$. The Hilbert space $\mathfrak{h}$ of the system can then be decomposed into a direct sum of the various *unitary irrep* (referred to hitherto as irrep) carrier spaces $\mathfrak{h}_\Lambda$ of $\mathcal{G}$,

$$\mathfrak{h}=\sum_{\Lambda} \oplus Y_\Lambda \mathfrak{h}_\Lambda . \tag{2.1}$$

In Eq. (2.1), Λ denotes an irrep of $\mathcal{G}$ (when $\mathcal{G}$ is a Lie group, Λ is the largest weight of this irrep) and Y_Λ the degeneracy of Λ in $\mathfrak{h}$. It can be shown that no correlations exist between various $\mathfrak{h}_\Lambda$'s. Thus, without any loss of generality, the study of dynamical properties of the system can be focussed on one particular irreducible subspace $\mathfrak{h}_\Lambda$ of $\mathfrak{h}$.

Since the dynamical groups of most quantum systems are Lie groups, only such groups will be discussed here. Suppose $\mathcal{G}$ is l-rank, n dimensional, and has λ subgroup chains $\{\mathcal{G}^1,\mathcal{G}^2,\ldots,\mathcal{G}^\lambda\}$:

$$\mathcal{G}\supset\begin{cases}\mathcal{G}^1\equiv(\mathcal{G}^1_{s^1}\supset\mathcal{G}^1_{s^1-1}\supset\cdots\supset\mathcal{G}^1_1)\\ \mathcal{G}^2\equiv(\mathcal{G}^2_{s^2}\supset\mathcal{G}^2_{s^2-1}\supset\cdots\supset\mathcal{G}^2_1)\\ \cdots\\ \mathcal{G}^\alpha\equiv(\mathcal{G}^\alpha_{s^\alpha}\supset\mathcal{G}^\alpha_{s^\alpha-1}\supset\cdots\supset\mathcal{G}^\alpha_1)\\ \cdots\\ \mathcal{G}^\lambda\equiv(\mathcal{G}^\lambda_{s^\lambda}\supset\mathcal{G}^\lambda_{s^\lambda-1}\supset\cdots\supset\mathcal{G}^\lambda_1),\end{cases} \tag{I-5}$$

then for each chain $\mathcal{G}\supset\mathcal{G}^\alpha$ there is a complete set of commuting operators[26,27] (CSCO) denoted as S^α:

$$S^\alpha:\ \{Q_i^\alpha|[Q_i^\alpha,Q_j^\alpha]=0;\ i,j=1,\ldots,d\} , \tag{2.2}$$

which will completely specify a basis set of $\mathfrak{h}_\Lambda,\{|\gamma^\alpha\rangle;\gamma^\alpha=(\gamma_1^\alpha,\gamma_2^\alpha,\ldots,\gamma_d^\alpha)\}$,

$$Q_i^\alpha|\gamma^\alpha\rangle=\gamma_i^\alpha|\gamma^\alpha\rangle, \quad i=1,\ldots,d^\alpha . \tag{2.3}$$

The dimension d^α of the set S^α is

$$d^{\alpha}=l+\frac{n-l}{2}=d\ . \tag{2.4}$$

Clearly d^{α} depends only on l and n and is independent of α [the subgroup chain index of Eq. (I-5)]:

$$d^{\alpha}=d\quad \text{for } \alpha=1,\ldots,\lambda\ . \tag{2.5}$$

Moreover, S^{α} is the sum of two subsets: a fully degenerate (FD) operator set S_F^{α} and a nonfully degenerate (NFD) operator set S_N^{α}:

$$S^{\alpha}=S_F^{\alpha}+S_N^{\alpha}\ . \tag{2.6}$$

The FD operator $Q_j^{\alpha}\in S_F^{\alpha}$ is defined by the relation

$$Q_j^{\alpha}|\gamma^{\alpha}\rangle=c|\gamma^{\alpha}\rangle,\quad \forall|\gamma^{\alpha}\rangle\in\mathfrak{h}_{\Lambda} \tag{2.7}$$

where c is a constant independent of the subscript of Q_j^{α}. Obviously, only the operators Q_j^{α} which *do not* satisfy Eq. (2.7) span the NFD set S_N^{α}.

It must be stressed that the dimension of S_N^{α} is also independent of α but depends on $\mathfrak{h}_{\Lambda}$. For example, let $\mathfrak{h}_{\Lambda}$ be the carrier space of the nondegenerate irrep of $\mathcal{G}$, the set S_F^{α} contains l Casimir operators of $\mathcal{G}$, and then the dimension M_{Λ} of S_N^{α} is

$$M_{\Lambda}=d-l=\frac{n-l}{2}\ , \tag{2.8}$$

which is obviously independent of α. For degenerate irreps of $\mathcal{G}$, although $M_{\Lambda}<n-l/2$, it nevertheless is still independent of α (see Appendix). Thus it is shown that for a given quantum system within a specific $\mathfrak{h}_{\Lambda}$ the number of NFD operators is unique. According to the definition in paper I: Suppose $S:\{Q_j|[Q_j,Q_{j'}]=0,\ j=1,\ldots,N\}$ is CSCO of a quantum system. A basis set $\{|\alpha\rangle\}$ of its Hilbert space $\mathfrak{h}$ can be completely labeled by M quantum numbers (α_i, $i=1,\ldots,M$) which are related to the eigenvalues of the NFD observables $\{Q_i,\ i=1,\ldots,M\ (M\le N)\}$, a subset of S. Then M is defined as the number of QDDF. The above discussion shows that this definition of QDDF is unique.

B. Structure of quantum phase-space $\mathfrak{p}$

From geometry, the global property of a manifold is its dimension.[28] If the manifold has sufficient physical implications, its dimension should be related to an intrinsic physical property, for example, the number of dynamical variables. In the preceding subsection, the number of the QDDF was obtained. Therefore, for a quantum system with M_{Λ} independent QDDF, the corresponding QPS should be a $2M_{\Lambda}$-dimensional topological space without any additional constraints. In this subsection, a realization of this space will be given.

To explicitly demonstrate this point, let us first illustrate it with a simple but nontrivial example: the SU(3) quark model. According to the coherent state theory, we know that if we choose an arbitrary weight state as the "initial" state $|x\rangle$, the coherent states will provide a phase space SU(3)/U(1)$\otimes$U(1) which is a six-dimensional compact manifold. When one restricts $|x\rangle$ to be the lowest (or highest) weight state, the coherent states will provide a SU(3)/U(1)$\otimes$U(1) phase space for the nondegenerate irrep, and SU(3)/U(2) phase space for the degenerate irrep. The latter is a four-dimensional compact space. A natural question to ask is how the QDDF can select from either SU(3)/U(1)$\otimes$U(1) or SU(3)/U(2) to be the QPS. From paper I, we see that for the baryon octet, which is a nondegenerate irrep space, the number of QDDF is 3: hypercharge, isospin, and its z component and therefore the QPS is a six-dimensional manifold, i.e., SU(3)/U(1)$\otimes$U(1). For the baryon decuplet, which is a degenerate irrep space, the number of QDDF is two because the hypercharge Y and the isospin T are not independent: $T=Y/2+1$. Thus the QPS is a four-dimensional manifold, i.e., SU(3)/U(2). However, by not choosing the lowest or the highest weight state as $|x\rangle$, we will obtain for the baryon decuplet SU(3)/U(1)$\otimes$U(1) as its QPS via the coherent state theory. The coordinates of this QPS must be constrained by $\langle T\rangle=\langle Y\rangle/2+1$ and therefore are not independent, where $\langle T\rangle$ and $\langle Y\rangle$ are the expectation values of the isospin and hypercharge operators in the coherent states of SU(3)/U(1)$\otimes$U(1). This shows that the QDDF dictates the global property, i.e., dimension, of the QPS.

Next we will construct the QPS from the QDDF for an arbitrary quantum system. Since the number of QDDF is defined from the CSCO and the Hilbert space structure (or more precisely, from $\mathcal{G}$ and $\mathfrak{h}_{\Lambda}$), the mathematical structure of QPS should also be related to $\mathcal{G}$ and $\mathfrak{h}_{\Lambda}$. Furthermore, the realization of the QDDF must be operators in quantum mechanics. We shall call these independent QDDF the elementary excitation operators of the system. Thus the first task is to seek explicitly the elementary excitation operators.

For a given system, the elementary excitation operators can be obtained from the structure of $\mathcal{G}$ and $\mathfrak{h}_{\Lambda}$ as follows. Let $\{X_i^{\dagger}\}$ be a subset of the generators of $\mathcal{G}$ such that any states $|\Psi\rangle$ of the system can be generated as follows:

$$|\Psi\rangle=F(X_i^{\dagger})|0\rangle,\quad \forall|\Psi\rangle\in\mathfrak{h}_{\Lambda} \tag{2.9}$$

where $F(X_i^{\dagger})$ is a polynomial of $\{X_i^{\dagger}\}$ and $|0\rangle$ ($\in\mathfrak{h}_{\Lambda}$) is a reference state. The requirement of the choice of $|0\rangle$ is that one can use a minimum subset of $\mathfrak{g}$ to generate the entire $\mathfrak{h}_{\Lambda}$ from $|0\rangle$ via Eq. (2.9). Such a minimum set of operators $\{X_i^{\dagger}\}$ is called the set of elementary excitation operators of QDDF. Then we find that *if $\mathcal{G}$ is compact, the state $|0\rangle$ is the lowest (highest) weight state $|\Lambda,-\Lambda\rangle$ ($|\Lambda,\Lambda\rangle$) of $\mathfrak{h}_{\Lambda}$; if $\mathcal{G}$ is noncompact, it is merely the lowest bound state of $\mathfrak{h}_{\Lambda}$.*[29] We shall refer to $|0\rangle$ in this paper as the fixed state.

Based on the above construction, we can conclude that *the number of $\{X_i^{\dagger}\}$ is identical to the number of QDDF* (a proof is given in the Appendix). Physically, this must be true since the elementary excitation operators are defined to be the QDDF. Thus $\{X_i^{\dagger}\}$ (QDDF or elementary excitation operators) and its Hermitian conjugate $\{X_i\}$ in $\mathfrak{g}_{\Lambda}$ form a *dynamical variable operator subspace $\mathfrak{p}$* of $\mathfrak{g}$:

$$\mathfrak{p}:\ \{X_i^{\dagger},X_i,\ i=1,\ldots,M_{\Lambda}\}\ . \tag{2.10}$$

With respect to $\mathfrak{p}$, there exists a manifold $\mathfrak{p}$ whose dimension is twice that of the number of the QDDF. To explicitly explore the structure of this manifold, we realize it by a unitary exponential mapping of the *dynamical variable operators subspace* $\not p$:

$$\sum_{i=1}^{M_\Lambda} (\eta_i X_i^\dagger - \text{H.c.}) \to \Omega = \exp\left[\sum_{i=1}^{M_\Lambda} (\eta_i X_i^\dagger - \text{H.c.})\right] \in \mathfrak{p} \tag{2.11}$$

where η_i, $i=1,\ldots,M_\Lambda$ are complex parameters. It can be proven that the element Ω is a unitary coset representative of $\mathcal{G}/\mathcal{H}$, where $\mathcal{H}$ ($\subset \mathcal{G}$) is generated by the subalgebra $\not k = \not g - \not p$. Thus Eq. (2.11) shows that $\mathfrak{p}$ is isomorphic to the $2M_\Lambda$*-dimensional* coset space $\mathcal{G}/\mathcal{H}$. From now on $\mathfrak{p}$ is denoted as $\mathcal{G}/\mathcal{H}$.

It must be pointed out that $\mathcal{G}/\mathcal{H}$ is a complex homogeneous space with a natural topology.[28] The complex structure is embedded by Eq. (2.11). The homogeneity can be easily verified. To be concrete and without loss of generality,[30] we shall confine our attention to just semisimple Lie groups whose $\not g$ satisfies the usual Cartan decomposition $\not g = \not k + \not p$:

$$[\not k, \not k] \subset \not k, \quad [\not k, \not p] \subset \not p, \quad [\not p, \not p] \subset \not k . \tag{2.12}$$

In most practical applications, $\mathcal{G}/\mathcal{H}$ does satisfy the decomposition of Eq. (2.12). Thus a group transformation

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{G}$$

acting on $\mathcal{G}/\mathcal{H}$ is a homomorphic mapping of $\mathcal{G}/\mathcal{H}$ onto itself with the explicit form given by[31(c)]

$$g\Omega \to z' = (Az+B)/Cz+D)^{-1} . \tag{2.13}$$

In Eq. (2.13), z' and z ($\in \mathcal{G}/\mathcal{H}$) are complex $k \times p$ matrices (k and p are the dimensions of $\not k$ and $\not p$, respectively) and are related to η_i as

$$z = \begin{cases} \eta \dfrac{\tan(\eta^\dagger \eta)^{1/2}}{(\eta^\dagger\eta)^{1/2}} & \text{for compact } \mathcal{G} \\ \eta \dfrac{\tanh(\eta^\dagger \eta)^{1/2}}{(\eta^\dagger\eta)^{1/2}} & \text{for noncompact } \mathcal{G}. \end{cases} \tag{2.14}$$

Here η represents the nonzero $k \times p$ block matrix of the operator $\sum_{i=1}^{M_\Lambda}(\eta_i X_i\text{-H.c.})$ in the faithful matrix representation.

Thus the Riemannian structure of the homogeneous space $\mathcal{G}/\mathcal{H}$ can be constructed[31] and the results are as follows:[28] the metric g_{ij} of $\mathcal{G}/\mathcal{H}$ is

$$g_{ij} = \frac{\partial^2 \ln K(z,z^*)}{\partial z_i \partial z_j^*} \tag{2.15}$$

where $K(z,z^*)$ is the so-called Bergmann kernel of $\mathcal{G}/\mathcal{H}$ and can be explicitly expressed as

$$K(z,z^*) = \sum_\lambda f_\lambda(z) f_\lambda^*(z) . \tag{2.16}$$

The function $f_\lambda(z)$ in Eq. (2.16) is an orthogonal basis of a closed linear subspace $\mathfrak{L}^2(\mathcal{G}/\mathcal{H})$ of $L^2(\mathcal{G}/\mathcal{H})$ (the square integrable Hilbert space with domain $\mathcal{G}/\mathcal{H}$):

$$\int_{\mathcal{G}/\mathcal{H}} f_\lambda(z) f_{\lambda'}^*(z) K^{-1}(z,z^*) d\mu(z,z^*) = \delta_{\lambda\lambda'} . \tag{2.17}$$

In Eq. (2.17) $d\mu(z,z^*)$ is the $\mathcal{G}$-invariant measure on $\mathcal{G}/\mathcal{H}$

$$d\mu(z,z^*) = \xi[\det(g_{ij})] \prod_{i=1}^{M_\Lambda} \frac{dz_i dz_i^*}{\pi} \tag{2.18}$$

where the coefficient ξ is a normalized measure factor given by the condition $\int_{\mathcal{G}/\mathcal{H}} d\mu(z,z^*) = 1$. Furthermore, there is a closed nondegenerate 2-form ω on $\mathcal{G}/\mathcal{H}$ which is given by

$$\omega = i\hbar \sum_{i,j} g_{ij} dz_i \wedge dz_j^* \tag{2.19}$$

and the corresponding Poisson bracket is then

$$\{f,g\} = \frac{1}{i\hbar} \sum_{i,j} g^{ij} \left[\frac{\partial f \partial g}{\partial z_i \partial z_j^*} - \frac{\partial f \partial g}{\partial z_j^* \partial z_i}\right] . \tag{2.20}$$

In Eq. (2.20), f and g are functions defined on $\mathcal{G}/\mathcal{H}$ and $g_{ij} g^{jk} = \delta_{ik}$. By introducing the canonical coordinates (q,p) of $\mathcal{G}/\mathcal{H}$:

$$\frac{1}{\sqrt{2\hbar\chi}}(q+ip) = Y(z) \tag{2.21}$$

where χ is a function of some inherent physical parameters (such as spin, particle number, and so on), the 2-form ω and the Poisson bracket of Eq. (2.20) can be transformed into the standard classical form:[32]

$$\omega = \sum_{i=1}^{M_\Lambda} dp_i \wedge dq_i , \tag{2.22}$$

$$\{f,g\} = \sum_{i=1}^{M_\Lambda} \left[\frac{\partial f \partial g}{\partial q_i \partial p_i} - \frac{\partial f \partial g}{\partial p_i \partial q_i}\right] . \tag{2.23}$$

When $\mathcal{G}$ is semisimple, the explicit form of $Y(z)$ is

$$Y(z) = \begin{cases} \dfrac{z}{(I+z^\dagger z)^{1/2}} = \eta \dfrac{\sin(\eta^\dagger\eta)^{1/2}}{(\eta^\dagger\eta)^{1/2}} & \text{for compact } \mathcal{G} \\ \dfrac{z}{(I-z^\dagger z)^{1/2}} = \eta \dfrac{\sinh(\eta^\dagger\eta)^{1/2}}{(\eta^\dagger\eta)^{1/2}} & \text{for noncompact } \mathcal{G}, \end{cases} \tag{2.24}$$

which is the independent matrix block in the faithful matrix representation of the coset representative Ω [Eq. (2.11)] in the faithful matrix representation. In other words, the canonical coordinates (p,q) of the coset representative of $\mathcal{G}/\mathcal{H}$ in terms of its faithful matrix representation provide a natural canonical form for the coordinates on $\mathcal{G}/\mathcal{H}$. It is worth pointing out that normally it is very difficult to construct a flat local coordinate on the nontrivial space $\mathcal{G}/\mathcal{H}$ which will preserve the global properties of $\mathcal{G}/\mathcal{H}$. However, the above systematic procedures of constructing the canonical coordinates on $\mathcal{G}/\mathcal{H}$ are generic and they may preserve the global properties (topological invariants) of $\mathcal{G}/\mathcal{H}$.

The above discussions reveal that for a quantum system (in terms of $\mathcal{G}$ and $\mathfrak{h}_\Lambda$), there is always an *inherent quantum phase space* which can be constructed from the intrinsic property: the quantum-dynamical degrees of freedom. In fact, it gives rise to a geometrical structure in a quantum system. *Just as classical mechanics can be built from a geometrical structure,*[32,33] *so can quantum mechanics.* It is also interesting that the geometrical nonintegrable phase, which was extensively discussed by Berry[34] to be an important quantum property, can also be naturally obtained from the nontrivial structure of the quantum phase space in certain cases.[35]

C. Phase-space representation of quantum systems

Based on the structure of $\mathcal{G}/\mathcal{H}$ constructed previously, the next question is, how can one describe the quantum dynamics on the QPS? To this end, it requires the existence of a phase-space representation for a given quantum system. Such a representation can be found if there exists an explicit mapping

$$\{\mathcal{g},\mathfrak{h}_\Lambda\}\rightarrow\{\mathfrak{g}(\mathcal{G}/\mathcal{H}),\mathfrak{L}^2(\mathcal{G}/\mathcal{H})\} \tag{2.25}$$

such that

$$A\longleftrightarrow\mathfrak{A}(p,q),\quad |\Psi\rangle\longleftrightarrow f(q+ip) \tag{2.26}$$

where A is given by Eq. (I-3), $\mathfrak{A}(p,q)\in\mathfrak{g}(\mathcal{G}/\mathcal{H})$, and $f(p,q)\in\mathfrak{L}^2(\mathcal{G}/\mathcal{H})$. For a quantum system with a QPS $\mathcal{G}/\mathcal{H}$, this mapping is uniquely realized by coherent states.[14-17] The coherent states of $\mathcal{G}$ and $\mathfrak{h}_\Lambda$, which are defined on $\mathcal{G}/\mathcal{H}$, can be constructed by choosing the fixed state $|0\rangle$ as its "initial" state

$$g|0\rangle=\Omega h|0\rangle=|\Lambda,\Omega\rangle e^{i\varphi(h)},\quad g\in\mathcal{G},\ h\in\mathcal{H},\ \Omega\in\mathcal{G}/\mathcal{H}\ . \tag{2.27}$$

It is obvious that $\mathcal{H}$ is just the maximal stability subgroup (or isotopy group) of $|0\rangle$, i.e., any $h\in\mathcal{H}$ acting on $|0\rangle$ will leave $|0\rangle$ invariant up to a phase factor:

$$h|0\rangle=|0\rangle e^{i\varphi(h)} \tag{2.28}$$

while $|\Lambda,\Omega\rangle$ are the coherent states which are isomorphic to $\mathcal{G}/\mathcal{H}$:

$$\begin{aligned}|\Lambda,\Omega\rangle&\equiv\Omega|0\rangle\\&=\exp\sum_{i=1}^{M_\Lambda}(\eta_iX_i^\dagger-\text{H.c.})|0\rangle &(2.29\text{a})\\&=K^{1/2}(z,z^*)\exp\left[\sum_{i=1}^{M_\Lambda}z_iX_i^\dagger\right]|0\rangle\\&\equiv K^{-1/2}(z,z^*)||\Lambda,z\rangle\ , &(2.29\text{b})\end{aligned}$$

and

$$\begin{aligned}K(z,z^*)&=\left\langle 0\left|\exp\left[\sum_{i=1}^{M_\Lambda}z_i^*X_i\right]\exp\left[\sum_{i=1}^{M_\Lambda}z_iX_i^\dagger\right]\right|0\right\rangle\\&=\langle\Lambda,z||\Lambda,z\rangle=|\langle 0|\Lambda,\Omega\rangle|^2=\sum_\lambda f_{\Lambda\lambda}(z)f^*_{\Lambda\lambda}(z)\end{aligned} \tag{2.30}$$

is the Bergmann kernel of Eq. (2.16). For a semisimple Lie group, the parameters $\{z_i\}$ are explicitly given by Eq. (2.14). The state $\|\Lambda,z\rangle$ in Eq. (2.29b) is an unnormalized form of $|\Lambda,\Omega\rangle$ and $f_{\Lambda\lambda}(z)$ in Eq. (2.30) is the orthogonal basis of the $\mathcal{L}^2(\mathcal{G}/\mathcal{H})$:

$$f_{\Lambda\lambda}(z)=\langle\Lambda\lambda||\Lambda z\rangle \tag{2.31}$$

where $|\Lambda\lambda\rangle$ is the basis of $\mathfrak{h}_\Lambda$. A very important and well-known property of Eq. (2.29) is that the coherent states of Eq. (2.29) are overcomplete:

$$\int_{\mathcal{G}/\mathcal{H}}|\Lambda,\Omega\rangle d\mu(z)\langle\Lambda,\Omega|=I \tag{2.32a}$$

or

$$\int_{\mathcal{G}/\mathcal{H}}\|\Lambda,z\rangle d\mu_H(z)\langle\Lambda,z\|=I \tag{2.32b}$$

where $d\mu(z)$ is given by Eq. (2.18) and $d\mu_H(z,z^*)=K^{-1}(z,z^*)d\mu(z,z^*)$ is the measure of the $\mathfrak{L}^2(\mathcal{G}/\mathcal{H})$.

We see that the coherent states (2.29) provide a bridge linking the Hilbert space to the quantum phase space $\mathcal{G}/\mathcal{H}$. Thus the phase-space representation of a quantum system can easily be constructed by closely following the procedure of Klauder's continuous representation theory.[14]

1. Phase-space representation of operators and wave functions

The phase-space representation of a wave function can uniquely be defined as

$$f(q+ip)\equiv f(z)=\langle\Psi||\Lambda,z\rangle\ . \tag{2.33}$$

The relationship between $q+ip$ and z is already given by Eq. (2.21). It can be shown that the uniqueness of $f(z)$ requires the following identity to be satisfied:

$$f(z)=\int_{\mathcal{G}/\mathcal{H}}K(z,z'^*)f(z')d\mu_H(z',z'^*)\ . \tag{2.34}$$

The scalar product of any two such functions, $f_1(z)$ and $f_2(z)$, is

$$(f_1\cdot f_2)=\int_{\mathcal{G}/\mathcal{H}}f_1^*(z)f_2(z)d\mu_H(z,z^*)<\infty\ . \tag{2.35}$$

In Klauder's continuous representation theory,[14] $K(z,z'^*)$ is called the reproduced kernel. Clearly $f_{\Lambda\lambda}(z)$ of Eq. (2.31) satisfies conditions of Eqs. (2.34) and (2.35). It can be shown that the closed subspace $\mathfrak{L}^2(\mathcal{G}/\mathcal{H})$ [$f(z)$ of Eq. (2.34) with Eq. (2.31) as a basis set] is isomorphic to $\mathfrak{h}_\Lambda$. Furthermore, with Eq. (2.33) one can directly generalize (which we will show later) the so-called Husimi representation[36,37] to arbitrary dynamical Lie groups.

The phase-space representation of an arbitrary operator A can also uniquely be expressed by its coherent state diagonal element (i.e., the Q representation,[17] or more generally, the covariant symbol[21(d)] of A):

$$\mathfrak{A}(p,q)\equiv\mathfrak{A}(z,z^*)=\langle\Lambda,\Omega|A|\Lambda,\Omega\rangle\ . \tag{2.36}$$

The operator function $\mathfrak{A}(z,z^*)$ acting on $f(z)$ and the phase-space representation of the product A_1A_2 have the standard forms[14,21(d)]

$$(\mathfrak{A}f)(z)=\int_{\mathcal{G}/\mathcal{H}}f(z')K^{1/2}(z',z'^*)\mathfrak{A}(z',z'^*;z,z^*)\times K^{1/2}(z,z^*)d\mu_H(z',z'^*) \tag{2.37}$$

and

$$A_1A_2\longleftrightarrow(\mathfrak{A}_1\mathfrak{A}_2)(z,z^*)=\int_{\mathcal{G}/\mathcal{H}}\mathfrak{A}_1(z,z^*;z',z'^*)\times\mathfrak{A}_2(z',z'^*;z,z^*)d\mu(z',z'^*) \tag{2.38}$$

where $\mathfrak{A}(z,z^*;z',z'^*)$ is an analytical continuation of $\mathfrak{A}(z,z^*)$ to $(\mathcal{G}/\mathcal{H})\times(\mathcal{G}/\mathcal{H})$:

$$\mathfrak{A}(z,z^*;z',z'^*)=\langle\Lambda,\Omega|A|\Lambda,\Omega'\rangle\ . \tag{2.39}$$

This means[21(d)] that there exists a one-to-one correspondence between operator A and its phase-space representation $\mathfrak{A}(z,z^*)$.

An important property of the phase-space representation of the operator is its algebraic structure, which is the necessary condition to obtain the classical limit of a quantum system and study explicitly the dynamical process from quantum to classical mechanics. We shall now discuss it.

2. Algebraic structure on $\mathcal{G}$ on $\mathcal{G}/\mathcal{H}$

Since the QPS possesses a natural symplectic structure, one can always define the Poisson bracket of the phase-space representation of physical operators on this space. However, one must ascertain *whether the Poisson bracket of the phase-space representation of two arbitrary physical operators satisfies the same algebraic structure of these operators.* The answer is as follows: if the operator A is the generator T_i of $\mathcal{G}$, then the commutation relations of Eq. (I-4) in the phase-space representation will (for explicit examples see Sec. V) preserve the same algebraic structure under Poisson bracket:

$$i\hbar\{\mathfrak{T}_i,\mathfrak{T}_j\}=\sum_{k=1}^{n}C_{ij}^k\mathfrak{T}_k \tag{2.40}$$

where

$$\mathfrak{T}_i=\langle\Lambda,\Omega|T_i|\Lambda,\Omega\rangle \text{ and } T_i\in\mathfrak{g}\ . \tag{2.41}$$

Hence the phase-space representation of the commutation relations of T_i and T_j, divided by $i\hbar$, is identical to the Poisson bracket of their phase-space representation. This is an important result to provide a universal classical analogy of *any* quantum system and is a starting point for realization of Dirac's approach.[20] More about this will be discussed in the next section.

However, when A is a nonlinear function of T_i, the first-order quantum correlations are included in general in its phase-space representation. Thus the algebraic structure is in general not preserved when we replace the commutator of two arbitrary operators by the Poisson bracket of their phase-space representation. Hence the preservation of the algebraic structure in the phase-space representation can be achieved only by having additional conditions (i.e., classical limit, see the next section).

3. Phase-space distribution

As we have pointed out previously, the phase-space representation of the wave function is given by Eq. (2.33). In fact, the behavior of the wave function can also be described in terms of the density operator ρ (except for a phase factor):[38]

$$\rho\equiv|\Psi\rangle\langle\Psi|,\quad |\Psi\rangle\in\mathfrak{h}_\Lambda\ . \tag{2.42}$$

Thus the phase-space representation of the wave function can be discussed in the same way as operators:

$$\rho(z)=\langle\Lambda,\Omega|\rho|\Lambda,\Omega\rangle=|\langle\Lambda,\Omega|\Psi\rangle|^2=|f(z)|^2/K(z,z^*)\ . \tag{2.43}$$

If the system is in the state $|\Psi\rangle$, then Eq. (2.43) is the phase-space distribution of the system. In fact, $\rho(z)$ of Eq. (2.43) is a generalization of the so-called Husimi representation[36] (extending the Husimi representation to a general quantum system with a nontrivial phase space $\mathcal{G}/\mathcal{H}$).[39]

Let us now summarize the above discussion for this subsection. It shows that the normalized constant of Eq. (2.30) can provide an explicit form of the Bergmann kernel of the phase space $\mathcal{G}/\mathcal{H}$ [from which the symplectic structure of $\mathcal{G}/\mathcal{H}$ can directly be calculated by Eq. (2.15)] and the phase-space representation of a quantum system discussed above has a complete classical framework:

$$A(T_i)\to\mathfrak{A}(p,q),\quad T_i\in\mathfrak{g}\ , \tag{2.44}$$

$$|\Psi\rangle\to\rho(p,q),\quad |\Psi\rangle\in\mathfrak{h}_\Lambda\ . \tag{2.45}$$

This gives rise in general to a one-to-one correspondence between classical and quantum frameworks for various $\mathcal{G}/\mathcal{H}$ as Dirac required.[20] Since the canonical coordinates (p,q) [see Eqs. (2.21) and (2.24)] have been systematically constructed, the above results also generalize the weak correspondence principle[40] in field theory to a general quantum system.

However, although the above one-to-one correspondence is generic for various quantum systems, it pertains only to kinematics. Whether a similar explicit correspondence can exist dynamically and can be found is still a question which will be discussed next.

III. CLASSICAL ANALOGY OF QUANTUM DYNAMICS

A. Correspondence principle and classical analogy

It is desirable that a quantum theory should contain a prescription for going over to the classical limit and for relating the quantum observables to those of the corresponding classical systems. However, for a realistic quantum system it is not always possible to find a one-to-one correspondence between $\mathfrak{h}_\Lambda$ and $L^2(\mathbb{R}^N)$, where $L^2(\mathbb{R}^N)$ is the square-integrable Hilbert space on $\mathbb{R}^N$ and N is the number of the classical degrees of freedom. This is because for such a system, (i) its quantum states are usually restricted by symmetries (or statistics) and only a part of the physical space $\mathfrak{h}_\Lambda$ covers a subspace of $L^2(\mathbb{R}^N)$, and (ii) there exist in general additional internal quantum degrees of freedom (spin and so on) such that the remaining parts of $\mathfrak{h}_\Lambda$ lie outside of $L^2(\mathbb{R}^N)$, as shown schematically in Fig. 2. These are the consequences of the so-called superselection rules of quantum mechanics.[27(b),41] This fact indicates that the usual manner of studying the dynamical correspondence from classical to quantum is *in practice* not straightforward.

In this paper, we have constructed the classical-like framework (i.e., quantum phase space) of a given quantum system (specified by $\mathcal{G}$ and $\mathfrak{h}_\Lambda$). In this framework, the $2M_\Lambda$-dimensional quantum phase space $\mathcal{G}/\mathcal{H}$ does possess all the necessary structures of a classical mechanics and therefore it is always possible to establish a classical dynamical theory in $\mathcal{G}/\mathcal{H}$. In other words, one can describe a dynamical system whose motion is confined on

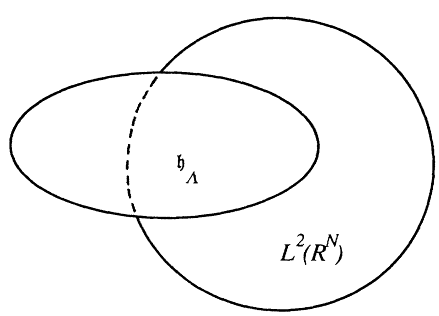

FIG. 2. An illustration of the difference between classical and quantum systems. $L^2(\mathbb{R}^N)$ (the circle part) is the square-integrable Hilbert space on $\mathbb{R}^N$, and $\mathfrak{h}$ (the elliptic part) represents a physical state space of a realistic dynamical system. For example, consider two electrons in a one-dimensional potential. The physical Hilbert space consists of the spatially symmetric and antisymmetric parts (shaded part inside the circle) of $L^2(\mathbb{R}^2)$ and spin antisymmetric and symmetric parts (shaded part outside the circle).

$\mathcal{G}/\mathcal{H}$ and is determined by the following equations of motion:

$$\frac{d\mathfrak{A}(q,p)}{dt}=\{\mathfrak{A}(q,p),\mathfrak{H}(q,p)\}; \quad q,p\in\mathcal{G}/\mathcal{H} . \tag{3.1}$$

Equivalently, Eq. (3.1) can be replaced by the Hamilton equations

$$\begin{aligned}\frac{dq_i}{dt}&=\frac{\partial\mathfrak{H}(q,p)}{\partial p_i} ,\\ \frac{dp_i}{dt}&=-\frac{\partial\mathfrak{H}(q,p)}{\partial p_i} .\end{aligned} \tag{3.2}$$

In Eqs. (3.1) and (3.2), $\mathfrak{H}(q,p)$ is the Hamiltonian of the system and $\mathfrak{A}(q,p)$ is a physical observable. The correspondence principle implied here is to find suitable conditions such that the quantum-dynamical Heisenberg equation can be written as Eq. (3.2). Based on the phase-space representation described in the preceding section [Eq. (2.44)], such a process is manifestly clear: if under suitable conditions the phase-space representation of the commutator of any two operators is equal to the Poisson bracket of the phase-space representation of these two operators, i.e.,

$$\frac{1}{i\hbar}\langle\Lambda\Omega|[A_H,B_H]|\Lambda\Omega\rangle=\{\mathfrak{A},\mathfrak{B}\} , \tag{3.3}$$

then the phase-space representation of the Heisenberg equation

$$\frac{dA_H}{dt}=\frac{1}{i\hbar}[A_H,H_H] \tag{3.4}$$

can directly be given by Eq. (3.1), and is therefore equivalent to Eq. (3.2). In Eqs. (3.3) and (3.4), A_H is the Heisenberg operator, i.e.,

$$A_H=UAU^{-1}, \quad U=e^{iHt/\hbar} \tag{3.5}$$

while A is time independent (in the Schrödinger picture). Correspondingly, the coherent state in the left-hand side of Eq. (3.3) is time independent. However, the observables in the right-hand side of Eq. (3.3) are the expectation values of the Schrödinger operators in the time-dependent coherent state (by regarding the coherent state parameters as time dependent; detailed discussions will be given later). Then, the classical theory is available in the above formulation. This formulation provides a quantum-classical correspondence. Based on the QPS which we have constructed, we shall refer to the phase-space representation [i.e., Eq. (2.44)] with dynamical equations of motion [Eq. (3.2)] as a *classical analogy* or semiquantal dynamics of the quantum system. Here QPS maintains most of the important quantum properties, such as *internal degrees of freedom, Pauli principle, statistical properties of microscopic particles, and dynamical symmetry*. Thus, although the equation of motion is formally "classical," the phase-space representation is based on the quantum framework of $\mathcal{G}/\mathcal{H}$.

The classical analogy of the quantum system discussed here in fact gives rise to the following three types of solutions.

1. The exact quantum solutions

According to Eq. (2.40), no additional requirement is needed for the generators of $\mathcal{G}$ to satisfy Eq. (3.3). Hence, when the Hamiltonian is a linear function of T_i, Eq. (3.4) can directly be reduced to Eq. (3.1). We can show this by using Eq. (3.5) and taking the expectation value of Eq. (3.4) in the time-independent coherent state which will arrive at

$$\left\langle \Lambda\Omega \left| \frac{dA_H}{dt} \right| \Lambda\Omega \right\rangle = \frac{d\langle \Lambda\Omega'(t)|A|\Lambda\Omega'(t)\rangle}{dt} = \frac{d\mathfrak{A}(q,p)}{dt} = \frac{1}{i\hbar}\langle \Lambda\Omega|[A_H,H_H]|\Lambda\Omega\rangle = \{\mathfrak{A}(q,p),\mathfrak{H}(q,p)\}, \quad q,p\in \mathcal{G}/\mathcal{H} . \tag{3.6}$$

This is just Eq. (3.1). In deriving Eq. (3.6), we have made use of Eq. (2.40). The observables $\mathfrak{A}(q,p)$ and $\mathfrak{H}(q,p)$ here are the expectation values of the Schrödinger operators A and H in the time-dependent coherent states:

$$U^{-1}|\Lambda\Omega\rangle = U^{-1}\Omega|0\rangle = \Omega'(t)h(t)|0\rangle = |\Lambda\Omega'(t)\rangle e^{i\varphi} \tag{3.7a}$$

with

$$h(t)|0\rangle = |0\rangle e^{i\varphi(h(t))} . \tag{3.7b}$$

The realization of Eq. (3.7) is based on the fact that the Hamiltonian is a linear function of T_i. In this condition, U of Eq. (3.5) is a group element of $\mathcal{G}$ and Eq. (3.7) can then be derived directly and exactly by using the Baker-Campbell-Hausdorff (BCH) formula.[17] Equation (3.6) is the equivalent of Hamilton's equation:

$$\frac{dq_i}{dt} = \frac{\partial \mathfrak{H}_l(q,p)}{\partial p_i} , \quad \frac{dp_i}{dt} = -\frac{\partial \mathfrak{H}_l(q,p)}{\partial q_i} , \tag{3.2'}$$

where $\mathfrak{H}_l(q,p)$ denotes the phase-space representation of a linear Hamiltonian function of T_i. In this case the solution of classical analogy [Eq. (3.2')] is an exact quantum solution. Furthermore, the quantum wave packet is just the coherent state of Eq. (3.7) and will not spread in time in QPS. This is a generalization of Schrödinger's quantum states of a harmonic oscillator's classical motion with minimum uncertainty to other quantum systems.[14,16,18] Clearly, in such cases, the classical and quantum motions are in one-to-one correspondence. It is worth mentioning that in this case the system is always integrable.

2. The mean-field solutions

Since $\mathfrak{H}(q,p)$ is the phase-space representation of the Hamiltonian operator H, then whether Eq. (3.3) holds true or not, the solutions of Eq. (3.2) will indeed provide a general result of the quantum mean-field dynamics of Eq. (3.4). This can be shown by the stationary phase approximation of the path integral[23,42] defined in the phase space $\mathcal{G}/\mathcal{H}$ as follows: Since the formal solution of Eq. (3.4) is the time evolution operator

$$U(t''-t') = \exp\left[\frac{1}{i\hbar}H(t''-t')\right] , \tag{3.8}$$

then its phase-space representation is

$$\mathfrak{U}(p''q'';p'q',t''-t') = \left\langle \Lambda\Omega'' \left| \exp\left[\frac{1}{i\hbar}H(t''-t')\right]\right| \Lambda\Omega' \right\rangle . \tag{3.9}$$

Using the overcomplete relation [Eq. (2.32)], we can formally express Eq. (3.9) in terms of a path integral:[23]

$$\mathcal{U}(p''q'';p'q',t''-t') = \int_{p'q'}^{p''q''} \mathfrak{D}\mu(p,q)\exp\left[\frac{i}{\hbar}\mathfrak{S}[p(t),q(t)]\right] \tag{3.10}$$

where

$$\mathfrak{D}\mu(p,q) = \prod_{t'\le t\le t''} d\mu[p(t),q(t)] \tag{3.11}$$

is the functional measure of the path integral and

$$\mathfrak{S}[p(t),q(t)] = \int_t^{t''} dt \left[\left\langle \Lambda\Omega \left| i\hbar\frac{\partial}{\partial t} \right| \Lambda\Omega\right\rangle - \mathfrak{H}(p(t),q(t))\right] \tag{3.12}$$

is the system's action functional. Again, $\mathfrak{H}(p(t),q(t))$ in Eq. (3.12) is the expectation value of the Schrödinger operators H evaluated for the time-dependent coherent states. An application of the variation principle to Eq. (3.12) (i.e., the stationary phase approximation of the path integral):

$$\delta\mathfrak{S}[p(t),q(t)] = 0 \tag{3.13}$$

will give the Hamilton equation of Eq. (3.2)[43]

$$\frac{dq_i}{dt} = \frac{\partial \mathfrak{H}(q,p)}{\partial p_i} , \quad \frac{dp_i}{dt} = -\frac{\partial \mathfrak{H}(q,p)}{\partial q_i} , \tag{3.2'}$$

where

$$\mathfrak{H}(p(t),q(t)) = \langle \Lambda\Omega|H|\Lambda\Omega\rangle \tag{3.14}$$

is the well-known mean-field approximation or mean-field dynamics. Thus the commonly used mean-field dynamics is merely a special case of the classical analogy.

It should be mentioned that in the local canonical coordinates expression of the path integral on $\mathcal{G}/\mathcal{H}$, the requirement of the canonical transformation invariant of the integral measure may add a surface term to the action in Eq. (3.10). This term will not have any effect in the mean-field dynamical equation of Eq. (3.2) since it vanishes under the variation of the action of Eq. (3.13) and therefore will not be discussed here. However, it must be considered carefully when one is concerned with the full quantum-mechanical problem.[44]

3. The "classical" limits

If the following factorization condition of the phase-space representation is found:[45]

$$\langle \Lambda\Omega | AB | \Lambda\Omega \rangle = \mathfrak{A}(p,q)\mathfrak{B}(p,q) \tag{3.15}$$

and is applied to the Hamiltonian function

$$\mathfrak{H}(p(t),q(t)) \rightarrow \mathfrak{H}_c(p(t),q(t)) = H(\mathfrak{T}\mathfrak{A}(p(t),q(t))) , \tag{3.16}$$

then the mean-field dynamics is reduced to what we call the "classical" limit:

$$\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial \mathfrak{H}_c(q,p)}{\partial p_i} , \\ \frac{dp_i}{dt} &= -\frac{\partial \mathfrak{H}_c(q,p)}{\partial q_i} . \end{aligned} \tag{3.2'''}$$

In fact, additional requirements must be imposed in order for Eq. (3.16) to hold. For example, one such requirement is for $\hbar \rightarrow 0$. But this is not in the usual limiting procedure quoted in textbooks, but in the sense that $\chi \rightarrow \infty$, where χ is a parameter related to the structure of $\mathfrak{h}$ [see Eq. (2.21)]. What one normally means by $\hbar \rightarrow 0$ is that there is some quantity in the system which must have the same dimension as $\hbar$ and should be large in comparison to $\hbar$. However, there does not exist a general criterion to search for such a quantity. Here we have explicitly provided such a quantity, i.e., χ, and it emerges naturally from the system's geometry. Such a quantity was discussed by Yaffe[22] but without explicitly constructing it. In a forthcoming paper,[46] we shall provide a systematic procedure to calculate χ and show that when $\chi \rightarrow \infty$, the quantum correlations must explicitly approach zero for which Eq. (3.16) is satisfied. In such a case, the quantum system truly approaches a classical limit. However, for some systems, χ cannot approach infinity and therefore has no classical limit.

The time evolution of the phase-space distribution $\rho(z)$ [Eq. (2.43)] is

$$\frac{d\rho(p,q)}{dt} = \frac{1}{i\hbar} \int_{G/H} [\mathfrak{H}(z,z')\rho(z',z) - \rho(z,z')\mathfrak{H}(z',z)] \times d\mu(z',z'^*) . \tag{3.17a}$$

This is the integral form of the quantum Liouville equation. In the classical limit, if it exists, the dynamical equation of $r(z)$ is reduced to

$$\frac{d\rho(p,q)}{dt} = \{\mathfrak{H}_c(p,q), \rho(p,q)\} . \tag{3.17b}$$

This is the Liouville equation in a classical theory.

Now, let us analyze the classical analogy defined by Eqs. (3.2′)–(3.2‴). The classical analogy consisted of three cases, i.e., the exact, the mean-field, and the "classical" limit cases. They are defined by Eqs. (3.2′), (3.2″), and (3.2‴), respectively. Indeed, when the Hamiltonian is a linear function of T_i, we have

$$\mathfrak{H}_l(p(t),q(t)) = \mathfrak{H}(p(t),q(t)) = \mathfrak{H}_c(p(t),q(t)) , \tag{3.18}$$

therefore all three cases are equivalent in this particular situation. Thus the classical analogy is reduced to only Eqs. (3.2″) and (3.2‴) and the latter is obviously only an approximation of the former. The difference between Eqs. (3.2″) and (3.2‴) is the quantum correlation $\Delta\mathfrak{H}$:

$$\Delta\mathfrak{H} = \mathfrak{H} - \mathfrak{H}_c . \tag{3.19}$$

For any quantum system, if the quantum fluctuation is fixed, its dynamical evolution should follow the trajectories determined by Eq. (3.2″) and not Eq. (3.2‴). For integrable systems, the global (topological) structures of the trajectories determined by $\mathfrak{H}$ and $\mathfrak{H}_c$ are not very different. However, for nonintegrable systems, the global structures of the trajectories determined by $\mathfrak{H}$ and $\mathfrak{H}_c$ can be, and generally are, completely different.[32,33] The quantum dynamics, however, is definitely closer to $\mathfrak{H}$ than $\mathfrak{H}_c$. Primary attention thus far is focused on obtaining the solutions for $\mathfrak{H}_c$ and its corresponding requantization problem in the study of both classical limit and "quantum chaos." Since $\Delta\mathfrak{H}$ is now an explicit function of dynamical variables and is consistently included in the semiquantal dynamical equations, we can use Eqs. (3.2″) and (3.2‴) to explicitly examine the difference between the classical and the semiquantal dynamics and to investigate how quantum correlation can alter the classical dynamical structure. This will provide a direct and dynamical quantum manifestation of chaos in the semiquantal sense. In Sec. V we will give an explicit example to illustrate this very important point which has, in fact, not been noticed in the study of "quantum chaos."

It is also worth pointing out that the above "classical" limit is different from the well-known Ehrenfest theorem which is derived by imposing an *ad hoc* factorization ansatz.[47] In Ehrenfest's theorem, there is no systematic way to take the $\hbar \rightarrow 0$ limit. On the other hand, a coherent state with designated fixed state [Eq. (2.9)] is itself a wave function with certain built-in minimum uncertainty and therefore *is* a state, best describing a "classical" particle. The factorization introduced via the coherent state will naturally reduce the mean-field approximation to the classical equations.

We are now in a position to show rigorously the consistency of quantum integrability in I with the classical theory and the relation between dynamical symmetry and integrability in classical mechanics.

B. Integrability and dynamical symmetry

In I, quantum integrability is defined as follows: a quantum system with M_Λ independent degrees of freedom is integrable if and only if one can simultaneously measure accurately the M_Λ NFD observables in the energy representation. In other words, there exist $M_\Lambda - 1$ independent NFD observables: $\{C_i,\ i=1,\ldots,M_\Lambda - 1\}$, which commute with each other and H:

$$[C_i, C_j] = 0 \text{ and } [C_i, H] = 0 . \tag{3.20}$$

By Eqs. (3.16) and (2.40), it is easy to show that in the "classical" limit of classical analogy we have

$$\{\mathfrak{C}_i, \mathfrak{C}_j\} = 0 \text{ and } \{\mathfrak{C}_i, \mathfrak{H}\} = 0 . \tag{3.21}$$

Together with Eq. (3.2'''), Eq. (3.21) also formally defines the classical integrability, and it is in this sense that the definition of quantum integrability is completely consistent with the classical theory.

We will now discuss the relation between the integrability and dynamical symmetry. This is straightforward within the context of the classical analogy. In the classical analogy, the group structure of the system is defined by Poisson brackets. Since the algebraic structure of T_i in the phase-space representation is preserved [see Eq. (2.40)], the concept of the dynamical symmetry in I is also naturally preserved in the classical analogy. Therefore the theorem on integrability and dynamical symmetry is also operational for the classical theory. To be more explicit, we will first show that if the Hamiltonian has the symmetry R, then its phase-space representation also has the same symmetry. This is because if

$$RHR^{-1}=H \tag{3.22}$$

then, in the phase-space representation, we have

$$\langle\Lambda\Omega|H|\Lambda\Omega\rangle=\langle\Lambda\Omega|RHR^{-1}|\Lambda\Omega\rangle=\langle\Lambda\Omega'|H|\Lambda\Omega'\rangle\ , \tag{3.23a}$$

i.e.,

$$\mathfrak{H}(p,q)=\mathfrak{H}(p',q') \tag{3.23b}$$

where

$$R^{-1}|\Lambda\Omega\rangle=R^{-1}\Omega|0\rangle=\Omega'h|0\rangle=|\Lambda\Omega'\rangle e^{i\varphi(h)}\ . \tag{3.24}$$

Furthermore, although the algebraic structure of two functions of generators in the phase-space representation may not be preserved, the phase-space representation of an *invariant operator* $C(T_i)$ can usually be expressed in the following form:

$$\langle\Lambda\Omega|C(T_i)|\Lambda\Omega\rangle=s(\chi)C(\langle\Lambda\Omega|T_i|\Lambda\Omega\rangle)+c(\chi) \tag{3.25}$$

where $s(\chi)$ and $c(\chi)$ are functions of χ, the physical parameter. In the "classical" limit, $s(\chi)$ approaches unity and $c(\chi)$ goes to zero. Explicit examples of Eq. (3.25) have been given for SU(2),[12] SU(6),[48] SO(8), and Sp(6) cases.[49] Equations (2.40) and (3.25) show that if C is an invariant operator, then its phase-space representation is also an invariant observable [obeying Eq. (3.21)]. This means that the symmetry and dynamical symmetry of the system are also preserved in the classical analogy. This also indicates that the integrability of the system is preserved even in the general classical analogy (not only the "classical" limit). Therefore the relationship between integrability and dynamical symmetry in classical analogy is the same as in the quantum case.

IV. A GENERAL ALGORITHM FOR SEEKING "QUANTUM CHAOS"

Thus far, there does not exist a generally accepted definition of "quantum chaos." However, two working definitions are often used: One is the so-called level-spacing distribution[2,50] and the other is the study of the semiclassical dynamics of the quantum system whose classical analogy is chaotic.[25] The first definition can be built firmly on the random matrix theory and it "predicts" that the Gaussian orthogonal ensemble (GOE) distribution is a possible generic property of the quantum counterpart of a classical chaotic system. However, whether such a definition is unique and whether it is consistent with the classical concept remain open. Furthermore, there is no transparent link of the GOE to any dynamical properties of the Hamiltonian system. The second working definition is to seek the semiclassical manifestation of classical chaos in a quantum system, where "semiclassical" means $\hbar\rightarrow 0$ but not equal to zero.[3] It is well known that the physical measurement of a quantum system is the properties of the observable operators in wave functions. Therefore this working definition demands that one should first study the time evolutionary semiclassical behaviors of expectation values of operators in the phase space, followed by the properties of the wave functions. In the preceding two sections, a generic phase-space framework and phase-space representation of the operators and the wave functions have been established for an arbitrary quantum system from which a classical analogy (semiclassically) is derived. In this section a general algorithm for seeking manifestation of quantum chaos via the classical analogy can be provided even when its classical version is absent. The algorithm is as follows.

(1) Determine the algebraic structure of the system from its dynamical properties:

$$\mathcal{g}:\ \left\{T_i;[T_i,T_j]=\sum_{k=1}^{n}C_{ij}^{k}T_k,\ i=1,\ldots,n\right\}.$$

The Hilbert space of the system is represented by one of the irrep carrier spaces $\mathfrak{h}_\Lambda$ of $\mathcal{g}$:

$$f(T_i)|\Psi\rangle\in\mathfrak{h}_\Lambda,\quad\forall|\Psi\rangle\in\mathfrak{h}_\Lambda\ .$$

(2) Ascertain the integrability of the system by locating the dynamical symmetry of the Hamiltonian which provides a stringent test as to whether the Hamiltonian can be expressed as a function of the Casimir operators of a particular subgroup chain $\mathcal{G}^\alpha$ of $\mathcal{G}$.

(3) Determine the number of quantum degrees of freedom M_Λ from the NFD operators of CSCO S. For any group chain $\mathcal{G}^\alpha$ of $\mathcal{G}$ with a specific $\mathfrak{h}_\Lambda$, this number is unique:

$$M_\Lambda=\text{dimension of } S^\alpha,\ S^\alpha\text{: }\{Q_i^\alpha\in S,Q_j^\alpha|\gamma^\alpha\rangle\neq c|\gamma^\alpha\rangle\}$$

where c is a constant.

(4) Construct the $2M_\Lambda$-dimensional QPS $\mathcal{G}/\mathcal{H}$:

$$\mathcal{G}/\mathcal{H}\ni\Omega=\exp\sum_{i=1}^{M_\Lambda}(\eta_iX_i^\dagger-\text{H.c.})$$

where

$$X_i^\dagger\in\mathcal{g}$$

is an elementary excitation operator. In practice, instead of seeking $X_i^\dagger$, it is simpler to find $\mathcal{H}$ for the fixed state $|0\rangle$.

(5) Calculate the phase-space representation of Hamiltonian operator H, i.e., evaluate the expectation value of H in the coherent states:

$$\mathfrak{H}(q,p)=\langle \Lambda\Omega|H|\Lambda\Omega\rangle$$

where

$$|\Lambda,\Omega\rangle=\exp\sum_{i=1}^{M_\Lambda}(\eta_i X_i^\dagger-\text{H.c.})|0\rangle .$$

The generalized coordinates and momenta (q_i,p_i) are related to z_i or η_i [cf. Eq. (2.19)].

(6) Solve Hamilton's equations [Eq. (3.2'')]:

$$\frac{dq_i}{dt}=\frac{\partial\mathfrak{H}(q,p)}{\partial p_i},$$

$$\frac{dp_i}{dt}=-\frac{\partial\mathfrak{H}(q,p)}{\partial q_i} .$$

(7) Explore the phase-space distribution of the wave function [Eq. (2.43)]:

$$\rho(q,p)=|\langle\Lambda\Omega|\Psi\rangle|^2=|f(q,p)|^2/K$$

where $|\Psi\rangle$ is the solution of the Schrödinger equation and K is the Bergmann kernel of $\mathcal{G}/\mathcal{H}$ [Eq. (2.30)].

Remarks. (i) Steps (1)–(5) do not involve any approximation or assumptions, and therefore the structure of the quantum theory is completely preserved. Thus a connection of the fundamental framework between quantum and classical mechanics is made. (ii) Step (6) provides a generic *classical analogy* of a quantum system. If *no* replacement of $\mathcal{H}(q,p)=\langle\Lambda\Omega|H(T_i)|\Lambda\Omega\rangle$ by $H(\langle\Lambda\Omega|T_i|\Lambda\Omega\rangle)$ is made, in other words, keeping the first-order quantum correlations in the equation of motion, then the above Hamilton's equations of the classical analogy give a semiquantal solution in which many important quantum properties (e.g., internal degrees of freedom, Pauli principle, statistical properties of microscopic particles, symmetry, dynamical symmetry, and minimum uncertainty relation) are strictly preserved. In fact, these solutions merely correspond to the usual mean-field dynamics. For most complicated quantum systems in solid-state physics, atomic and molecular physics, and nuclear physics, the quantum dynamics of such systems are only well investigated at this level. Only under the "classical" limit $\hbar\to 0$ in the sense $\chi\to\infty$, are the semiquantal solutions reduced to the classical cases. Thus, comparing with the semiquantal and classical solutions in step (6), one can dynamically and quantitatively carry out the effect of the quantum correlations to the classical trajectories, and therefore provide a way to test the manifestation of "quantum chaos" in the sense of the second working definition. (iii) Step (7) provides a generic form of one of the three [Q (Husimi), P, and Wigner] phase-space distributions of wave function in coherent state (phase) space.[17] This phase-space distribution has widely been used to reveal the classical invariant structure in wave functions and has become the hottest topic in the study of "quantum chaos." The classical analogy presented in this paper shows that under "classical" limit the wave function should be reduced to the coherent states [Eq. (2.29)]. Furthermore, the equation of motion for step (7), i.e., Eq. (2.46), can provide an understanding of the dynamical process for the classical reduction of wave functions and the patterns of classical trajectories in wave function. To focus the main motivation of this paper, the dynamics of step (7) will be discussed in detail in the next paper.

V. EXAMPLES

To illustrate the general procedure and the conclusions we have reached in the preceding sections, let us consider several fundamental examples in quantum physics. It should be emphasized that these examples, which are by no means exhaustive, will serve only to illustrate the theoretical underpinnings of the study we have introduced here.

A. Harmonic oscillator system (H_4)

It was shown in I that as in the classical case the quantum harmonic oscillator which has a dynamical group H_4 is a one-degree-of-freedom system. Here we shall study its phase-space structure, phase-space distribution, and classical analogy.

(1) QPS structure. Since the dynamical group is H_4 with algebra h_4: $\{a^\dagger,a,a^\dagger a,I\}$ and the corresponding Hilbert space is the Fock space V^F: $\{|n\rangle, n=1,2,\ldots\}$, the QPS can easily be constructed. By Eq. (I-11), the fixed state is just the ground state $|0\rangle$. Also, according to Eq. (2.9), the elementary excitation is $a^\dagger$. This result is consistent with the definition of the number of QDDF.[1] Then the QPS is constructed from the unitary exponential mapping of the subspace $\not{k}$: $\{a^\dagger,a\}$ of h_4:

$$\Omega(z)=\exp(za^\dagger-z^*a)\in H_4/\mathrm{U}(1)\otimes\mathrm{U}(1) \tag{5.1}$$

where $\mathrm{U}(1)\otimes\mathrm{U}(1)$ (with generators $a^\dagger a$ and I) is the maximal stability subgroup of $|0\rangle$. Since $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ is isomorphic to the one-dimensional complex plane C^1, the structure of the QPS is rather trivial and the metric and measure are, respectively,

$$ds=dzdz^*\ (\text{i.e., } g_{ij}=\delta_{ij})\text{ and } d\mu(z)=\frac{dzdz^*}{\pi} . \tag{5.2}$$

It is obvious that the space $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ is noncompact because of the infiniteness of V^F. This agrees with our remark in I. Furthermore, the existence of symplectic structure on complex plane is well known and the Poisson bracket of two functions $\mathfrak{F}_1,\mathfrak{F}_2$ defined on C^1 is

$$\{\mathfrak{F}_1,\mathfrak{F}_2\}=\frac{1}{i\hbar}\left[\frac{\partial\mathfrak{F}_1\partial\mathfrak{F}_2}{\partial z\partial z^*}-\frac{\partial\mathfrak{F}_1\partial\mathfrak{F}_2}{\partial z^*\partial z}\right] . \tag{5.3}$$

When we introduce the usual canonical position and momentum coordinates

$$z=\frac{1}{\sqrt{2\hbar}}(q+ip)\text{ and } z^*=\frac{1}{\sqrt{2\hbar}}(q-ip) \tag{5.4}$$

into Eq. (5.3), it will take on the standard form of Eq. (2.25).

(2) Phase-space distribution. The phase-space represen-

tation of the quantum harmonic oscillator can be realized by the well-known Glauber coherent states $|z\rangle$.[19] The set $\{|z\rangle\}$ is isomorphic to $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ and can be constructed by $\Omega(z)$ acting on $|0\rangle$:

$$\begin{aligned}|z\rangle \equiv \Omega(Z)|0\rangle &= \exp(za^\dagger - z^* a)|0\rangle \\ &= e^{-zz^*/2}\exp(za^\dagger)|0\rangle \equiv e^{-zz^*/2}\|z\rangle \ .\end{aligned} \tag{5.5}$$

The normalization constant of $\|z\rangle$ is the Bergmann kernel $K(z,z^*)$:

$$K(z,z^*)=\exp(zz^*) \ . \tag{5.6}$$

By using Eqs. (2.15), it can easily be verified that Eq. (5.6) provides the mathematical structure of $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$, i.e., Eq. (5.2). Based on the general theory given in Sec. II, the phase-space representation of the wave function $|\Psi\rangle \in V^F$ is

$$f(z)=\langle\Psi\|z\rangle = \sum_{n=0}^{\infty} f_n \frac{z^n}{\sqrt{n!}} \tag{5.7}$$

or

$$\rho(z)=|f(z)|^2 K(z,z^*) \tag{5.8}$$

where $f(z)\in L^2(C)$. On the other hand, with the aid of Wick's theorem,[51] it is always possible to express an observable operator A in terms of the normal product form:

$$A = A(a^\dagger,a)=\sum_{k,l} A^n_{kl}(a^\dagger)^k(a)^l \ . \tag{5.9}$$

Then the phase-space representation of A is just

$$\mathfrak{A}(z,z^*)=\langle z|A|z\rangle = \sum_{k,l} A^n_{kl}(z^*)^k(z)^l \ . \tag{5.10}$$

When A is restricted to the generator of H_4, we have

$$\begin{aligned}&\mathfrak{a}^\dagger=\langle z|a^\dagger|z\rangle = z^*, \mathfrak{a}=\langle z|a|z\rangle = z \ , \\ &\mathfrak{a}^\dagger\mathfrak{a}=\langle z|a^\dagger a|z\rangle=|z|^2, \quad \mathfrak{l}=\langle z|I|z\rangle = 1 \ .\end{aligned} \tag{5.11}$$

Hence the corresponding algebraic structure of H_4 in the phase-space represented is given by the Poisson brackets:

$$i\hbar\{\mathfrak{a},\mathfrak{a}^\dagger\}=1, \quad i\hbar\{\mathfrak{a}^\dagger\mathfrak{a},\mathfrak{a}\}=-\mathfrak{a}, \quad i\hbar\{\mathfrak{a}^\dagger\mathfrak{a},\mathfrak{a}^\dagger\}=\mathfrak{a}^\dagger \ . \tag{5.12}$$

This shows that the algebraic structure of the H_4 generators is preserved when the commutators of operators are replaced by Poisson brackets in the phase-space representation. Using Eq. (5.4), we immediately obtained the Dirac quantization condition[20]

$$[q,p]=i\hbar\{q,p\} \ . \tag{5.13}$$

(3) Classical analogy. According to the general discussion of Sec. III, the classical analogy of the system is governed by the Hamilton equations:

$$\begin{aligned}\frac{dq}{dt} &= \frac{\partial \mathfrak{H}(q,p)}{\partial p} \ , \\ \frac{dp}{dt} &= -\frac{\partial \mathfrak{H}(q,p)}{\partial q} \ ,\end{aligned} \tag{5.14}$$

where

$$\mathfrak{H}(q,p)=\langle z|H|z\rangle \ . \tag{5.15}$$

For instance, consider the forced harmonic oscillator system of Eq. (I-14). The classical analogy of the Hamiltonian is

$$\mathfrak{H}(q,p)=\frac{\omega}{2}(p^2+q^2)+\sqrt{2}\,\mathrm{Re}[\lambda(t)q]-\sqrt{2}\,\mathrm{Im}[\lambda(t)p] \tag{5.16}$$

and the solution of Eq. (5.14) with $\mathfrak{H}(q,p)$ given by Eq. (5.16) is

$$\begin{aligned}q(t)+ip(t)&=\left[q(0)+ip(0)-i\sqrt{2}\int_0^t \lambda^*(\tau)e^{i\omega\tau}d\tau\right]e^{-i\omega t} \\ &= z(t)\sqrt{2\hbar} \ .\end{aligned} \tag{5.17}$$

If the initial state is $|0\rangle$ or a coherent state $|z(0)\rangle$, then we can show that the exact quantum solution of Eq. (I-14) is

$$|\psi(t)\rangle = |z(t)\rangle e^{i\varphi(t)} \tag{5.18}$$

where $z(t)$ is given by Eq. (5.17). The phase factor $\varphi(t)$ in Eq. (5.18) is a quantum effect determined by $z(t)$:

$$\varphi(t)=-\tfrac{1}{2}\omega t - \int_0^t \mathrm{Re}[\lambda(\tau)z(\tau)]d\tau \ . \tag{5.19}$$

This shows that the classical analogy does indeed provide an exact quantum solution if the Hamiltonian is a linear function of the generators of $\mathcal{G}$.

B. Spin systems [SU(2)]

In this subsection, we will construct the phase-space structure of the spin system and its phase-space distribution and classical analogy.

(1) QPS structure. Since the dynamical group of spin system is SU(2) and its Hilbert space is V^{2j+1}: $\{|jm\rangle\}$, where $m=-j,-j+1,\ldots,j$, and spin j is integer or half-integer, the fixed state of Eq. (I-15) is $|j-j\rangle$ which is the lowest weight state of V^{2j+1}. Thus the elementary excitation operator of the spin system is J_+. The explicit form of $|jm\rangle$ is

$$|jm\rangle = \frac{1}{(j+m)!}\begin{bmatrix}2j\\ j+m\end{bmatrix}^{-1/2}(J_+)^{j+m}|j-j\rangle \ . \tag{5.20}$$

It is obvious that any state $|\Psi\rangle=\sum_{m=-j}^{j} f_m|jm\rangle \in V^{2j+1}$ can be generated by a polynomial of J_+ acting on $|j-j\rangle$. This again shows that the number of QDDF equals the number of elementary excitation operators. The QPS can then be obtained by mapping the subspace $\mathscr{p}$: $\{J_+,J_-\}$ to the coset space SU(2)/U(1):

$$(\eta J_+ - \eta^* J_-) \to \exp(\eta J_+ - \eta^* J_-) \tag{5.21}$$

where $\eta=(\theta/2)e^{-i\varphi}$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$, i.e., the coset space SU(2)/U(1) (the QPS) is isomorphic to a two-dimensional sphere S^2. The differential structure on S^2 can be evaluated from the coherent states of SU(2)/U(1) via Eqs. (2.30) and (2.17). The coherent states of

SU(2)/U(1) in V^{2j+1} are well known:[52]

$$|j\Omega\rangle \equiv \exp(\eta J_+ - \eta^* J_-)|j\,-j\rangle = (1+zz^*)^{-j}\exp(zJ_+)|j\,-j\rangle \equiv (1+zz^*)^{-j}||jz\rangle \tag{5.22}$$

and

$$z=\tan\frac{\theta}{2}e^{-i\varphi} . \tag{5.23}$$

By Eq. (2.30), the generalized Bergmann kernel on S^2 is

$$K(z,z^*)=(1+zz^*)^{2j} . \tag{5.24}$$

Thus its metric g_{ij} and measure $d\mu$ are

$$g_{ij}=\delta_{ij}\frac{2j}{(1+zz^*)^2} \text{ and } d\mu=\frac{(2j+1)}{\pi}\frac{dzdz^*}{(1+zz^*)^2} , \tag{5.25}$$

respectively. The Poisson bracket can be expressed as follows:

$$\{\mathfrak{F}_1,\mathfrak{F}_2\}=\frac{(1+zz^*)^2}{i2j\hbar}\left[\frac{\partial\mathfrak{F}_1\partial\mathfrak{F}_2}{\partial z\partial z^*}-\frac{\partial\mathfrak{F}_1\partial\mathfrak{F}_2}{\partial z^*\partial z}\right] . \tag{5.26}$$

By introducing the canonical coordinates

$$\frac{1}{\sqrt{4j\hbar}}(q+ip)=\frac{z}{\sqrt{1+zz^*}}=\sin(\theta/2)e^{-i\varphi} \tag{5.27}$$

we have

$$\{\mathfrak{F}_1,\mathfrak{F}_2\}=\frac{\partial\mathfrak{F}_1\partial\mathfrak{F}_2}{\partial q\partial p}-\frac{\partial\mathfrak{F}_1\partial\mathfrak{F}_2}{\partial p\partial q} \tag{5.28}$$

where $p^2+q^2\le 4j\hbar$. This shows that the phase space of a spin system is compact.

(2) Phase-space representation. Using the overcomplete relation of the coherent states $|j\Omega\rangle$ (or $||jz\rangle$):

$$\int_{S^2}|j\Omega\rangle d\mu\langle j\Omega|=I \tag{5.29a}$$

or

$$\int_{S^2}||jz\rangle d\mu_H\langle jz||=I , \tag{5.29b}$$

the phase-space representation of the spin system is easily realized. Explicitly, the phase-space representation of the wave function $|\Psi\rangle\in V^{2J+1}$ is

$$f(z)=\langle\Psi||j\Omega\rangle=\sum_{n=0}^{\infty}f_n\begin{bmatrix}2j\\ j+m\end{bmatrix}^{1/2}z^{j+m} \tag{5.30}$$

where $f(z)\in L^2(S^2)$. The phase-space representation of an operator $A=A(J_i)$ is

$$\mathfrak{A}(z,z^*)=\langle j\Omega|A(J_i)|j\Omega\rangle . \tag{5.31}$$

When A is J_+, J_- or J_0, the results are

$$\begin{aligned}\mathfrak{J}_+&=\langle j\Omega|J_+|j\Omega\rangle\\&=\frac{2jz^*}{1+zz^*}=\frac{1}{2\hbar}(q-ip)(4j\hbar-p^2-q^2)^{1/2} ,\\ \mathfrak{J}_-&=\langle j\Omega|J_-|j\Omega\rangle\\&=\frac{2jz}{1+zz^*}=\frac{1}{2\hbar}(q+ip)(4j\hbar-p^2-q^2)^{1/2} ,\\ \mathfrak{J}_0&=\langle j\Omega|J_0|j\Omega\rangle=j\frac{zz^*-1}{1+zz^*}=\frac{1}{2\hbar}(p^2+q^2)-j\hbar .\end{aligned} \tag{5.32}$$

The corresponding algebraic structure of SU(2) in the phase-space representation is determined by Poisson brackets:

$$i\hbar\{\mathfrak{J}_-,\mathfrak{J}_+\}=-2\mathfrak{J}_0,\quad i\hbar\{\mathfrak{J}_0,\mathfrak{J}_\pm\}=\pm\mathfrak{J}_\pm . \tag{5.33}$$

Since $\mathfrak{J}_\pm=\langle j\Omega|(J_x\pm iJ_y)/\hbar|j\Omega\rangle=(\mathfrak{J}_x\pm i\mathfrak{J}_y)/\hbar$, $\mathfrak{J}_0=\mathfrak{J}_z/\hbar$,[38] we have

$$\{\mathfrak{J}_i,\mathfrak{J}_j\}=\varepsilon_{ijk}\mathfrak{J}_k,\quad i,j,k=x,y,z . \tag{5.34}$$

These are just the well-known Poisson bracket of angular momentum. The dynamical properties of the system in the phase-space representation are formally represented by Eq. (2.46).

(3) Classical analogy. The classical analogy of an observable $A(J_i)$ is given by the following equation:

$$\mathfrak{A}(q,p)=\langle j\Omega|A(J_i)|j\Omega\rangle . \tag{5.35a}$$

Comparing Eq. (5.27) and Eq. (2.21), we conclude that $\chi=2j$. Thus the classical limit is $j\to\infty$ and the classical Hamiltonian function is

$$\mathfrak{A}_c(q,p)=A(\langle j\Omega|J_i|j\Omega\rangle)=A(\mathfrak{J}_+,\mathfrak{J}_-,\mathfrak{J}_0) . \tag{5.35b}$$

If one keeps $j\hbar$ finite, then $j\to\infty$ implies $\hbar\to 0$. This is the common understanding of the classical limit. Its time evolution is determined by dynamical equation of Eq. (3.2'''). The solutions may be obtained by solving Hamilton's equation with the Hamiltonian

$$\mathfrak{H}(q,p)\langle j\Omega|H|j\Omega\rangle = H(\mathfrak{J}_+(q,p),\mathfrak{J}_-(q,p),\mathfrak{J}_0(q,p)) \quad \text{as } \hbar\to 0, j\to\infty . \tag{5.36}$$

For example, let us consider the two-level Lipkin model:[53]

$$H=\varepsilon J_0+\frac{W}{2}(J_+J_-+J_-J_+)+\frac{V}{2}(J_+^2+J_-^2) . \tag{5.37}$$

This Hamiltonian does not have dynamical symmetry. However, as we have pointed out in I, the energy conservation of the system restores the dynamical symmetry of the one-degree-of-freedom system. Thus this system is still integrable as in the classical theory. The phase-space representation of the Hamiltonian is

$$\mathfrak{H}(q,p)=\begin{cases}\varepsilon x+\left[1-\dfrac{1}{2j}\right](Wx-Vy)(2j\hbar-x)+(W\hbar-\varepsilon)j\hbar\\ \varepsilon x+(Wx-Vy)(2j\hbar-x)-\varepsilon j\hbar\quad(\hbar\to 0, j\to\infty)\end{cases} \tag{5.38}$$

where

$$x=\tfrac{1}{2}(p^2+q^2),\quad y=\tfrac{1}{2}(p^2-q^2)\ . \tag{5.39}$$

It is manifestly clear from Eq. (5.38) that the difference between the "classical" limit and the mean-field dynamics is the quantum correlations. Usually, the solutions of the "classical" limit and mean-field dynamics qualitatively behave in a similar manner. However, quite often the "classical" limit is not available. For instance, if $j=\frac{1}{2}$, the "classical" limit clearly breaks down and yet the mean-field theory still provides the same qualitative behavior as the exact quantum solutions. This can be verified from Eq. (5.38). In the exact quantum solution, when $j=\frac{1}{2}$, the contribution of the W term is $W\tilde{\hbar}^2/2$ while the contribution from the V term is zero. Likewise, the same results are obtained by substituting $j=\frac{1}{2}$ into the mean-field equation of Eq. (5.38). Solving the mean-field dynamics and requantizing it, the exact quantum solution can be recovered in this case. This indicates that the mean-field dynamics is at least qualitatively close to the quantal solution.

C. Two-level atomic system interacting with an external quantum field

The above two examples are standard textbook examples, based on which the basic assumptions and conclusions of our theory are tested. However, both systems, having only one degree of freedom, are integrable if the system is autonomous. In order to apply the theory to the nonintegrable case, we now consider some examples. One such example, the two spin coupled system, has already been studied in a previous paper[10] and the result is consistent with our present formulation. We will not repeat the discussion here. Another example is the spin-harmonic coupled system which simulates, for instance, the two-level atomic system interacting with an external quantum field. The Hamiltonian of this system is

$$\begin{aligned} H&=\omega_0\tilde{\hbar}S_0+\tilde{\hbar}\omega b^\dagger b+\frac{\alpha}{2}\tilde{\hbar}(b^\dagger+b)(S_++S_-)\\ &=H_0+H_1 \end{aligned} \tag{5.40}$$

where S_i are the spin operators and $b^\dagger$, b the creation and annihilation operators of the external field. This system can be regarded as a simplified model of the matter-field interaction (e.g., electron-radiation field interacting or electron-phonon interacting) in which the general form of Hamiltonian is

$$\begin{aligned} H=&\sum_{q;s}e_q a_{qs}^\dagger a_{qs}+\sum_k \omega_k(b_k^\dagger b_k+\tfrac{1}{2})\\ &+\sum_{k;q;s}\alpha(k)a_{q-k,s}^\dagger a_{qs}(b_k^\dagger+b_{-k})\ , \end{aligned} \tag{5.41}$$

where e_q is the one-body energy of electrons with wave number q, spin s ($+$ or $-$) with the corresponding creation (annihilation) operator $a_{qs}^\dagger$ (a_{qs}). The coupling constant $\alpha(k)$ is the strength of the electron-field interaction. When we restrict ourselves to the problem of two electronic states ($q=1$ and 2), the Hamiltonian can be reduced to the form of Eq. (5.40), in which

$$\begin{aligned} &S_0=\tfrac{1}{2}\sum_s(a_{2s}^\dagger a_{2s}-a_{1s}^\dagger a_{1s})\ ,\\ &S_+=\sum_s a_{2s}^\dagger a_{1s},\quad S_-=(S_+)^\dagger\ . \end{aligned} \tag{5.42}$$

This is also a schematic nuclear model if one integrates away the field part (which, of course, is a truly difficult task in practice) and keeps the matter interaction up to two-body interaction terms. In such a scheme, the Hamiltonian is reduced to the Lipkin model of Eq. (5.37).

The dynamical group of Eq. (5.40) is $SU(2)\otimes H_4$ with the Hilbert space $\mathfrak{h}=V^F\otimes V^{2j+1}$:

$$\mathfrak{h}=\{|n\rangle|jm\rangle\}\ . \tag{5.43}$$

No dynamical symmetry exists for the entire Hamiltonian of Eq. (5.40). However, the interacting term of Eq. (5.40) can be separated into two parts:

$$\begin{aligned} \frac{\alpha}{2}\tilde{\hbar}(b^\dagger+b)(S_++S_-)&=\frac{\alpha}{2}\tilde{\hbar}(bS_++b^\dagger S_-)\\ &\quad+\frac{\alpha}{2}\tilde{\hbar}(b^\dagger S_++bS_-)\\ &=H_{11}+H_{12}\ . \end{aligned} \tag{5.44}$$

Both parts (H_{11} and H_{12}) have dynamical symmetry $SU(1,1)\otimes H_4\supset\cdots\supset U(1)$, where

$$U^1(1):\ S_0+b^\dagger b\ , \tag{5.45a}$$

$$U^2(1):\ S_0-b^\dagger b\ . \tag{5.45b}$$

Normally, by neglecting the H_{12}, that is, making the rotational wave approximation (RWA), the remaining Hamiltonian $H'=H_0+H_{11}$ is integrable. Such a Hamiltonian is widely used in quantum optics.[54] However, the RWA is valid only for very small values of α. By increasing the value of α, our purpose here is to study the effect of H_{12} on mean-field dynamics.

Since the quantum phase space of this system is $\mathbb{C}^1\otimes S^2$, the classical analogy is given by a two-degree-of-freedom Hamiltonian system with the following Hamiltonian function:

$$\begin{aligned} \mathfrak{H}(q,p)=&\frac{\omega}{2}(p_1^2+q_1^2)+\frac{\omega_0}{2}(p_2^2+q_2^2-2j\tilde{\hbar})\\ &+\frac{\alpha}{2\sqrt{2}}q_1q_2(4j\tilde{\hbar}-p_2^2-q_2^2)^{1/2}\ , \end{aligned} \tag{5.46}$$

where $j=N/2$, which is twice the total number of atoms. From the concept of dynamical symmetry and its breaking, the system possesses a dynamical phase transition: when $\alpha=0$, the system has dynamical symmetry $SU(1,1)\otimes H_4\supset U(1)\otimes U(1)$. When α increases, the dynamical symmetry is broken such that the system becomes nonintegrable and eventually chaos will occur. The numerical calculations are shown in Figs. 3–5.

In Fig. 3, we present the Poincaré surfaces of a section of the mean-field dynamics for the RWA Hamiltonian. The result shows that the trajectories are periodic, consistent with the prediction of integrability. In Fig. 4, we set $\omega_0=\omega$, which is the resonance case. It is commonly assumed that in the resonance case H_{12} can be neglected.[54] However, the mean-field calculation shows that the RWA is only valid for very small α. In general, H_{12} is very important since it breaks the integrability of the RWA system, especially when α is sufficiently large. Indeed, in that case, the dynamical structure of the RWA system is completely broken and the dynamics becomes globally chaotic. In Fig. 5, we have considered the nonresonance case: $\omega_0\neq\omega$. In this case the mean-field

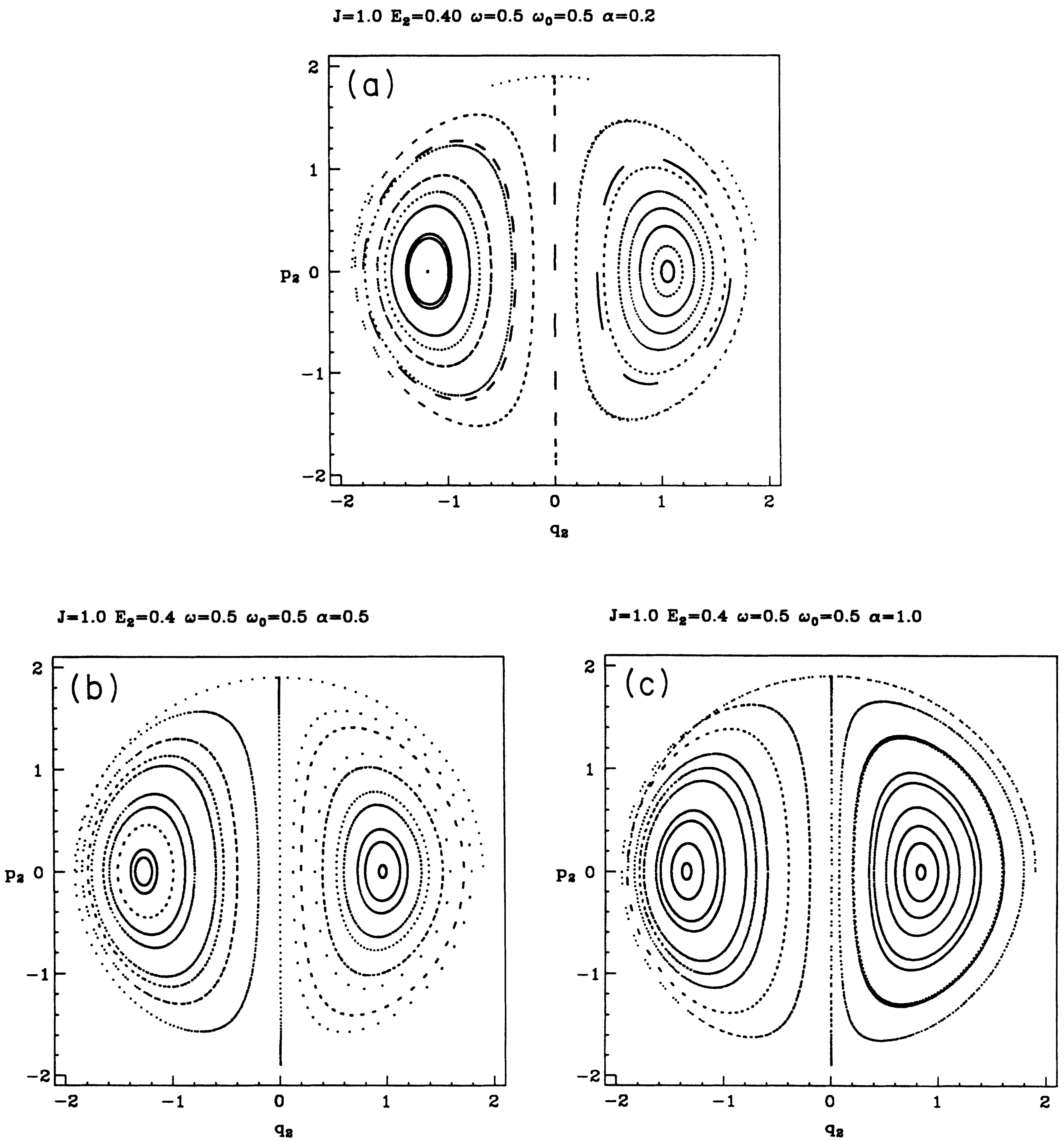

FIG. 3. The Poincaré surfaces of a section of the RWA Hamiltonian $H=H_0+H_{11}$. (a) $\alpha=0.2$; (b) $\alpha=0.5$; (c) $\alpha=1.0$, with $J=j\hbar=1$. The results show that in the RWA, the system is integrable, as we have predicted from its dynamical symmetry.

dynamical calculations show that there are essentially no differences between the resonance and the nonresonance cases.

It is also worth noting that for this example, the mean-field calculations and the classical limit[55] are the same. This is because the dynamical group is a direct product of two subgroups and the Hamiltonian is a linear function of the generators of these two subgroups (but not those of the total dynamical group). Thus the Hamiltonian functions in Eqs. (3.2″) and (3.2‴) are the same and the quantum correlation terms vanish. In order to search for the effect of quantum correlation in classical

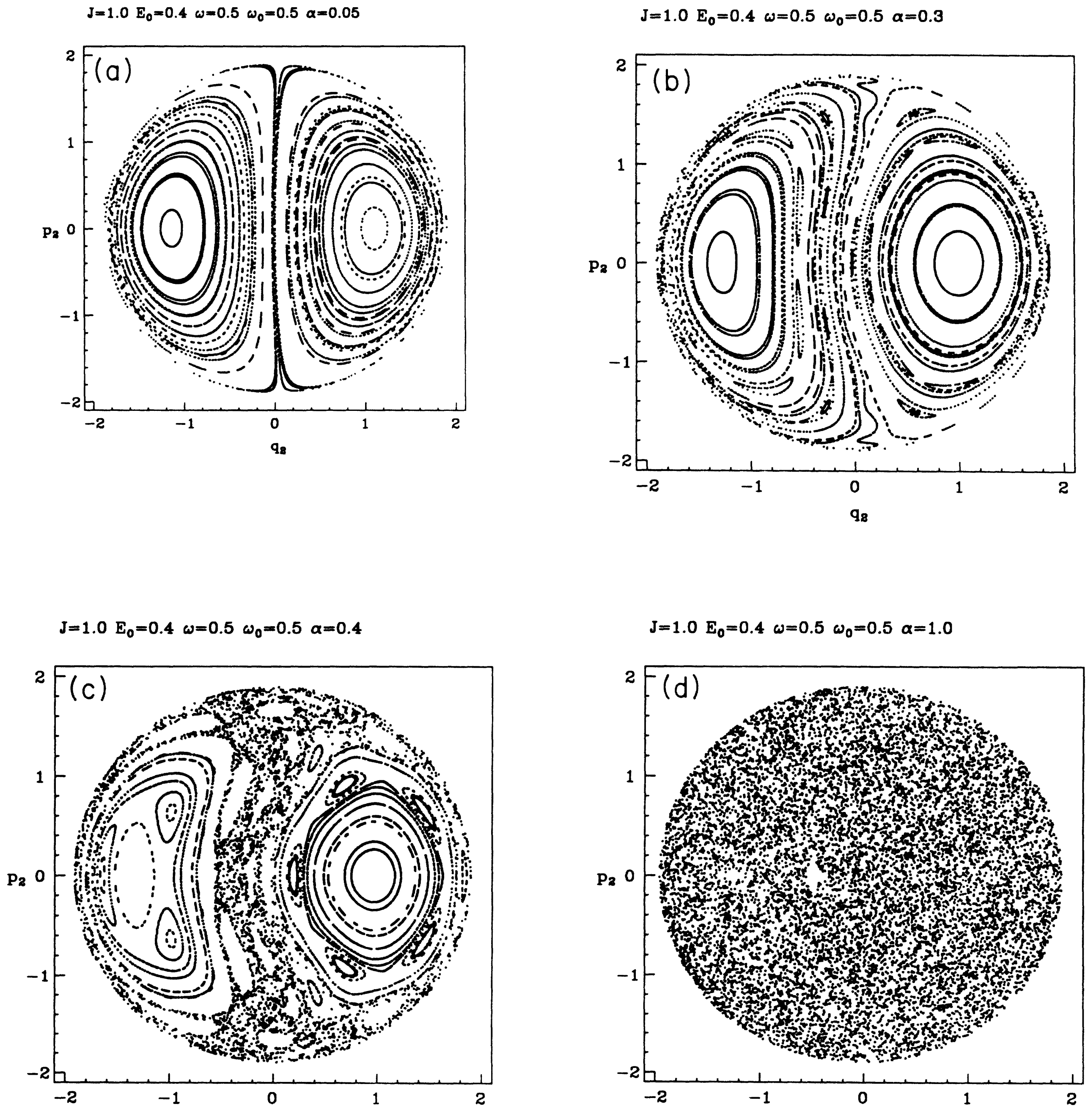

FIG. 4. The Poincaré surfaces of a section of the two-level atomic-external field interacting system of Eq. (5.40) in resonance, $\omega_0=\omega=0.5$. (a) $\alpha=0.01$; (b) $\alpha=0.3$; (c) $\alpha=0.4$, and (d) $\alpha=1.0$, with $J=j\hbar=1$. The results show that the system is globally chaotic when the coupling constant is large.

dynamics and to explore the behavior of the intrinsic quantum properties for chaos, we will consider below a system with much richer structure.

D. SU(3) model

As we have discussed in I, compared to the SU(2), the SU(3) case (its generators are denoted by E_{ij}, $i,j=0,1,2$ [cf. Eq. (I-20)]) has a richer Hilbert space structure. For example, unlike the SU(2) case, the QDDF now depends explicitly on the irrep space. For the fully symmetric irrep, the number of the QDDF is two. Otherwise, it is three. Therefore the QPS which depends sensitively on the structure of Hilbert space has different properties. In this subsection, we will discuss these properties.

(1) QPS structure. Any irrep of SU(3) can be denoted by its highest weight $\Lambda=\mu_1 f_1+\mu_2 f_2$, where f_1 and f_2 are the highest weights of the two fundamental representations of SU(3): (1,0) and (0,1). The lowest weight state

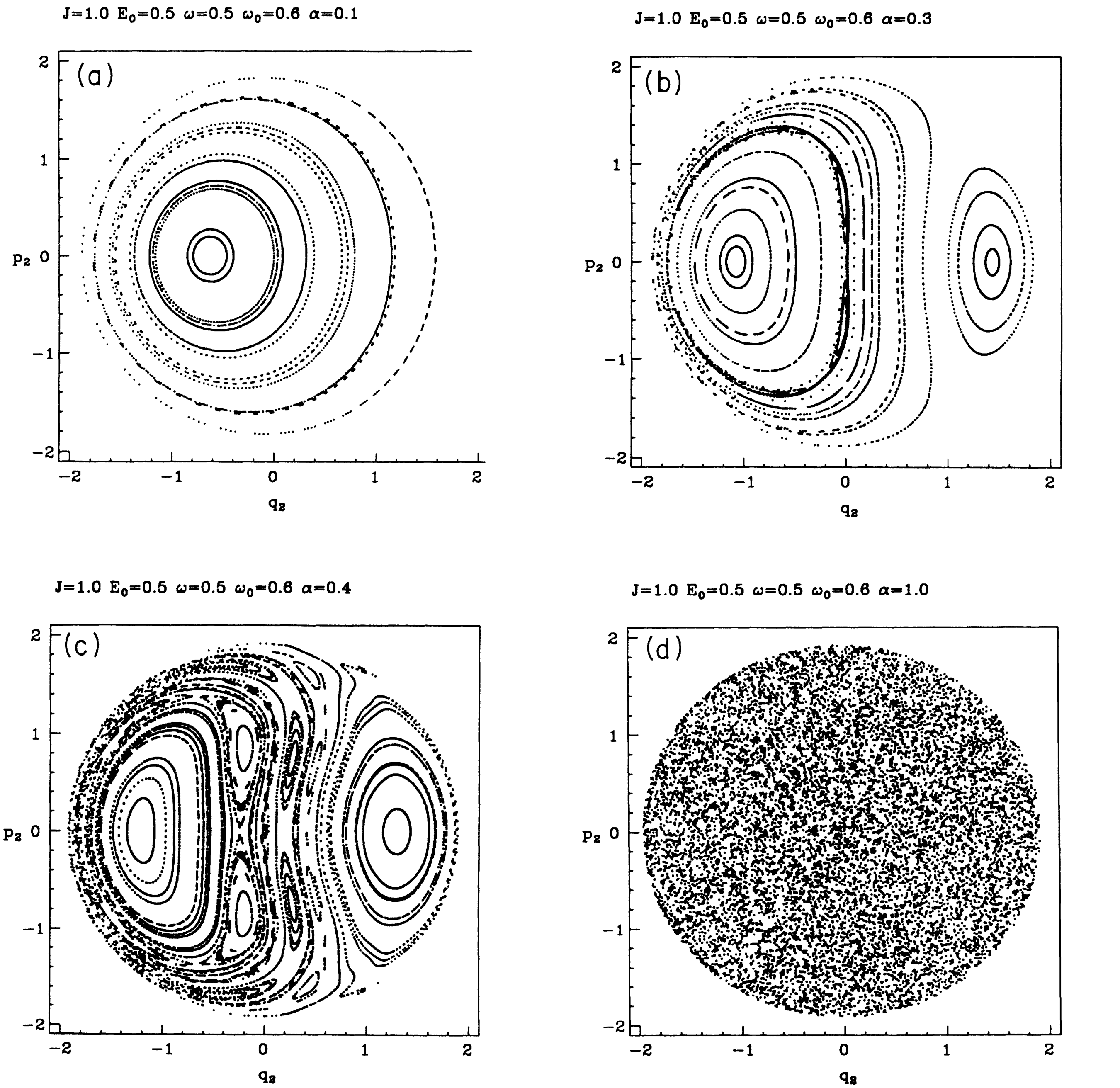

FIG. 5. The Poincaré surfaces of a section of the two-level atomic-external field interacting system of Eq. (5.40) with nonresonance, $\omega_0=0.5$ and $\omega=0.6$. (a) $\alpha=0.1$; (b) $\alpha=0.3$; (c) $\alpha=0.4$, and (d) $\alpha=1.0$, with $J=j\tilde{\hbar}=1$. The results do not deviate much from the resonance case of Fig. 4.

is $|0\rangle=|\Lambda,-\Lambda\rangle$. Thus the diagonal generators E_{ii} acting on $|0\rangle$ must be invariant and all the lowering generators E_{ij} $(i<j)$ acting on $|0\rangle$ are zero. For the fully symmetric irrep, only two raising generators E_{i0}, $i=1,2$ for $\mu_2=0$ (or E_{2i}, $i=0,1$ for $\mu_1=0$) are the elementary excitation operators:

$$E_{ij}|0\rangle\neq|0\rangle \text{ (or 0) for } i>j \text{ and } \begin{cases} i=2 & \text{if } \mu_1=0 \\ j=0 & \text{if } \mu_2=0\ . \end{cases} \tag{5.47}$$

However, for the nonfully symmetric irrep, all three raising generators E_{ij} $(i>j)$ are the elementary excitation operators. This shows again that the number of such operators is identical to the number of QDDF. The QPS of fully symmetric and nonsymmetric irrep spaces of SU(3) can be obtained by the exponential mapping of the subspace $\not{p}$ of $\mathscr{g}$: $\{E_{01,E_{02}},E_{10},E_{20}\}$ (only $\mu_2=0$ is considered here although the same conclusion is reached for the $\mu_1=0$ case) and $\{E_{ij},\ i\neq j\}$, respectively:

$$\sum_{\substack{i,j\\ i>j}}\eta_{ij}E_{ij}-\text{H.c.}\rightarrow\exp\left[\sum_{\substack{i,j\\ i>j}}\eta_{ij}E_{ij}-\text{H.c.}\right] \tag{5.48}$$

where for the fully symmetric irrep $j=0$. Then the corresponding geometrical spaces on the right-hand side of Eq. (5.48) are the coset spaces SU(3)/U(2) and SU(3)/U(1)⊗U(1). The differential geometrical structures of SU(3)/U(2) or SU(3)/U(1)⊗U(1) can simply be computed from the respective coherent states:[17]

$$\begin{aligned}|\Lambda\Omega\rangle&\equiv\exp\left[\sum_{\substack{i,j\\ i>j}}\eta_{ij}E_{ij}-\text{H.c.}\right]|\Lambda,-\Lambda\rangle\\ &=K^{-1/2}(z,z^*)\exp\left[\sum_{\substack{i,j\\ i>j}}z_{ij}E_{ij}\right]|\Lambda,-\Lambda\rangle\\ &\equiv K^{-1/2}(z,z^*)\|\Lambda z\rangle\ .\end{aligned} \tag{5.49}$$

According to Eq. (2.30), the generalized Bergmann kernels on SU(3)/U(2) or SU(3)/U(1)⊗U(1) is

$$K(z,z^*)=\langle\Lambda z\|\Lambda z\rangle \tag{5.50}$$

and the metric g_{ij} and measure $d\mu$ are determined by Eqs. (2.17) and (2.20), respectively. For instance, consider the fully symmetric irrep ($\mu_2=0$) case. The explicit form of the generalized Bergmann kernel is

$$K(z,z^*)=(1+z^\dagger z)^{\mu_1} \tag{5.51}$$

where

$$z_i\equiv z_{i0}=\eta_{i0}\frac{\tan\eta}{\eta}\quad(i=1,2) \tag{5.52}$$

and

$$\eta=(\eta_{10}\eta_{10}^*+\eta_{20}\eta_{20}^*)^{1/2}\ . \tag{5.53}$$

By transforming Eq. (5.52) into the canonical coordinates (and letting $\hbar=1$):

$$\frac{1}{(2\mu_I)^{1/2}}(q_i+ip_i)=\frac{z_i}{(1+zz^\dagger)^{1/2}}=\eta_{i0}\frac{\sin\eta}{\eta} \tag{5.54}$$

we have

$$\{\mathfrak{F}_1,\mathfrak{F}_2\}=\sum_{i=1}^{2}\left[\frac{\partial\mathfrak{F}_1\partial\mathfrak{F}_2}{\partial q_i\partial p_i}-\frac{\partial\mathfrak{F}_1\partial\mathfrak{F}_2}{\partial p_i\partial q_i}\right] \tag{5.55}$$

where $p_1^2+q_1^2+p_2^2+q_2^2\leq 2\mu_1$.

(2) Phase-space representation. In the same manner as the SU(2) case, the phase-space representation of the SU(3) system is easily realized by using the overcomplete relation of the coherent states $|\Lambda\Omega\rangle$ (or $\|\Lambda z\rangle$):

$$\int|\Lambda\Omega\rangle d\mu\langle\Lambda\Omega|=I \tag{5.56a}$$

or

$$\int\|\Lambda z\rangle d\mu_H\langle\Lambda z\|=I \tag{5.56b}$$

The formal expressions of the phase-space representation of wave functions and operators are given by Eqs. (2.33) and (2.36). For simplicity, we only present the phase-space representation of the generator of SU(3) in its fully symmetric irrep ($\mu_2=0$ case). The results are

$$\begin{aligned}&\mathfrak{E}_{00}=\tfrac{1}{2}[2\mu_1-(p_1^2+q_1^2+p_2^2+q_2^2)]\ ,\\ &\mathfrak{E}_{11}=\tfrac{1}{2}(p_1^2+q_1^2),\quad \mathfrak{E}_{22}=\tfrac{1}{2}(p_2^2+q_2^2)\ ,\\ &\mathfrak{E}_{10}=\tfrac{1}{2}(q_1-ip_1)(2\mu_1-p_1^2-q_1^2-p_2^2-q_2^2)^{1/2}\ ,\\ &\mathfrak{E}_{20}=\tfrac{1}{2}(q_2-ip_2)(2\mu_1-p_1^2-q_1^2-p_2^2-q_2^2)^{1/2}\ ,\\ &\mathfrak{E}_{32}=\tfrac{1}{2}(q_1+ip_1)(q_2-ip_2),\quad \mathfrak{E}_{ij}=\mathfrak{E}_{ji}^\dagger\ .\end{aligned} \tag{5.57}$$

The corresponding algebraic structure of SU(3) in the phase-space representation is determined by Poisson brackets:

$$\{\mathfrak{E}_{ij},\mathfrak{E}_{kl}\}=-i(\delta_{jk}\mathfrak{E}_{il}-\delta_{il}\mathfrak{E}_{kj})\ . \tag{5.58}$$

(3) Classical analogy. The classical analogy of an observable $A(\mathbf{E})$ is given by the following equation:

$$\mathfrak{A}(q,p)=\langle\Lambda\Omega|A(\mathbf{E})|\Lambda\Omega\rangle \tag{5.59}$$

while its time evolution is determined by Hamilton's dynamical equations of Eq. (3.2″) with $\mathfrak{H}(q,p)$ as the Hamiltonian.

In order to use the above results to explicitly illustrate the relationship between nonintegrability and dynamical symmetry breaking in its classical analogy, we shall study the three-level Lipkin model,[56,57] where each level has $\mathcal{N}$-fold degeneracy. The model Hamiltonian is taken to be

$$H=\sum_{i=0}^{2}\varepsilon_iE_{ii}+\tfrac{1}{2}\sum_{\substack{i,j\\ i\neq j}}V_{ij}(E_{ij})^2+\tfrac{1}{2}\sum_{i=0}^{2}W_{ii}(E_{ii})^2 \tag{5.60}$$

where

$$E_{ij}=\sum_{k=1}^{\mathcal{N}}a_{ik}^\dagger a_{jk},\quad i,j=0,1,3\ . \tag{5.61}$$

Unlike Refs. 56 and 57, we have added here the self-interaction terms $(E_{ii})^2$ in Eq. (5.60). Thus the system now has the following explicit dynamical symmetries.

When $V=W=0$, it is $SU(3)\supset SU(2)\otimes U(1)\supset U(1)\otimes U(1)$. If $V=W\gg\varepsilon_i$, it approaches the dynamical symmetry $SU(3)\supset SO(3)\supset SO(2)$. Therefore the system can undergo a transition from integrable to nonintegrable and back to integrable by the variation of the parameters V and W. Also we have generalized the Lipkin model to the case where the particle number is variable.

In order to find its quantum phase space, let us first assume that the total particle number $N\leq\mathcal{N}$. Thus we will find that the quantum phase space of Eq. (5.60) is U(3)/U(2) for both fermions and bosons. This is because each level has $\mathcal{N}$-fold degeneracy and therefore the fixed states have the same form for fermions and bosons:

$$|0\rangle=\prod_{k=1}^{N}a_{0k}^{\dagger}|\underline{0}\rangle \tag{5.62}$$

where $|\underline{0}\rangle$ denotes the bar vacuum. The phase-space representation of the generators is given by Eq. (5.57) and the quantum correlations of the quadratic function of the generators are

$$\Delta(E_{ij}^2)=-\frac{1}{N}\mathfrak{E}_{ij}^2+c,\quad c=\begin{cases}N & \text{for } i=j=0\\ 0 & \text{otherwise}\ .\end{cases} \tag{5.63}$$

Now, let us set $e_0=0$ and

$$\begin{aligned}V_{ij}&=V\ ,\\ W_{ii}&=W\end{aligned} \tag{5.64}$$

in Eq. (5.60). Thus in the classical analogy of Eq. (3.2″) the Hamiltonian function is

$$\begin{aligned}\mathfrak{H}(q,p)=&\frac{e_1}{2}(p_1^2+q_1^2)+\frac{e_2}{2}(p_2^2+q_2^2)\\ &+\frac{V}{4}(1-1/N)[(p_1^2+p_2^2)^2-(q_1^2+q_2^2)^2+(q_1^2-p_1^2)(q_2^2-p_2^2)+4q_1q_2p_1p_2+2N(q_1^2+q_2^2-p_1^2-p_2^2)]\\ &+\frac{W}{4}(1-1/N)[(p_1^2+p_2^2)^2+(q_1^2+q_2^2)^2+(q_1^2+q_2^2)(p_1^2+p_2^2)-2N(q_1^2+q_2^2+p_1^2+p_2^2)+2N^2]\ .\end{aligned} \tag{5.65}$$

By taking the limiting case of $N\ (=\chi)\to\infty$, Eq. (5.65) is reduced to the same result as given in Ref. 57 with $W=0$. This is not a surprising result since the "classical limit" is obtained for $N\to\infty$.

In order to regard the Lipkin model as a "schematic" shell model, the shell degeneracy should be finite, then the "classical limit" cannot exist. Yet, viewed as a "realistic" shell model, the classical analogy we present in this paper can still exhibit the inherent dynamical behavior of this model. To illustrate this, let us consider the case where the model has only one fermion and each of the three levels can accommodate only one particle (Pauli principle), i.e., $N=\mathcal{N}=1$. The exact quantum calculation of the matrix elements for the operators E_{ij}^2 are are

$$\langle i|E_{i'j'}^2|j\rangle=\begin{cases}1 & i=i'=j=j'\\ 0 & \text{otherwise}\ ,\end{cases} \tag{5.66}$$

i.e., the V term in Eq. (5.60) has zero contribution to the dynamics and the W term is equivalent to a constant. In this case, Eq. (5.60) is integrable for any values of V and W. In the classical analogy theory, when $N=1$, according to Eq. (5.65), one immediately obtains the same result as Eq. (5.66), i.e., the V term in Eq. (5.60) has zero contribution to the dynamics and the W term is equivalent to a constant. Also, Eq. (5.65) becomes integrable for any V and W values for this case. In fact, one can easily prove that in this case, the wave packets of Eq. (5.60) will follow the mean-field trajectory of Eq. (5.65). From this discussion, it appears that not much significance can be attached when one compares the classical phenomena with quantum calculations.

Now let us consider the system with $\mathcal{N}\geq N>1$. In this case, Eq. (5.60) is nonintegrable and the classical analogy can describe its chaotic behavior. The numerical calculations presented in Figs. 6 and 7 are the Poincaré sections with $\mathcal{N}=10$. From Eq. (5.65), we see that the quantum correlation is not negligible and the classical limit calculation is still not valid.

Figure 6 corresponds to the $V=W$ case, where Eq. (5.65) can be rewritten as

$$\begin{aligned}\mathfrak{H}(p,q)=&\frac{e_1}{2}(p_1^2+q_1^2)+\frac{e_2}{2}(p_2^2+q_2^2)\\ &+\frac{V}{2}(1-1/N)[(p_1^2+p_2^2-N)^2\\ &+(p_1q_1+p_2q_2)^2]\ .\end{aligned} \tag{5.67}$$

Although Fig. 6 seems to suggest that Eq. (5.67) represents an integrable system, the precise determination of its integrability is by no means obvious in classical mechanics. However, from the dynamical symmetry point of view, it is very easy to find the existence of the second constant of motion (besides the energy). Since the first two terms in Eq. (5.60) are functions of the Cartan subgroup generators of SU(3), and the interaction term has SO(3) symmetry, then the operator L_0 (the z component of the angular momentum) is a constant of motion in the spherical basis. In other words, the system has partial dynamical symmetry of $SU(3)\supset SO(2)$. Therefore, together with energy conservation, such a two-degree-of-freedom system is integrable. Explicitly, the classical analogy of L_0:

$$\mathfrak{L}_0(p,q)=p_1(2N-p_1^2-q_1^2-p_2^2-q_2^2)^{1/2} \tag{5.68}$$

is a constant of motion which can be verified by a direct

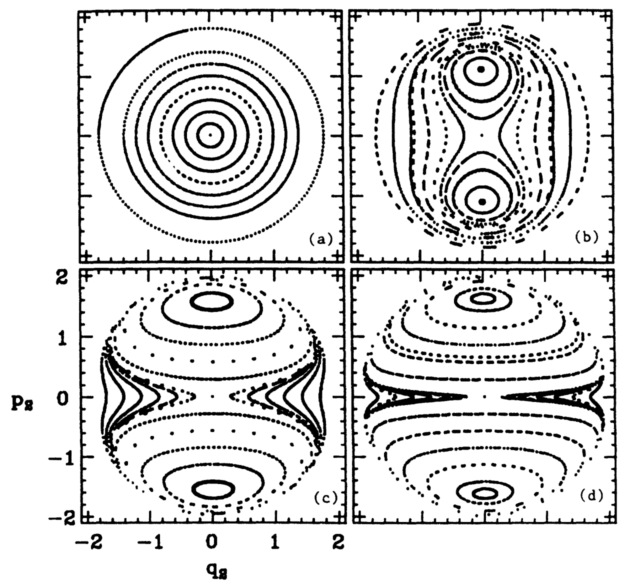

FIG. 6. (a), (b), (c), and (d) are the Poincaré sections of trajectories for the three-level Lipkin model in the p_2-q_2 plane with $p_1=0$ and $q_1>0$ for $V=W=0.015$, 0.15, 1.5, and 15.0. It is shown that the system is integrable in this case. The constant of motion is found by using the concept of dynamical symmetry. In all the calculations, we have set $N=4$, and $E_0=\mathfrak{H}(q,p)-WN^2/2$. The phase space has been scaled to two units in the calculations.

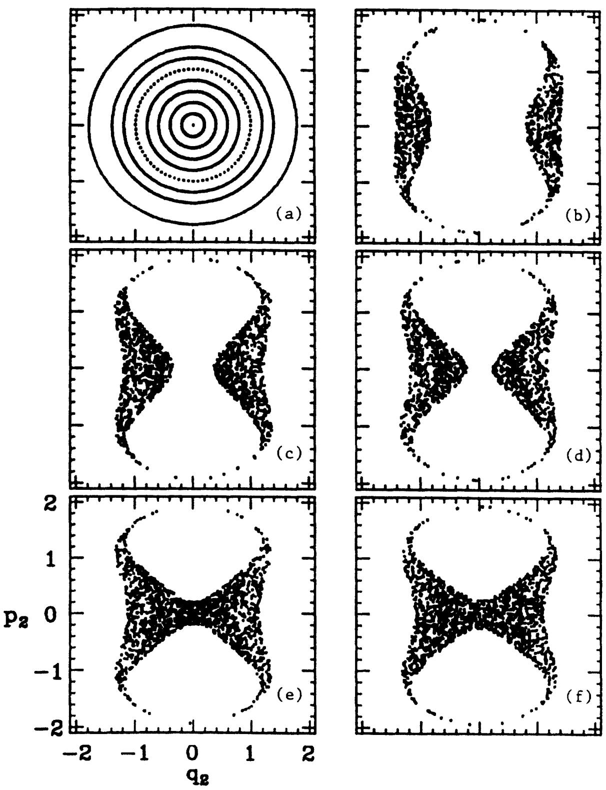

FIG. 7. The Poincaré sections of trajectories for the three-level Lipkin model in the p_2-q_2 plane with $p_l=0$ and $q_l>0$ for $V=0.8$ and $W=0$. The system is chaotic. The mean-field dynamics show that when N is small, the quantum correlation is very strong and suppresses chaos. With increasing N, the quantum correlation is reduced and the topological structure of the chaotic motion in the mean-field dynamics is altered. (a) $N=1$; (b) $N=2$; (c) $N=4$; (d) $N=5$; (e) $N=8$; and (f) $N=10$. The phase space has been scaled to two units in the calculations.

calculation via Poisson bracket of Eq. (5.55). If $V\neq W$, the SO(2) symmetry is also broken and the system becomes nonintegrable and can be chaotic in a certain energy and parameter (V) regions as shown in Fig. 7 (with $W=0$). In Fig. 7, we consider the case of $N=1-10$. The dynamical effect of quantum correlation is exhibited for the first time. For $N=1$, the system is integrable as we have predicted and the mean-field trajectories are periodic [see Fig. 7(a)]. In this case, the mean-field solution is the exact quantum solution and the wave packets evolve in time along the mean-field trajectories. When $N\geq 2$, the system is nonintegrable [see Figs. 7(b)–7(f)]. From Figs. 7(b)–7(f), we see that by changing N, the topology of the phase portraits is altered. This alteration is only due to the effect of quantum correlations [see the quantum correlation factor $(1-1/N)$ in Eq. (5.65)]. Again, these properties cannot be explored in classical limit calculations. A study of the generic behavior of quantum correlation affecting chaos will be published elsewhere.[46]

The above results also indicate that the concepts of nonintegrability and dynamical symmetry breaking and their relationship are very useful in predicting the dynamics of the quantum system and its classical analogy. It also provides a useful way to find the constants of motion in classical mechanics via the concept of dynamical symmetry. The classical analogy provides an explicit way to explore the dynamical effect of quantum correlation in topological structure of chaotic motion.

Furthermore, when $2\mathcal{N}>N>\mathcal{N}$ the fixed state which is a key to determining the quantum phase space is not Eq. (5.62) but

$$|0\rangle=\prod_{k=1}^{N-\mathcal{N}}\prod_{l=1}^{\mathcal{N}}a_{1k}^{\dagger}a_{0l}^{\dagger}|\underline{0}\rangle\ . \tag{5.69}$$

Correspondingly, the elementary excitation operators are E_{ij}, $i>j$ and the quantum phase space for such a case is given by the coset space $U(3)/U(1)\otimes U(1)\otimes U(1)$ which is a six-dimensional manifold and the QDDF is 3. This is of course *totally* different from the $N\leq\mathcal{N}$ case. This conclusion can also be directly obtained from the irrep space of U(3). For the $2\mathcal{N}>N>\mathcal{N}$ case, the irrep space is the nondegenerate irrep space: $(\mathcal{N},N-\mathcal{N},0)$. As we have discussed in example 3 of I, there are three NFD operators in the CSCO of U(3). Hence the number of the QDDF is 3. However, in the $N\leq\mathcal{N}$ case, the irrep space corresponds to the degenerate irrep of U(3): $(N,0,0)$ and there are only two NDF operators for this irrep. Thus the number of the QDDF is 2 and the geometrical space must be a four-dimensional manifold, i.e., the coset space $U(3)/U(2)\otimes U(1)$. When $3\mathcal{N}\geq N\geq 2\mathcal{N}$ the structure of quantum phase space is the same as $N\leq\mathcal{N}$.

From our procedure of constructing quantum phase space, it is clear that the change of the quantum phase space as a function of N is the geometrical manifestation of the Pauli principle. In fact, this manifestation has already been revealed[58] in a dynamical truncated shell model[59] and its experimental evidence has also been extensively discussed.[60]

E. Atomic hydrogen system [SU(1,1)]

Atomic hydrogen is not only the first realistic quantum example to be analytically solved by algebraic methods but also is a quantum system which possesses one of the richest dynamical group structures. For the radial motion of the hydrogen atom, the dynamical group is SU(1,1).[61] In the presence of a magnetic field or a linearly polarized electromagnetic external field, the dynamical group is SO(2,2).[62] A larger dynamical group of the hydrogen atom is SO(4,2).[63] In this paper, we shall only concentrate on the radial motion without the presence of external fields.

For the radial motion of the hydrogen atom, the Hamiltonian has SU(1,1) dynamical group with generators

$$K_1=\tfrac{1}{2}(r\mathbf{p}^2-r),\quad K_2=\mathbf{r}\cdot\mathbf{p}-i,\quad K_3=\tfrac{1}{2}(r\mathbf{p}^2+r)\ , \tag{I-32}$$

which satisfy the commutation relations

$$[K_1,K_2]=-iK_3,\quad [K_2,K_3]=iK_1,\quad [K_3,K_1]=iK_2\ . \tag{I-33}$$

The Hilbert space corresponds to its positive discrete irrep $D^+(k)$ with $k>0$. In this case, the discussion is similar to the discussion of the SU(2) except for its noncompactness. The basis vectors of $D^+(k)$ are $|kn\rangle$, which are eigenstates of K_3: $K_3|kn\rangle=(n+k)|kn\rangle$, $n=0,1,2,\ldots$. The label k is determined by the SU(1,1) Casimir operator: $K^2=K_3^2-K_1^2-K_2^2$, and $K^2|kn\rangle=k(k-1)|kn\rangle$. In the following, we will construct the QPS structure of the hydrogen atom's radial motion, its phase-space representation as well as classical analogy.

(i) Phase-space structure. First of all, the Hamiltonian of Eq. (I-31) in the Hilbert space of $D^+(k)$ has a ground state $|k0\rangle$ which is the lowest bound state of $D^+(k)$. Since

$$|kn\rangle=\left[\frac{\Gamma(2k)}{n!\Gamma(2k+n)}\right]^{1/2}(K_+)^n|k0\rangle \tag{5.70}$$

any state $|\Psi\rangle=\sum_{m=0}^{2j}f_n|kn\rangle$ $[\in D^+(k)]$ can be generated by a polynomial of K_+ acting on $|k0\rangle$, where $K_\pm=K_1\pm iK_2$. Thus the elementary excitation operator of the SU(1,1) is only the raising operator K_+. The QPS is obtained via the general procedure of mapping the subspace $\not{h}$: $\{K_+,K_-\}$ of SU(1,1) onto the coset space SU(1,1)/U(1):

$$(\eta K_+-\eta^*K_-)\rightarrow\exp(\eta K_+-\eta^*K_-) \tag{5.71}$$

where $\eta=(\theta/2)e^{-i\varphi}$, $-\infty\le\theta\le\infty$, $0\le\varphi\le2\pi$. This means that SU(1,1)/U(1) is isomorphic to a two-dimensional hyperboloid H^2.[16] Therefore the QPS of radial motion of the hydrogen atom is H^2. Similarly, the differential structure of H^2 can be calculated from the coherent states of SU(1,1)/U(1) via Eqs. (2.17). The coherent states of SU(1,1)/U(1) are[15]

$$\begin{aligned}|k\Omega\rangle&\equiv\exp(\eta K_+-\eta^*K_-)|k0\rangle\\&=(1-zz^*)^k\exp(zK_+)|k0\rangle\equiv(1-zz^*)^k\|kz\rangle\end{aligned} \tag{5.72}$$

and

$$z=\tanh(\theta/2)e^{-i\varphi}\ . \tag{5.73}$$

Thus the so-called generalized Bergmann kernel is

$$K(z,z^*)=(1-zz^*)^{-2k}\ . \tag{5.74}$$

The metric g_{ij} and the measure $d\mu$ of SU(1,1)/U(1) are

$$g_{ij}=\delta_{ij}\frac{2k}{(1-zz^*)^2}\ \text{and}\ d\mu=\frac{2k-1}{\pi}\frac{dzdz^*}{(1-zz^*)^2}\ , \tag{5.75}$$

respectively. The Poisson bracket of the functions $\mathfrak{F}_1,\mathfrak{F}_2$ defined on SU(1,1)/U(1) can be expressed as follows:

$$\{\mathfrak{F}_1,\mathfrak{F}_2\}=\frac{(1-zz^*)^2}{2ik}\left[\frac{\partial\mathfrak{F}_1\partial\mathfrak{F}_2}{\partial z\partial z^*}-\frac{\partial\mathfrak{F}_1\partial\mathfrak{F}_2}{\partial z^*\partial z}\right]\ . \tag{5.76}$$

The canonical coordinates are ($\tilde{h}=1$)

$$\frac{1}{\sqrt{4k}}(q+ip)=\frac{z}{(1-zz^*)^{1/2}}=\sinh(\theta/2)e^{-i\varphi}\ . \tag{5.77}$$

By using Eq. (5.77), Eq. (5.76) becomes the standard form of Eq. (2.25).

(2) Phase-space representation. Since the overcomplete relation of the coherent states $|k\Omega\rangle$ is

$$\int_{D^2}|k\Omega\rangle d\mu\langle k\Omega|=I\left[\int_{D^2}\|kz\rangle d\mu_H\langle kz\|=I\right] \tag{5.78}$$

the phase-space representation of wave function $|\Psi\rangle$ of Eq. (5.70) is

$$f(z)=\langle\Psi\|kz\rangle=\sum_{n=0}^{\infty}f_n\left[\frac{\Gamma(2k+n)}{n!\Gamma(2k)}\right]^{1/2}z^n \tag{5.79}$$

where $f(z)\in L^2(H^2)$. The phase-space representation of any operator $A=A(K_i)$ is

$$\mathfrak{A}(z,z^*)=\langle k\Omega|A(K_i)|k\Omega\rangle\ . \tag{5.80}$$

When the operators A are generators of SU(1,1), their classical analogies are

$$\begin{aligned}\mathfrak{K}_+&=\langle k\Omega|K_+|k\Omega\rangle\\&=\frac{2kz^*}{1-zz^*}=\tfrac{1}{2}(q-ip)(4k+p^2+q^2)^{1/2}\ ,\end{aligned} \tag{5.80a}$$

$$\begin{aligned}\mathfrak{K}_-&=\langle k\Omega|K_-|k\Omega\rangle\\&=\frac{2kz}{1-zz^*}=\tfrac{1}{2}(q+ip)(4k+p^2+q^2)^{1/2}\ ,\end{aligned} \tag{5.80b}$$

$$\mathfrak{K}_3=\langle k\Omega|K_3|k\Omega\rangle=k\frac{1+zz^*}{1-zz^*}=k+\frac{p^2+q^2}{2}\ . \tag{5.80c}$$

The corresponding algebraic structure of SU(1,1) in the phase-space representation is

$$i\{\mathfrak{K}_-,\mathfrak{K}_+\}=2\mathfrak{K}_3,\quad i\{\mathfrak{K}_3,\mathfrak{K}_\pm\}=\pm\mathfrak{K}_\pm\ . \tag{5.81}$$

Similar to the angular momentum, $\mathfrak{K}_\pm \equiv \langle k\Omega|K_1\pm iK_2|k\Omega\rangle=\mathfrak{K}_1\pm i\mathfrak{K}_2$, we have

$$\{\mathfrak{K}_1,\mathfrak{K}_2\}=-\mathfrak{K}_3,\quad \{\mathfrak{K}_2,\mathfrak{K}_3\}=\mathfrak{K}_1,\quad \{\mathfrak{K}_3,\mathfrak{K}_1\}=\mathfrak{K}_2\ . \tag{5.82}$$

Thus the dynamical properties of the system in the phase-space representation may formally be described by the equations of Eq. (2.46).

(3) Classical analogy. The classical analogy of the quantum problem is determined by Eq. (3.2) with the Hamiltonian

$$\mathfrak{H}(q,p)=H(\mathfrak{K}_+(q,p),\mathfrak{K}_-(q,p),\mathfrak{K}_3(q,p))\ . \tag{5.83}$$

Here we only consider the case of "classical" limit in order to compare it with its classical version. The Hamiltonian of hydrogen atom in terms of SU(1,1) generators (in atomic units) can be rewritten as

$$H=(K_3+K_1-2)/2(K_3-K_1)\ . \tag{5.84}$$

Using Eqs. (5.77) and (5.80), we have

$$\begin{aligned}\mathfrak{H}(q,p)&=\left[k+\frac{p^2+q^2}{2}+\frac{q}{2}(4k+p^2+q^2)^{1/2}-2\right]\Big/2\left[k+\frac{p^2+q^2}{2}-\frac{q}{2}(4k+p^2+q^2)^{1/2}\right]\\&=\left[k+\frac{p^2+q^2}{2}+\frac{q}{2}(4k+p^2+q^2)^{1/2}-2\right]\Big/2\mathfrak{r}(p,q)\end{aligned} \tag{5.85}$$

where $\mathfrak{r}(p,q)$ is the classical analogy of the radial position operator:

$$\begin{aligned}\mathfrak{r}(p,q)&=\langle k\Omega|r|k\Omega\rangle\\&=k+\frac{p^2+q^2}{2}-\frac{q}{2}(4k+p^2+q^2)^{1/2}\ .\end{aligned} \tag{5.86}$$

Then the equations of motion are

$$\begin{aligned}\frac{dq}{dt}&=\frac{p}{\mathfrak{r}(p,q)}[\tfrac{1}{2}-\mathfrak{H}(q,p)]\\&\quad+\frac{pq}{2\mathfrak{r}(p,q)(4k+p^2+q^2)^{1/2}}[\tfrac{1}{2}+\mathfrak{H}(q,p)]\ ,\end{aligned} \tag{5.87a}$$

$$\begin{aligned}\frac{dp}{dt}&=-\frac{q}{\mathfrak{r}(p,q)}[\tfrac{1}{2}-\mathfrak{H}(q,p)]\\&\quad-\frac{4k+p^2+2q^2}{2\mathfrak{r}(p,q)(4k+p^2+q^2)^{1/2}}[\tfrac{1}{2}+\mathfrak{H}(q,p)]\ .\end{aligned} \tag{5.87b}$$

The time evolution of $\mathfrak{r}(p,q)$ can be obtained from Eq. (5.87):

$$\begin{aligned}\frac{d\mathfrak{r}(p,q)}{dt}&=-\frac{p}{2\mathfrak{r}(p,q)}(4k+p^2+q^2)^{1/2}\\&=\left[2\left[E-\frac{1}{\mathfrak{r}(p,q)}-\frac{k^2}{2\mathfrak{r}^2(p,q)}\right]\right]^{1/2}\end{aligned} \tag{5.88}$$

where $E=\mathfrak{H}(q,p)$ is the energy and $k=l$ the angular momentum. It is obvious that Eq. (5.88) is the exact classical equation of radial motion. The above discussion demonstrates the statement made in Sec. III that the "classical" limiting case of classical analogy is identical to its classical version if the latter exists. The above derivation shows explicitly the difference from Ehrenfest's theorem.[47]

VI. SUMMARY AND CONCLUSIONS

The following points summarize our work in this paper.

(i) The explicit structures of the quantum phase space are explored. The quantum phase space is an inherent geometry for an arbitrary quantum system and possesses naturally the symplectic and complex structures. Such a quantum phase space is controlled by the number of quantum dynamical degrees of freedom given in paper I and includes various inherent properties of the quantum theory, such as Pauli principle, quantum internal degrees of freedom, and quantum-statistical properties of microscopic particles. We have provided a general procedure of constructing this quantum phase space from the QDDF, and have calculated explicitly its canonical coordinates for all *semisimple* dynamical Lie groups with Cartan decomposition.

(ii) Based on the constructed $\mathcal{G}/\mathcal{H}$, one can define the associated coherent states which provide a natural bridge to link the physical Hilbert space and the quantum phase space. Then the explicit phase-space representation of the quantum system can be obtained by closely following the procedure of Klauder's continuous representation theory.[14]

(iii) The algebraic structure of the phase-space representation of observables has been studied. It was found that the algebraic structure of operators is preserved in the phase space if the operators are the generators of the dynamical group $\mathcal{G}$. This property results in an explicit realization of classical limit of quantum systems.

(iv) A classical analogy of quantum mechanics was developed for an arbitrary quantum system which is independent of the existence of the classical counterpart. The results show that the classical limit of the quantum system can be explicitly obtained if it exists. Furthermore, the classical analogy contains the first-order quantum correlation and can describe the semiquantal dynamics. It is shown in this paper that the theorem about the relationship of dynamical symmetry and integrability[9,10] is also valid in classical mechanics.

(v) From the classical analogy theory, a general algorithm for seeking the quantum manifestation of chaos was constructed. This is consistent with Berry's definition of the quantum manifestation of chaos, i.e., *the study of semiclassical, but nonclassical, behavior characteristic of systems whose classical motion exhibits chaos.*[25] It provided, for the first time, a general procedure to explicitly examine the dynamical effect of quantum correlation on classical chaos.

In Sec. V, many quantum-mechanical examples were discussed in order to verify the theoretical underpinnings given in paper I and this paper. The first two examples: the driven radiation field (testing exact quantum solution via classical analogy) and the two-level Lipkin model (testing the difference between the mean-field dynamics and "classical" limit) are integrable systems. The next two examples: the two-level atomic system interacting with an external field and the three-level Lipkin model are nonintegrable systems and were used to test the effects of inherent quantum structures and quantum correlation on systems whose mean-field motion is periodic or chaotic. The last is the hydrogen atom which was to explicitly test the agreement of the "classical" limit with classical mechanics. It shows clearly as a limiting case that the classical analogy of a quantum system includes the classical mechanics. These analytical as well as numerical calculations illustrate the utility and applicability of the theory.

It is also worth pointing out that before numerical computations were implemented, the concepts of dynamical symmetry and dynamical symmetry breaking could be employed to analyze the integrability and nonintegrability of quantum systems. All the results we have presented showed that by using such concepts the general behavior of integrability and nonintegrability can be determined *a priori*.

Two open questions remain. One is how to provide a general *systematic procedure* to construct explicit canonical coordinates (i.e., their Poisson brackets satisfy $\{q_i,p_j\}=\delta_{ij}$) of quantum phase space when the system possesses a *nonsemisimple* dynamical Lie group, although we know what the quantum phase space is in such cases and in fact some special cases have been constructed.[17,64] Another is, what are the generic dynamical behaviors of quantum correlation in classical chaos. We have examined the quantum correlation effects in classical chaos when the quantum phase space is compact. However, when the QPS is noncompact, we suspect that the dynamical behaviors of quantum correlations may very well be different. Research is now underway to address these questions.

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APPENDIX: DIMENSION THEOREM

Theorem. For an irrep of Lie group $\mathcal{G}$, the number M_Λ of the NFD operators in CSCO which specify the basis of its carrier space $\mathfrak{h}_\Lambda$ is identical to the number of positive roots $\{\nu_i\}\in R_+$ of its Lie algebra $\mathfrak{g}$, which satisfy the condition $\Lambda-\nu_i\in R$. Here Λ is the highest weight and R is root space of $\mathfrak{g}$.

Proof. For an N-dimensional configuration space, the number of quantum number, i.e., the number of the NFD operators (for definition see paper I) in CSCO which specify the basis of its Hilbert space, is N [an explicit example is (I-2) in I].[26,27,41] Consider an n-dimensional, l-rank Lie group $\mathcal{G}$. Its generators in the Cartan basis are

$$\{H_k,E_{\nu_i},E_{-\nu_i},\ k=1,\ldots,l;\nu_i\in R_+\} \tag{A1}$$

where $R_+\equiv\{\nu_i,\ i=1,\ldots,n-l/2\}$ is the set of positive roots. For any state $|\Psi\rangle\in\mathfrak{h}_\Lambda$, we have

$$|\Psi\rangle=F(E_{\nu_i})|\Lambda,-\Lambda\rangle \tag{A2}$$

where $F(E_{\nu_i})$ is a polynomial of E_{ν_i} with ν_i satisfying the following condition:

$$\Lambda-\nu_i\in R\ . \tag{A3}$$

Let the number of ν_i which satisfies Eq. (A3) be M_Λ, then it is obvious that

$$M_\Lambda\le\frac{n-l}{2}\ . \tag{A4}$$

On the other hand, from Eq. (A2), the configuration space corresponding to $\mathfrak{h}_\Lambda$ is an M_Λ-dimensional manifold.[41] Therefore the number M_Λ of NFD operators in CSCO is identical to the number of positive roots $\{\nu_i\}\in R_+$, which satisfy the condition $\Lambda-\nu_i\in R$.

Explicitly, for nondegenerate irrep space $\mathfrak{h}_\Lambda$, one has

$$(\nu_i\Lambda)\neq0\quad\text{for all }\nu_i\in R_+,\ i=1,\ldots,\frac{n-l}{2}\ , \tag{A5}$$

i.e.,

$$\Lambda-\nu_i\in R\quad\text{for all }\nu_i\in R_+,\ i=1,\ldots,\frac{n-l}{2}\ . \tag{A6}$$

This shows that the number of NFD operators is

$$M_\Lambda=\frac{n-l}{2}\ . \tag{A7}$$

For the degenerate irrep the highest weight Λ is singular, i.e., for some ν_i $(i=1,\ldots,m<n-l/2)$

$$\nu_i\Lambda=0\ . \tag{A8}$$

In other words, there exist one or several positive roots ν_i which satisfy the following equation:

$$\Lambda-\nu_i\notin R,\quad i=1,\ldots,m<\frac{n-l}{2}\ . \tag{A9}$$

Correspondingly, the number of NFD operators in CSCO is

$$M_\Lambda = \frac{n-l}{2} - m < \frac{n-l}{2} . \tag{A10}$$

These results are independent of which subgroup chain of Eq. (I-5) one is referring to and only depend on the structures of $\mathcal{G}$ and $\mathfrak{h}_\Lambda$.

Furthermore, from the definition of Eq. (2.9), it can directly be shown that the number of elementary excitation operators is the same as the number of NFD operators in CSCO (as well as the number of QDDF). The reason is that if $\Lambda - \nu_i \in R$, then one has

$$E_{\nu_i} |\Lambda, -\Lambda\rangle \neq 0 \text{ or } |\Lambda, -\Lambda\rangle . \tag{A11}$$

According to the definition of Eq. (2.9), $E_{\nu_i}(\Lambda - \nu_i \in R)$ is the elementary excitation operator. Thus, it is obvious that for the nondegenerate irrep space, all the $n-l/2$ generators $\{E_{\nu_i}\}$ satisfy Eq. (2.9) and are the elementary excitation operators; for degenerate irreps, they are $\{E_{\nu_i}\}$ with $\Lambda - \nu_i \in R$. Some detailed examples are presented in Sec. V.

A similar theorem (about the dimensionality of phase space of the Lie group) is recently provided by Faddeev, who obtained a similar result for the U(r) group via the Gelf'and pattern.[66]

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