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Size of an inflated vesicle in two dimensions

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A two-dimensional vesicle of N monomers of diameter a (or a closed, planar, self-avoiding walk of N links) subject to an internal pressure increment $\Delta p \geq 0$ is analyzed. In the large-inflation scaling limit $x \propto \Delta p a^2 N^{2\nu}/k_B T \gg 1$, the mean area and mean-square radius of gyration vary as $\Delta p^{2\omega} N^{2\nu^+}$ with $\omega=(1-\nu)/(2\nu-1)=\frac{1}{2}$ and $\nu^+=\nu/(2\nu-1)=\frac{3}{2}$, where $\nu=\frac{3}{4}$ is the self-avoiding-walk size exponent. The results are related to Pincus's expression for the size of a stretched polymer chain. Monte Carlo simulations confirm the behavior and yield the corresponding scaling functions.

I. INTRODUCTION

A vesicle is a closed membrane typified, in the real world, by red blood cells which, as temperature, solution pH, etc., change, exhibit a variety of characteristic but fluctuating shapes. The statistical mechanics of vesicles is of interest as one aspect of the general theoretical study of interfaces, random surfaces, membranes, and their interactions.[1] One would like to understand how the mean sizes and shapes of vesicles in thermal equilibrium depend (i) on the size of the enveloping membrane, measured by N, the number of constitutive units (e.g., lipid molecules) in the surface, each of characteristic linear dimensions a; (ii) on the osmotic pressure difference, say,

$$\Delta p = p_{\text{in}} - p_{\text{out}} , \tag{1.1}$$

measured between interior and exterior; and (iii) on specific membrane properties such as the *rigidity* κ.

Real vesicles exist in ($d=3$)-dimensional space and have ($d' \equiv d-1=2$)-dimensional surfaces. It is instructive, however, to study initially the simpler case of ($d=2$)-dimensional vesicles enclosed by ($d'=1$)-dimensional surfaces. Such a program was broached by Leibler, Singh, and Fisher,[2,3] (LSF) who performed Monte Carlo simulations of two-dimensional vesicles modeled by closed chains of N "hard", self-avoiding beads (or disks) of diameter a linked together by loose tethers[4] of maximum extension $l_0 < 2a$ between centers of adjacent beads. (In the simulation of LSF, which we extend somewhat here, the value $l_0=\frac{9}{5}a$ was adopted.) For zero pressure increment $\Delta p=0$, this model simply represents a two-dimensional "pearl necklace" of the sort traditionally used to model polymeric molecules. [Of course, Δp now denotes a ($d=2$)-dimensional pressure.] Also discussed by LSF was a bending energy of magnitude fixed by the rigidity κ and in a form proportional to the square of the local radius of curvature of the vesicle surface (or perimeter) as determined by the relative positions of three adjacent beads. However, for our present considerations, which are restricted to $\Delta p \gtrsim 0$, the rigidity κ will play no special role (and may be taken as vanishing or as fixed but not large relative to Nak_BT).[2]

It was concluded by LSF that the mean vesicle area $\langle A \rangle$ and the radius of gyration $\langle R_G^2 \rangle$ obey scaling laws in terms of the scaling variable

$$x = D\bar{p}N^{2\nu} \quad \text{with } \bar{p} = \Delta p a^2/k_BT , \tag{1.2}$$

where D is a nonuniversal metrical factor specified below, and ν is the size (or "correlation length") exponent for self-avoiding random walks. For $d=2$ dimensions one may accept as exact the value[5]

$$\nu = \tfrac{3}{4} \quad (d=2) . \tag{1.3}$$

Then as $N \to \infty$ with x fixed one has[2,6,7]

$$\langle A \rangle_{N,\bar{p}} \approx A_0 N^{2\nu} Y(x) \ , \tag{1.4}$$

$$\langle R_G^2 \rangle_{N,\bar{p}} \approx R_0^2 N^{2\nu} X(x) \ , \tag{1.5}$$

where the normalization equations

$$Y(0)=X(0)=1, \quad Y_1 \equiv \left[\frac{dY}{dx} \right]_{x=0} = 1 \ , \tag{1.6}$$

prove convenient and serve to specify A_0, R_0^2, and D for any particular model. Note that these laws embody the natural scalings: $A \sim R^2$, $\Delta p \sim A^{-1} \sim R^{-2}$, and, by definition, $R \sim N^{\nu}$.

The scaling laws (1.4) and (1.5) were checked by LSF via Monte Carlo simulations using values of N up to 96 and found to be well verified. For large *negative* Δp, corresponding to *deflated* vesicles, the scaling functions $X(x)$ and $Y(x)$ approach asymptotic power-law behavior. The observed exponents were successfully interpreted in terms of a collapse of the vesicles into the form of branched polymers.[2,6]

As observed by LSF, definite asymptotic power laws should also be observed for $\Delta p > 0$, which describes *inflated vesicles.* For large Δp it is intuitively clear that the vesicles should approach circular shape so that $R_G^2 \sim A$; thus we may write

$$X(x) \approx X_+ x^{2\omega}, \quad Y(x) \approx Y_+ x^{2\omega} \tag{1.7}$$

as $x \to \infty$. Furthermore, if the picture of a circular vesicle is correct, one should, with the normalizations (1.4)–(1.6), have

$$\tilde{\pi} \equiv \frac{\langle A \rangle}{\langle R_G^2 \rangle} \to \frac{A_0 X_+}{R_0^2 Y_+} = \pi \quad \text{as } x \to \infty \ . \tag{1.8}$$

The checking of these relations and, in particular, the evaluation of the exponent ω are the principal objectives of this paper. The results prove of significance for further studies of planar vesicles.[7,8]

The investigation of these issues by LSF was only exploratory since they felt that computer limitations prevented attainment of the proper asymptotic regime. Nevertheless, they remarked on an analogy with the problem of an *open* chain subject to a *tensile force* $\mathbf{f}$ applied to its ends. This situation had been studied much earlier in an important seminal paper by Pincus.[9] He concluded that the mean projection $\langle Z_N \rangle$ of the end-to-end vector onto the axis parallel to $\mathbf{f}$ should obey the scaling law

$$\langle Z_N \rangle \approx Z_0 N^{\nu} W(z) \ , \tag{1.9}$$

where

$$z = \bar{f} N^{\nu}, \quad \bar{f} = |\mathbf{f}| a / k_B T \ . \tag{1.10}$$

Pincus further argued that in the *stretched regime* of large z one should have

$$W(z) \approx W_+ z^{\chi} \quad \text{with } \chi = \frac{1}{\nu} - 1 \ , \tag{1.11}$$

in which case one obtains[8] $\langle Z_N \rangle \sim N f^{\chi}$. Although Pincus focused on $d=3$, his conclusion applies more generally. For $d=2$ one has $\chi=\frac{1}{3}$ and LSF confirmed this prediction for *open,* planar tethered chains. [Webman, Lebowitz, and Kalos[10] had checked the behavior for stretched ($d=3$)-dimensional chains.]

As regards vesicles, on the other hand, LSF stated, for reasons explained later, that $\langle A \rangle \sim N^2 \bar{p}^{\chi}$ should be expected in the inflated regime; this would correspond to $\omega = \frac{1}{6}$ in (1.7). (The LSF simulations for $\Delta p > 0$, however, actually suggested that $\langle A \rangle$ increases more rapidly than N^2.) We shall see below that the LSF expectation is actually incorrect. Rather one should find $\langle A \rangle \sim N^3 \bar{p}$ corresponding to $\omega = \frac{1}{2}$; indeed this is well borne out by our more extensive Monte Carlo simulations described below; see Fig. 1.

In Sec. II we present a systematic derivation of the value of ω based on a result of Fisher[11] for the decay of the scaling function describing the distribution of end-to-end distances R_N of general self-avoiding walks. In Sec. III we revisit Pincus's theory[9] of a stretched open chain (which also utilized Fisher's work) and rederive the vesicle result from that perspective. Some other aspects, including the "blob" picture are discussed in Sec. IV.

II. INFLATED VESICLE REGIME

In order to estimate the size of a vesicle in the highly inflated regime we will accept the extremely plausible conclusion that the limiting shape closely approximates a circle of radius R, circumference $2\pi R$, and area $A_R = \pi R^2$. This is, in fact, confirmed by simulations:[7] see below. (However, an approach to any other fixed geometric shape will yield the same value for ω.) We then aim to construct a constrained free-energy function $\mathcal{F}(R; N, \Delta p)$ whose minimization on R will yield the desired mean value $\bar{R}(N, \Delta p)$. Since the Boltzmann factor associated with a vesicle of area A is[2,7] $\exp(\Delta p A / k_B T)$, one term contributing to $\mathcal{F}$ is simply

$$\Delta \mathcal{F}_p = -\Delta p A_R \approx -\pi \Delta p R^2 \ . \tag{2.1}$$

The opposing term represents the stretching free energy of the closed self-avoiding walk constituting the vesicle perimeter. In the highly inflated regime this may be regarded as composed of m almost linear and more-or-less independent segments of $M = N/m$ beads and stretched length

$$R_M = 2\pi R / m \ . \tag{2.2}$$

One may consider, for example, $m = 12$. Now the probability distribution $\mathcal{P}(\mathbf{R}_M)$ for the end-to-end vector $\mathbf{R}_M$ of a self-avoiding chain of M beads obeys the scaling law[11,12]

$$\mathcal{P}(\mathbf{R}_M) \approx M^{-d\nu} P(R_M / \bar{a} M^{\nu}) \ , \tag{2.3}$$

where $P(y)$ is a scaling function which is universal if appropriately normalized while $\bar{a} = O(a)$ is a nonuniversal metrical factor. For large arguments, Fisher[11] demonstrated that this scaling function decays as

$$P(y) \sim \exp(-c_d y^{\delta}) \quad \text{with } \delta = 1/(1-\nu) \ , \tag{2.4}$$

where c_d is a constant. Note that there are power-law (and, possibly, other) prefactors omitted here but they are logarithmically negligible for $y \gg 1$. When $\nu = \frac{1}{2}$ the relation (2.4) reduces to the usual Gaussian law. Renormalization-group calculations[13] also confirm (2.4). If the end-to-end vector $\mathbf{R}_M$ is fixed, the result corresponds to a stretching free energy

$$\mathcal{F}_M^0(R_M) \approx c_d k_B T R_M^\delta / \bar{a}^\delta M^{\delta\nu} . \quad (2.5)$$

This nonlinear spring free energy applies, of course, only for $y = R_M / M^\nu \gg 1$.

Finally, we can use (2.5) to estimate the total vesicle stretching free energy as

$$\Delta\mathcal{F}_N \approx m \mathcal{F}_M^0 \approx \tilde{c}_d k_B T R^\delta / a^\delta N^{\delta\nu} , \quad (2.6)$$

where $\tilde{c}_d$ is *independent* of m, a striking result. One actually finds $\tilde{c}_d \approx (2\pi a/\bar{a})^\delta c_d$, but should not be surprised if this relation should actually be modified by some fixed factor to allow for the circular topology. To complete the work we take $\mathcal{F}(R) = \Delta\mathcal{F}_N + \Delta\mathcal{F}_p$ and minimize on R to find

$$\bar{R}(N, \Delta p) \approx \tilde{a} N^{\nu^+} \bar{p}^{\,\omega} , \quad (2.7)$$

with exponents

$$\nu^+ = \frac{\nu}{2\nu - 1}, \quad \omega = \frac{1-\nu}{2\nu-1} , \quad (2.8)$$

while $\tilde{a} = (2\pi/\delta\tilde{c}_d)^\omega a$. For $d=2$ the value $\nu = \frac{3}{4}$ yields $\nu^+ = \frac{3}{2}$ and $\omega = \frac{1}{2}$, confirming the behavior quoted in Sec. I.

More generally one may contemplate closed self-avoiding walks in $d > 2$ dimensions *spanned* by a *minimal surface* of (two-dimensional) area A and subject to a corresponding pressure Δp. The same scaling theory applies: thus, under high inflation, a circular disk shape will be attained of radius given by (2.7). For $d=3$ a recent estimate is[14] $\nu = 0.592 \pm 0.002$; this yields $\nu^+ = 3.22 \pm 0.06$ and $\omega = 2.22 \pm 0.06$.

When $\nu \to \frac{1}{2}$ (as $d \to 4-$) one sees that ν^+ and ω diverge. This reflects the fact that a Gaussian chain, yielding a harmonic stretching force, cannot resist any inflation ($\Delta p > 0$). A tethered chain would exhibit no scaling regime but rather expand directly to a limit set by the tethers.

By a similar token, it is clear that when Δp increases at *fixed* N, the system must eventually leave the scaling regime, within which (1.4) and (1.5) are valid. If we suppose $l_0/a = O(1)$ or, more generally, that the intrinsic nonlinear bonds which couple the beads have a scale length $l_0 = O(a)$, the limit on scaling will be reached when $2\pi R \approx Na$, if not before. On using (2.7) and (2.8) this yields

$$p \ll N^{-1} \quad \text{or } x \ll N^{2\nu-1} , \quad (2.9)$$

as equivalent necessary (but possibly not sufficient) criteria for the validity of vesicle scaling when $\Delta p > 0$.

Finally, it is clearly of interest to test the conclusion $\langle A \rangle \sim \pi \langle R^2 \rangle \sim N^3 \bar{p}$ for planar vesicles by simulation. The Monte Carlo data presented in Fig. 1 were obtained along previous lines[2,7,8] taking, as before, appropriate

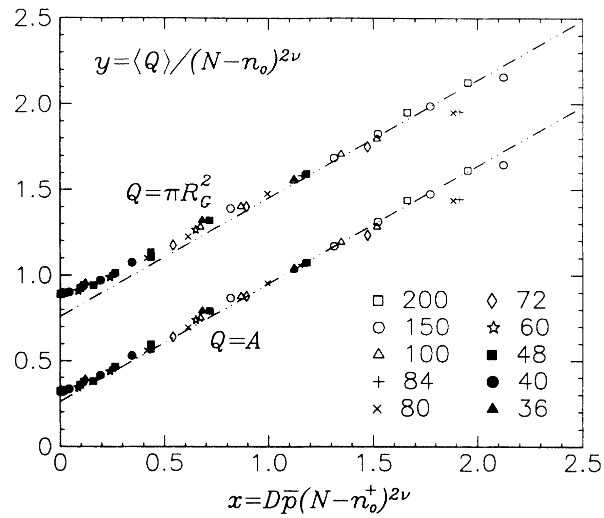

FIG. 1. Scaling plot of data for the area and square radius of gyration of two-dimensional vesicles of N beads demonstrating the asymptotically linear behavior of the scaling functions $X(x)$ and $Y(y)$ for $\pi\langle R_G^2 \rangle$ and $\langle A \rangle$, respectively. Units of a^2 have been used for R_G^2 and A. Note that the plot for $y = \langle \pi R_G^2 \rangle / (N - n_0)^{2\nu}$ has been shifted upwards by $\Delta y = \frac{1}{2}$ for clarity. The asymptotically negligible N shifts n_0 and n_0^+ are explained in the text. The dash-dotted lines correspond to the estimated large-x asymptote; see Eq. (2.10).

precautions regarding equilibration. However, in the well-inflated regime it is not necessary to wait for a full rotational diffusion time in order to obtain good data since the close-to-circular shape ensures sufficient sampling of the phase space on much shorter time scales.

In essence Fig. 1 represents linear plots of the scaling function $X(x)$ and $Y(x)$ for πR_G^2 and A, respectively; but note that for clarity the former has been shifted vertically upwards by $\Delta y = \frac{1}{2}$. Data for which the criteria (2.9) are clearly starting to be violated, so that they depart sharply from the general trend (as in Fig. 1 of LSF), have not been included in the plot. The line of reasoning used to derive (2.9) readily provides the asymptotic estimate $y_{\max}(n) \approx \pi (l_0/2\pi a)^2 N^{1/2}$ for the maximum ordinate in Fig. 1 at fixed N. This gives a feel for the breakaway points which are found to occur roughly at $y \simeq 0.6 y_{\max}$. More concretely one has $y_{\max} \simeq 1.6$, 1.8, 2.2, and 2.6 for $N = 36$, 48, 72, and 100, respectively. The "N shifts" $n_0 \simeq 0.5$ and $n_0^+ = 9$ have been incorporated, following detailed studies,[7] to hasten convergence when $N \to \infty$ at fixed x. These parameters effectively allow for the leading corrections to scaling which, on general renormalization-group-theory grounds, one knows must be present. The value $D = 0.0175 \pm 0.0015$ has been used in the plot; it is based on a careful analysis[7] of the behavior near $x = 0$.

The good data collapse for different N values seen in Fig. 1 evidently confirms the scaling law. The linear behavior, which sets in surprisingly soon as x increases, validates the prediction $2\omega = 1$; the result $\nu^+ = 3$ is then implied via scaling. In fact, when the shift $\Delta y = \frac{1}{2}$ is removed the two data sets coincide within the scatter for $x \gtrsim 0.8$ and indicate a common linear asymptote, thence confirming (1.8), which is well described by

$$\pi R_0^2 X(x),\ A_0 Y(x) \approx 0.69x + 0.26\ , \tag{2.10}$$

when a separate analysis yields[7] $R_0^2 \simeq 0.123_2 \pm 0.001_5$ and $A_0 \simeq 0.314 \pm 0.004$. This estimate for the asymptote is represented by the dash-dotted lines in Fig. 1. We may note that further theoretical analysis[7] shows that $\bar{\pi}(x) \approx \langle A \rangle / \langle R_G^2 \rangle$ should approach its limit π (from below) as $1/x^2$. This is consistent with leading correction terms in (2.10) of the form b_X/x and b_Y/x with $b_X > b_Y$. The approach of $\bar{\pi}$ to π, observed in the simulations[7] serves, of course, to support the hypothesis that the highly inflated shape is circular.

III. STRETCHED OPEN CHAINS AND VESICLES

If, on the basis of the result (2.4),[11] we accept (2.5) for the free energy of an open chain with fixed end points at separation $\mathbf{R}_M$, we can address the problem of a chain under a tension $\mathbf{f}$ simply by adding the term

$$\Delta \mathcal{F}_f = -\mathbf{f} \cdot \mathbf{R}_M\ . \tag{3.1}$$

For the mean projection of $\mathbf{R}_M$ on $\mathbf{f}$, minimization then yields the result

$$\bar{Z} \equiv \langle Z_N \rangle \approx a_f N (fa/k_B T)^\chi = a_f N \bar{f}^{\,\chi}\ , \tag{3.2}$$

where $\chi = \nu^{-1} - 1$, as before, while

$$a_f = a(\bar{a}/a)^{1/\nu} (\delta c_d)^{-\chi}\ .$$

This is just the result found by Pincus;[9] if the scaling ansatz (1.9) is accepted, it implies the asymptotic behavior (1.11) for the scaling function. This argument[15] appears to us more direct and mathematically transparent than the two original derivations of Pincus, which, however, certainly merit further discussion.

In his first approach, Pincus[9] accepts scaling with the self-avoiding exponent ν for low distortions as $f \to 0$. Then he argues that "as the chain stretches, its average monomer density decreases leading to a weakening of the excluded volume effect. Thus for sufficiently large forces, we expect to eventually recover ideal behavior with $\bar{Z} \propto N$ rather than $\bar{Z} \propto N^{2\nu}$" (where the last relation follows from scaling and the Hooke's law assumption $\bar{Z} \propto f$ for *small* f). The behavior $\bar{Z} \sim N$ for large f together with scaling leads directly to (3.2) with $\chi = \nu^{-1} - 1$.

This argument is elaborated by de Gennes who envisages the chain breaking into a series of N_b weakly stretched "blobs" of linear size R_b proportional to the *tensile length*[9]

$$\xi_f = k_B T / f\ . \tag{3.3}$$

It is then argued that the blobs become independent when $Z_N \gg \xi_f$. If there are $M_b = N/N_b$ monomers in a blob, one expects

$$R_b \approx \langle R_{M_b}^2 \rangle^{1/2} \approx a_0 M_b^\nu\ , \tag{3.4}$$

with $a_0 = O(a)$. Finally, one has $\bar{Z} \sim (N/M_b) R_b$, which then yields (3.2).

These arguments are appealing and, as we have seen, their conclusions are surely correct. Nonetheless, on reflection they give rise to various questions. The main one concerns the "weakening" of the self-avoidance and the "independence" of the blobs. Technically, *in* renormalization-group terms, this amounts to the claim that self-avoidance is *irrelevant* about the stretched chain limit. For a *highly* stretched chain with $Z \simeq Na$ this is reasonable (and can be argued more precisely in various ways). However, the highly stretched limit lies beyond the scaling regime. Although plausible, it is not obvious that the irrelevance should be maintained in the stretched *scaling* regime. The blobs, after all, are fuzzy and strongly coupled to their neighbors. Said differently, there might conceivably be a distinct, stretched self-avoiding fixed point.[16] Another potential pitfall connected with the simple Pincus argument is that it makes it tempting to suppose that the mean perimeter $2\pi\bar{R}$ of an inflated vesicle should, like $\bar{Z}$, also become proportional to N. This supposition leads to $\langle R_G^2 \rangle \sim \langle A \rangle \sim N^2 \bar{p}^{\,\chi}$ for vesicles, as proposed by LSF; but, as we have seen this is definitely *not* valid!

It is interesting, however, that the correct result for vesicles can be obtained from the *full* Pincus result (3.2) by a more careful argument. If an inflated vesicle in d dimensions is regarded as a spherical bubble of radius $\bar{R}$ enclosed in a surface of tension Σ, the standard arguments[17] give

$$\Delta p = (d-1)\Sigma / \bar{R}\ . \tag{3.5}$$

Now, for $d=2$, we may identify the surface tension with the stretching tension f and take $\bar{Z} \simeq 2\pi\bar{R}$ so that, using (3.2), we have

$$\Sigma = f = \frac{k_B T}{a} \left[\frac{2\pi \bar{R}}{a_f N} \right]^{\nu/(1-\nu)}\ . \tag{3.6}$$

Combining this with (3.5) and solving for $\bar{R}$ yields precisely our original result (2.7).

A second argument[9] aims at an analytic derivation of (3.2) for $d=3$ which starts with an intermediate result in Fisher's argument,[11] namely, that the self-avoiding-walk analog of the two-point correlation function at a bulk critical point decays as $e^{-r/\xi}/r$ for $T > T_c$.[16] In principle, this approach should yield the desired result; however, the necessary analysis is nontrivial and the development in Ref. 9 seems incomplete.[18] Using the full result (2.4) as in our derivation above,[15] obviates the need for further detailed calculation.

IV. BLOB PICTURE

It is interesting to pursue the blob picture a little further for vesicles. Indeed, the size of a blob R_b may be regarded as measuring the *intrinsic thickness* or *width* of the vesicle wall. From (3.1)–(3.6) we find[19] explicitly for vesicles

$$R_b \approx a_0 / k^{1/\chi} (\bar{p} N)^{\nu^+}\ , \tag{4.1}$$

where $k = 2\pi\tilde{a}/a_0$; here $\tilde{a}$ is the amplitude entering in the expression (2.7) for the mean radius $\bar{R}$ of an inflated vesicle. Note that R_b decreases as N increases, but becomes large when $\bar{p} \to 0$. The number of blobs making up the perimeter can similarly be written

$$N_b = k^{1/(1-\nu)}(x/D)^{1/(2\nu-1)} = N_0 x^2 , \tag{4.2}$$

where, as before, $x = D\bar{p}N^{2\nu}$ is the scaled pressure variable and $\nu = \frac{3}{4}$ in the last equality.

Now the vesicle perimeter of mean length $L = 2\pi\bar{R}$ will undergo transverse thermal wandering,[7,9] of a magnitude[20] $u_{\text{rms}} = (k_B TL/f)^{1/2} = (2\pi k_B T/\Delta p)^{1/2}$. These thermal fluctuations are responsible for the leading deviation of the vesicle shape from a perfect circle.[7] Relative to the radius of the vesicle they have a magnitude[7]

$$u_{\text{rms}}/\bar{R} \sim N^{-\nu^+} \Delta p^{-1/2(2\nu-1)} \sim x^{-1/2(2\nu-1)} , \tag{4.3}$$

where for $\nu = \frac{3}{4}$ the final exponent is just -1. On the other hand, the ratio of the intrinsic width to the amplitude of the wandering is found to be

$$\frac{R_b}{u_{\text{rms}}} \approx \frac{(a_0^2/2\pi a)^{1/2}}{k^{1/\chi}(x/D)^{1/2(2\nu-1)}} . \tag{4.4}$$

Surprisingly, this has precisely the same form as $u_{\text{rms}}/\bar{R}$, decaying as $1/x$ when $\nu = \frac{3}{4}$.

These considerations also apply to the model, mentioned after (2.8), of a self-avoiding loop in d dimensions spanned by a minimal surface subject to a two-dimensional pressure. The decay in (4.3) and (4.4) is then approximately like $1/x^{2.7}$ for $d = 3$.

Finally, by employing the estimate (2.10) for the variation of $\bar{R}$ at large inflations we find $\bar{a}/a \simeq 0.062_0$. Here we have used the value of D quoted above Eq. (2.10).[7] If, in (3.4), we identify R_b in terms of the root-mean-square end-to-end distance of an open chain, we find[7] $a_0 = r_0 \simeq 0.86_8 a$. Thence the constant in (4.2) is found to be $N_0 \simeq 132$. For $x \simeq 0.6-1.3$, which, as seen in Fig. 1, is where the inflated region is established, this means the number of blobs lies in the range 50–220. Since our typical simulation entailed only $N = 60-150$ beads (or monomers) the conclusion is that the blobs consist of at most two or three beads. In these circumstances the picture of blobs is hardly realistic. Its success must rather be interpreted as a convenient way of envisaging in concrete terms the basic scaling properties. Evidently, as happens not infrequently, the asymptotic behavior sets in sooner than might be anticipated.

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[1]See, e.g., the reviews and references quoted in *Statistical Mechanics of Membranes and Surfaces,* edited by D. R. Nelson, T. Piran, and S. Weinberg (World Scientific, Singapore, 1989).

[2]S. Leibler, R. R. P. Singh, and M. E. Fisher, Phys. Rev. Lett. **59**, 1989 (1987); denoted LSF.

[3]See also N. Ostrowsky and J. Peyraud, J. Chem. Phys. **77**, 2081 (1982); G. C. Barker and M. J. Grimson, J. Phys. (Paris) **48**, 465 (1987).

[4]Y. Kantor, M. Kardar, and D. R. Nelson [Phys. Rev. Lett. **57**, 791 (1986); Phys. Rev. A **35**, 3056 (1987)] have introduced and studied ($d' = 2$)-dimensional tethered membranes.

[5]B. Nienhuis, Phys. Rev. Lett. **49**, 1062 (1982); B. Duplantier, *ibid.* **64**, 493 (1990).

[6]See also M. E. Fisher, Nucl. Phys. B (Proc. Suppl.) **5A**, 165 (1988); Physica D **38**, 112 (1989).

[7]C. J. Camacho and M. E. Fisher, Phys. Rev. Lett. (to be published).

[8]A. C. Maggs and S. Leibler, Europhys. Lett. (to be published).

[9]P. Pincus, Macromolecules **9**, 386 (1976).

[10]I. Webman, J. L. Lebowitz, and M. H. Kalos, Phys. Rev. A **23**, 316 (1981).

[11]M. E. Fisher, J. Chem. Phys. **44**, 616 (1966).

[12]P.-G. de Gennes, *Scaling Concepts in Polymer Physics* (Cornell University Press, Ithaca, 1979), Chap. 1.

[13]See, e.g., Y. Oono, T. Ohta, and K. F. Freed, Macromolecules **14**, 880 (1980).

[14]A. J. Guttmann, J. Phys. A **20**, 1839 (1987).

[15]This derivation is more or less explicit in the treatment of de Gennes (Ref. 12). However, what he actually does is (i) accept Pincus's result (1.11) on the basis of the first derivation criticized below; (ii) compare this with the result (3.2), with $\chi = (\delta - 1)^{-1}$ regarded as unknown; and (iii) thereby deduce the Fisher relation (2.4) between δ and ν.

[16]This issue is reflected in bulk critical phenomena by the observation that the scaling function for the pair correlations $G(\mathbf{r}; T)$ near criticality reproduces the Ornstein-Zernike decay laws when $R/\xi \to \infty$ (for fixed $T > T_c$). This fact has been checked by exact calculations for various models and via renormalization-group ϵ expansions. It also underlies Fisher's analysis in Ref. 11.

[17]See, e.g., J. S. Rowlinson and B. Widom, *Molecular Theory of Capillarity* (Oxford University Press, Oxford, England, 1982), Sec. 2.4.

[18]For example, the origin of Eq. (II.5) of Ref. 9 is unclear and this equation entails a *sum* over a *continuous* variable ρ of unspecified character.

[19]We take $R_b = c\xi_f$ and eventually eliminate the proportionality constant c in favor of $\bar{a}$.

[20]See, e.g., M. E. Fisher, J. Chem. Soc. Faraday Trans. 2 **82**, 1569 (1986) (Faraday Symp. 20).