Relativistic wave equations for bound states of two scalar particles from scalar quantum electrodynamics

Lixin Xiao, Jurij W. Darewych, and Marko Horbatsch
Department of Physics, York University, Toronto, Ontario, Canada M3J 1P3
(Received 29 January 1990)

Coupled integral equations for bound states of two spin-0 bosons are derived variationally from scalar quantum electrodynamics, using an *ansatz* which incorporates the transverse photon degrees of freedom of the electromagnetic field. The equations are decoupled approximately and then solved perturbatively as well as numerically. A perturbative eigenenergy formula for particles of arbitrary masses, including all the corrections of order α^4, is obtained. A variational solution of the wave equation is used to determine eigenvalues and eigenfunctions at arbitrary coupling for the $1s$ and $2p$ states of the equal-mass and m_K/m_π cases. Radial s-state excitations are calculated using the basis-state expansion method. In the limit where one particle becomes infinite our equation turns out to be the same as the weak-field limit of the Klein-Gordon-Coulomb equation.

I. INTRODUCTION

In a recent paper[1] (hereafter to be referred to as I), Darewych and Horbatsch have discussed various relativistic bound-state systems of two particles interacting electromagnetically using the variational method within the Hamiltonian formalism of quantum field theory. Since the approximate variational *ansatz* used in I is not sensitive to the transverse photon degrees of freedom of the electromagnetic field, the resulting equations do not contain effects of transverse photon exchange. Those equations describe the relativistic bound-state systems of two particles interacting via a pure Coulomb force. For real physical systems their applicability is, thus, restricted. For example, for positronium and high Z ions the incorporation of transverse photon effects is important and has been discussed by Darewych and Horbatsch[2] and Dykshoorn and Koniuk[3] for two-fermion systems.

As pointed out in paper I, one can incorporate, in principle, all interaction effects by constructing systematically improved *ansätze*. In this paper we use an *ansatz* with the minimal modification that is necessary to sample the transverse photon degrees of freedom, to obtain and solve modified bound-state equations for systems of two scalars (spin-0 bosons) interacting electromagnetically. Even though no observations of electromagnetically bound states of oppositely charged pions and kaons have been reported so far, they are believed to be formed if sufficient amounts of exotic matter are accumulated. The lifetimes of free pions and kaons are 2.6×10^{-8} and 1.2×10^{-8} sec., respectively, and give bounds on the lifetimes of such bound states.

The plan of the paper is as follows. In Sec. II we derive the coupled variational bound-state equations for systems of two scalars. In Sec. III we approximately decouple these equations and renormalize the particle masses. Within this decoupling scheme, the nonrelativistic limit and the limit where the mass of one particle becomes infinite are compared with the Schrödinger and Klein-Gordon equations, respectively. Perturbative and numerical solutions of the decoupled equations are presented in Secs. IV and V. Conclusions are summarized in Sec. VI.

II. COUPLED BOUND-STATE EQUATIONS

We derive variational bound-state equations for two different systems of two scalars: one system consists of two distinct scalars and the other of a scalar and its antiparticle. As in paper I, the Hamiltonians are constructed by the canonical prescription from the covariant Lagrangians. In the radiation gauge $\nabla\cdot\mathbf{A}=0$, the Hamiltonian density for the system of two distinct scalars of masses M and m, and charges (absolute values) e_1 and e_2, takes the form

$$\mathcal{H}(x)=\mathcal{H}_\phi(x)+\mathcal{H}_\psi(x)+\lambda_c'\phi^*\phi\psi^*\psi+\mathcal{H}_\gamma(x)+\mathcal{H}_C(x)\ , \tag{2.1}$$

where

$$\mathcal{H}_\phi(x)=\pi_\phi\pi_{\phi^*}+[(\nabla+ie_1\mathbf{A})\phi]^*\cdot[(\nabla+ie_1\mathbf{A})\phi] +M_0^2\phi^*\phi\ , \tag{2.2}$$

$$\mathcal{H}_\gamma(x)=\tfrac{1}{2}[\dot{\mathbf{A}}^2+(\nabla\times\mathbf{A})^2]\ , \tag{2.3}$$

$$\mathcal{H}_C(x)=\frac{1}{8\pi}\int d^3y\frac{\rho(x)\rho(y)}{|\mathbf{x}-\mathbf{y}|}\ , \tag{2.4}$$

with

$$\rho(x)=ie_1(\phi^*\pi_{\phi^*}-\phi\pi_\phi)+ie_2(\psi^*\pi_{\psi^*}-\psi\pi_\psi)\ , \tag{2.5}$$

and $\mathcal{H}_\psi(x)$ takes the same form as $\mathcal{H}_\phi(x)$ but with e_1 and M_0 replaced by e_2 and m_0. In Eqs. (2.2) and (2.5), π_ϕ and π_{ϕ^*} are the conjugate momenta of ϕ and ϕ^*, respectively. The Hamiltonian density for the system of a scalar particle and its antiparticle is given by

$$\mathcal{H}(x)=\mathcal{H}_\phi(x)+\lambda_c(\phi^*\phi)^2+\mathcal{H}_\gamma(x)+\mathcal{H}_C(x)\ , \tag{2.6}$$

where $\mathcal{H}_\phi(x)$, $\mathcal{H}_\gamma(x)$, and $\mathcal{H}_C(x)$ are all as before but with

$$\rho(x)=ie_1(\phi^*\pi_{\phi^*}-\phi\pi_\phi)\ . \tag{2.7}$$

λ_c and λ_c' are coupling constants for the contact interaction that arises in scalar QED.[13]

We Fourier decompose the field operators at time $t=0$.[4] For the scalar fields we have

$$\phi(\mathbf{x})=\int d^3q[(2\pi)^3 2\Omega_q]^{-1/2}[A_M(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}+B_M^\dagger(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}}]\ , \tag{2.8a}$$

$$\phi^*(\mathbf{x})=\int d^3q[(2\pi)^3 2\Omega_q]^{-1/2}[A_M^\dagger(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}}+B_M(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}]\ , \tag{2.8b}$$

$$\pi_\phi(\mathbf{x})=i\int d^3q\left[\frac{\Omega_q}{2(2\pi)^3}\right]^{1/2}[A_M^\dagger(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}}-B_M(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}]\ , \tag{2.8c}$$

$$\pi_{\phi^*}(\mathbf{x})=-i\int d^3q\left[\frac{\Omega_q}{2(2\pi)^3}\right]^{1/2}[A_M(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}-B_M^\dagger(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}}]\ , \tag{2.8d}$$

where $\Omega_p=(\mathbf{p}^2+M^2)^{1/2}$. Similar decompositions hold for $\psi(\mathbf{x})$, $\pi_\psi(\mathbf{x})$, etc., but with M and Ω_p replaced by m and ω_p [$\omega_p=(\mathbf{p}^2+m^2)^{1/2}$]. For the electromagnetic field we have

$$\mathbf{A}(\mathbf{x})=\sum_{\lambda=1}^{2}\int d^3k\,\boldsymbol{\epsilon}(\mathbf{k},\lambda)[(2\pi)^3 2|\mathbf{k}|]^{-1/2}\times[a(\mathbf{k},\lambda)e^{i\mathbf{k}\cdot\mathbf{x}}+a^\dagger(\mathbf{k},\lambda)e^{-i\mathbf{k}\cdot\mathbf{x}}]\ , \tag{2.9a}$$

$$\mathbf{A}(\mathbf{x})=-i\sum_{\lambda=1}^{2}\int d^3k\,\boldsymbol{\epsilon}(\mathbf{k},\lambda)\left[\frac{|\mathbf{k}|}{2(2\pi)^3}\right]^{1/2}\times[a(\mathbf{k},\lambda)e^{i\mathbf{k}\cdot\mathbf{x}}-a^\dagger(\mathbf{k},\lambda)e^{-i\mathbf{k}\cdot\mathbf{x}}]\ , \tag{2.9b}$$

where the transverse polarization vectors, $\boldsymbol{\epsilon}(\mathbf{k},\lambda)$ ($\lambda=1,2$), satisfy $\boldsymbol{\epsilon}(\mathbf{k},\lambda)\cdot\mathbf{k}=0$. The momentum-space operators, A, B, a, etc., obey the usual commutation relations. The nonvanishing ones are

$$[A_M(\mathbf{p}),A_M^\dagger(\mathbf{q})]=[B_M(\mathbf{p}),B_M^\dagger(\mathbf{q})]=\delta^3(\mathbf{p}-\mathbf{q})\ , \tag{2.10a}$$

$$[A_m(\mathbf{p}),A_m^\dagger(\mathbf{q})]=[B_m(\mathbf{p}),B_m^\dagger(\mathbf{q})]=\delta^3(\mathbf{p}-\mathbf{q})\ , \tag{2.10b}$$

$$[a(\mathbf{k},\lambda),a^\dagger(\mathbf{k}',\lambda')]=\delta^3(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}\ , \tag{2.11}$$

while all other commutators vanish.

Since we are not interested in the vacuum energy problem, we normal order the Hamiltonian operator $H=\int d^3x\mathcal{H}(\mathbf{x})$ after having expressed it in terms of A, B, a, etc. The operators $A^\dagger$ and A ($B^\dagger$ and B) are identified with the creation and annihilation operators of positively (negatively) charged free scalar particles, and $a^\dagger$ and a with these operators for free transverse photons.[4] At this stage m and M in the Fourier expansions of scalar fields are still adjustable parameters, and m_0 and M_0 in the Hamiltonian density are the "bare" masses. In Sec. III we renormalize the masses and identify m and M with the physical particle masses.

For the system of two distinct scalar particles, the improved two-particle *ansatz* is taken to be[1]

$$|2\rangle=\int d^3pF(\mathbf{p})B_M^\dagger(-\mathbf{p})A_m^\dagger(\mathbf{p})|0\rangle+\sum_{\lambda=1}^{2}\int d^3p_1d^3p_2G(\mathbf{p}_1,\mathbf{p}_2,\lambda)B_M^\dagger(\mathbf{p}_2)A_m^\dagger(\mathbf{p}_1)\times a^\dagger(-\mathbf{p}_1-\mathbf{p}_2,\lambda)|0\rangle\ , \tag{2.12}$$

where F and G are variational coefficients to be determined from the variational principle. $|0\rangle$ is the trial vacuum state, which has the defining property $a(\mathbf{p},\lambda)|0\rangle=0$, and similarly for the A and B annihilation operators. For the system of a scalar particle and its antiparticle $A_m^\dagger$ is replaced by $A_M^\dagger$ in Eq. (2.12).

In the case of two distinct scalars, the variational principle

$$\delta\langle 2|{:}H-E{:}|2\rangle=0 \tag{2.13}$$

and functional differentiations with respect to F and G lead to the following coupled integral eigenvalue equations corresponding to the eigenenergy E:

$$\left[\omega_p+\Omega_p+\frac{m_0^2-m^2}{2\omega_p}+\frac{M_0^2-M^2}{2\Omega_p}-E\right]F(\mathbf{p})=\frac{e_1e_2}{(2\pi)^3}\int d^3q\frac{F(\mathbf{q})}{|\mathbf{q}-\mathbf{p}|^2}\frac{(\Omega_p+\Omega_q)(\omega_p+\omega_q)-(\lambda_c'/e_1e_2)(\mathbf{q}-\mathbf{p})^2}{4(\Omega_p\Omega_q\omega_p\omega_q)^{1/2}}$$
$$-\frac{e_1}{(4\pi)^{3/2}}\sum_{\lambda=1}^{2}\int d^3qG(\mathbf{p},-\mathbf{q},\lambda)\frac{\boldsymbol{\epsilon}(\mathbf{q}-\mathbf{p},\lambda)\cdot(\mathbf{p}+\mathbf{q})}{(\Omega_p\Omega_q|\mathbf{q}-\mathbf{p}|)^{1/2}}$$
$$-\frac{e_2}{(4\pi)^{3/2}}\sum_{\lambda=1}^{2}\int d^3qG(\mathbf{q},-\mathbf{p},\lambda)\frac{\boldsymbol{\epsilon}(\mathbf{p}-\mathbf{q},\lambda)\cdot(\mathbf{p}+\mathbf{q})}{(\omega_p\omega_q|\mathbf{p}-\mathbf{q}|)^{1/2}} \tag{2.14}$$

and

$$\left[\omega_{p_1}+\Omega_{p_2}+|\mathbf{p}_1+\mathbf{p}_2|+\frac{m_0^2-m^2}{2\omega_{p_1}}+\frac{M_0^2-M^2}{2\Omega_{p_2}}-E\right]G(\mathbf{p}_1,\mathbf{p}_2,\lambda)$$
$$=\frac{1}{(4\pi)^{3/2}}\frac{\epsilon(-\mathbf{p}_1-\mathbf{p}_2,\lambda)\cdot(\mathbf{p}_2-\mathbf{p}_1)}{|\mathbf{p}_1+\mathbf{p}_2|^{1/2}}\left[\frac{e_1F(\mathbf{p}_1)}{(\Omega_{p_1}\Omega_{p_2})^{1/2}}+\frac{e_2F(-\mathbf{p}_2)}{(\omega_{p_1}\omega_{p_2})^{1/2}}\right]$$
$$+\frac{e_1e_2}{(2\pi)^3}\int d^3p_1'd^3p_2'\delta^3(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1-\mathbf{p}_2)\frac{G(\mathbf{p}_1',\mathbf{p}_2',\lambda)}{|\mathbf{p}_1-\mathbf{p}_1'|^2}\frac{(\omega_{p_1}+\omega_{p_1'})(\Omega_{p_2}+\Omega_{p_2'})-(\lambda_c'/e_1e_2)(\mathbf{p}_1-\mathbf{p}_1')^2}{4(\omega_{p_1}\omega_{p_1'}\Omega_{p_2}\Omega_{p_2'})^{1/2}}$$
$$-\frac{e_1^2}{(2\pi)^3}\sum_{\lambda'=1}^{2}\int d^3qG(\mathbf{p}_1,\mathbf{q},\lambda')\frac{\epsilon(-\mathbf{p}_1-\mathbf{p}_2,\lambda)\cdot\epsilon(-\mathbf{p}_1-\mathbf{q},\lambda')}{(4\Omega_{p_2}\Omega_q|\mathbf{p}_1+\mathbf{p}_2||\mathbf{p}_1+\mathbf{q}|)^{1/2}}$$
$$-\frac{e_2^2}{(2\pi)^3}\sum_{\lambda'=1}^{2}\int d^3qG(\mathbf{q},\mathbf{p}_2,\lambda')\frac{\epsilon(-\mathbf{p}_1-\mathbf{p}_2,\lambda)\cdot\epsilon(-\mathbf{p}_2-\mathbf{q},\lambda')}{(4\omega_{p_1}\omega_q|\mathbf{p}_1+\mathbf{p}_2||\mathbf{p}_2+\mathbf{q}|)^{1/2}}\ . \tag{2.15}$$

For the scalar particle-antiparticle system, we note that the two coupled equations are the same as above but contain an additional term in each equation. The additional terms are

$$-\frac{e_1^2}{(2\pi)^3}\int d^3qF(\mathbf{q})\frac{\mathbf{p}\cdot\mathbf{q}}{12\Omega_p^2\Omega_q^2} \tag{2.16}$$

and

$$-\frac{e_1^2}{(2\pi)^3}\int d^3p_1'd^3p_2'\delta^3(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')$$
$$\times\frac{G(\mathbf{p}_1',\mathbf{p}_2',\lambda)}{|\mathbf{p}_1+\mathbf{p}_2|^2}\frac{(\Omega_{p_2}-\Omega_{p_1})(\Omega_{p_2'}-\Omega_{p_1'})}{4(\Omega_{p_1}\Omega_{p_2}\Omega_{p_1'}\Omega_{p_2'})^{1/2}} \tag{2.17}$$

for the right-hand sides of Eqs (2.14) and (2.15), respectively, and λ_c', m, and e_2 are to be replaced by $4\lambda_c$, M, and e_1. The two additional terms represent the interaction effects from virtual particle-antiparticle pair annihilation into a Coulomb photon without and with the presence of a transverse photon, respectively.

III. APPROXIMATE DECOUPLING AND MASS RENORMALIZATION

The coupled equations obtained above are, evidently, not easy to solve. We can, however, use the following approximate method to simplify them: in Eq. (2.15), to the lowest approximation, i.e., $O(\alpha^2)$, where $\alpha=e_1e_2/4\pi$, we may take

$$E=\omega_{p_1}+\Omega_{p_2}\ , \tag{3.1a}$$

$$m=m_0 \quad \text{and} \quad M=M_0\ , \tag{3.1b}$$

and omit the terms that involve integrals of the function G. Therefore, we have

$$G(\mathbf{p}_1,\mathbf{p}_2,\lambda)\simeq\frac{1}{(4\pi)^{3/2}}\frac{\epsilon(-\mathbf{p}_1-\mathbf{p}_2,\lambda)\cdot(\mathbf{p}_2-\mathbf{p}_1)}{|\mathbf{p}_1+\mathbf{p}_2|^{3/2}}$$
$$\times\left[\frac{e_1F(\mathbf{p}_1)}{(\Omega_{p_1}\Omega_{p_2})^{1/2}}+\frac{e_2F(-\mathbf{p}_2)}{(\omega_{p_1}\omega_{p_2})^{1/2}}\right]\ . \tag{3.2}$$

Substituting Eq. (3.2) into Eq. (2.14), we obtain a single decoupled bound-state equation:

$$F(\mathbf{p})\left[(\omega_p+\Omega_p-E)+\frac{m_0^2-m^2}{2\omega_p}+\frac{M_0^2-M^2}{2\Omega_p}-\frac{1}{(2\pi)^3}\int d^3q\sum_{\lambda=1}^{2}\frac{[\epsilon(\mathbf{p}-\mathbf{q},\lambda)\cdot\mathbf{p}]^2}{|\mathbf{p}-\mathbf{q}|^2}\left[\frac{e_1^2}{2\Omega_p\Omega_q}+\frac{e_2^2}{2\omega_p\omega_q}\right]\right]$$
$$=\frac{e_1e_2}{(2\pi)^3}\int d^3q\frac{F(\mathbf{q})}{|\mathbf{p}-\mathbf{q}|^2}K(\mathbf{p},\mathbf{q})\ , \tag{3.3a}$$

where

$$K(\mathbf{p},\mathbf{q})=\frac{(\omega_p+\omega_q)(\Omega_p+\Omega_q)-\dfrac{\lambda_c'(\mathbf{p}-\mathbf{q})^2}{e_1e_2}+4\displaystyle\sum_{\lambda=1}^{2}[\epsilon(\mathbf{p}-\mathbf{q},\lambda)\cdot\mathbf{p}]^2}{4(\omega_p\omega_q\Omega_p\Omega_q)^{1/2}}\ . \tag{3.3b}$$

This approximate decoupling procedure is justifiable to order α^4 from the perturbative point of view. The fact that we have used a limited Fock-space *ansatz* [Eq. (2.12)] means that we cannot reproduce perturbative results exactly beyond α^4, irrespective of whether Eqs. (2.14) and (2.15) are solved exactly or approximately [as is done below in Eq. (3.3)]. To obtain the correct higher-order terms would require the inclusion of additional Fock-

space states in the *ansatz* (2.12).

We note that the left-hand side of Eq. (3.3a) does not have the expected rest-plus-kinetic-energy form, $(\mathbf{p}^2+M^2)^{1/2}+(\mathbf{p}^2+m^2)^{1/2}$, unless we choose the adjustable parameter M such that

$$M_0^2-M^2=\frac{e_1^2}{(2\pi)^3}\int d^3q\frac{\sum_{\lambda=1}^{2}[\epsilon(\mathbf{p}-\mathbf{q},\lambda)\cdot\mathbf{p}]^2}{|\mathbf{p}-\mathbf{q}|^2\Omega_q} \tag{3.4}$$

and similarly for m and m_0. These are just the mass renormalization conditions that arise from the transverse photon degrees of freedom. Their imposition reduces (3.3a) to a form in which m and M can be identified with the physical particle masses. We should mention that the formally divergent integral in (3.4) is controlled by a cutoff procedure (for example, in the finite domain $0<|\mathbf{q}|\le\Lambda$). Any physically observable quantities that are derived must, of course, be independent of the cutoff and indeed of the renormalization procedure.

We also note that the same renormalization conditions are obtained if a variational one-particle *ansatz* analogous to the two-particle *ansatz* (2.12) is considered. With

$$|1\rangle=CB_M^\dagger(\mathbf{p})|0\rangle+\sum_{\lambda=1}^{2}\int d^3qg(\mathbf{p},\mathbf{q},\lambda)B_M^\dagger(\mathbf{q})a^\dagger(\mathbf{p}-\mathbf{q},\lambda)|0\rangle\ , \tag{3.5}$$

the variational principle

$$\delta\langle 1|:H-E_1:|1\rangle=0 \tag{3.6}$$

yields the coupled eigenvalue equations for the variational coefficients C and g and corresponding eigenergy E_1, viz.,

$$(E_1-\Omega_p)C=C\frac{M_0^2-M^2}{2\Omega_p}-\frac{e_1}{(4\pi)^{3/2}}\sum_{\lambda=1}^{2}\int d^3q\frac{\epsilon(\mathbf{p}-\mathbf{q},\lambda)\cdot(\mathbf{p}+\mathbf{q})}{[\Omega_p\Omega_q|\mathbf{p}-\mathbf{q}|]^{1/2}}g(\mathbf{p},\mathbf{q},\lambda) \tag{3.7a}$$

and

$$\left[\Omega_q+\frac{M_0^2-M^2}{2\Omega_q}+|\mathbf{p}-\mathbf{q}|-E_1\right]g(\mathbf{p},\mathbf{q},\lambda)=\frac{e_1}{(4\pi)^{3/2}}C\frac{\epsilon(\mathbf{p}-\mathbf{q},\lambda)\cdot(\mathbf{p}+\mathbf{q})}{[\Omega_p\Omega_q|\mathbf{p}-\mathbf{q}|]^{1/2}}-\frac{e_1^2}{(2\pi)^3}\sum_{\lambda'=1}^{2}\int d^3q'g(\mathbf{p},\mathbf{q}',\lambda')\frac{\epsilon(\mathbf{p}-\mathbf{q},\lambda)\cdot\epsilon(\mathbf{p}-\mathbf{q}',\lambda')}{[4\Omega_q\Omega_{q'}|\mathbf{p}-\mathbf{q}||\mathbf{p}-\mathbf{q}'|]^{1/2}}\ . \tag{3.7b}$$

Making the lowest-order approximation in Eq. (3.7b), we have

$$g(\mathbf{p},\mathbf{q},\lambda)\simeq\frac{e_1}{(4\pi)^{3/2}}\frac{\epsilon(\mathbf{p}-\mathbf{q},\lambda)\cdot(\mathbf{p}+\mathbf{q})}{[\Omega_p\Omega_q|\mathbf{p}-\mathbf{q}|^3]^{1/2}}C\ . \tag{3.8}$$

Substituting (3.8) into Eq. (3.7a) yields

$$E_1-\Omega_p=\frac{M_0^2-M^2}{2\Omega_p}-\frac{e_1^2}{2(2\pi)^3}\sum_{\lambda=1}^{2}\int d^3q\frac{[\epsilon(\mathbf{p}-\mathbf{q},\lambda)\cdot\mathbf{p}]^2}{|\mathbf{p}-\mathbf{q}|^2\Omega_p\Omega_q}\ . \tag{3.9}$$

This procedure is justifiable perturbatively, just like the approximate decoupling of the two-particle equations.

We require that the one-particle energy E_1 be of the form

$$E_1=\Omega_p=(\mathbf{p}^2+M^2)^{1/2}\ , \tag{3.10}$$

where M is the physical one-particle mass. Then from (3.9) one obtains the same equation as (3.4), i.e., the mass renormalization condition.

Using these mass renormalization conditions, we obtain, from Eq. (3.3), the following bound-state equation for the system of two distinct scalars:

$$(\omega_p+\Omega_p-E)F(\mathbf{p})=\frac{e_1e_2}{(2\pi)^3}\int d^3qF(\mathbf{q})\frac{K(\mathbf{p},\mathbf{q})}{|\mathbf{p}-\mathbf{q}|^2}\ , \tag{3.11}$$

where

$$K(\mathbf{p},\mathbf{q})=K_C(\mathbf{p},\mathbf{q})+K_T(\mathbf{p},\mathbf{q})\ . \tag{3.12}$$

Here,

$$K_C(\mathbf{p},\mathbf{q})=\frac{(\Omega_p+\Omega_q)(\omega_p+\omega_q)-(\lambda_c'/e_1e_2)(\mathbf{p}-\mathbf{q})^2}{4(\Omega_p\Omega_q\omega_p\omega_q)^{1/2}} \tag{3.13}$$

and

$$K_T(\mathbf{p},\mathbf{q})=\frac{\mathbf{p}^2-[\mathbf{p}\cdot(\mathbf{p}-\mathbf{q})]^2/(\mathbf{p}-\mathbf{q})^2}{(\Omega_p\Omega_q\omega_p\omega_q)^{1/2}}\ . \tag{3.14}$$

Likewise, for the system of a scalar particle and its antiparticle, we obtain, by using the above approximate decoupling scheme,

$$(2\Omega_p-E)F(\mathbf{p})=\frac{e_1^2}{(2\pi)^3}\int d^3qF(\mathbf{q})\times\left[\frac{K(\mathbf{p},\mathbf{q})}{|\mathbf{p}-\mathbf{q}|^2}-\frac{\mathbf{p}\cdot\mathbf{q}}{12\Omega_p^2\Omega_q^2}\right], \tag{3.15}$$

where $K(\mathbf{p},\mathbf{q})$ takes the same form as above but with λ_c' replaced by $4\lambda_c$.

We note that Eqs. (3.11) and (3.15) are of the same forms as those presented in paper I. Furthermore, setting $K_T=0$ leads to the equations for the case of the purely Coulombic interaction given in that paper. This shows

that K_T represents the transverse photon contribution to the interaction energy.

In addition, in the nonrelativistic limit (where $\Omega_p = M + \mathbf{p}^2/2M$ and $\omega_p = m + \mathbf{p}^2/2m$), expanding the kernels in powers of p/m and p/M, we find that, in the lowest-order approximation, both (3.11) and (3.15) reduce to the momentum-space Schrödinger equation for a nonrelativistic system of two particles interacting via the Coulomb potential, viz.,

$$\left[m+M+\frac{1}{2}\left[\frac{1}{m}+\frac{1}{M}\right]\mathbf{p}^2-E\right]F(\mathbf{p}) = \frac{e_1e_2}{(2\pi)^3}\int d^3q\frac{F(\mathbf{q})}{|\mathbf{p}-\mathbf{q}|^2} . \quad (3.16)$$

In the limit where the mass of one particle (say M) becomes infinite, Eq. (3.12) reduces to

$$(\omega_p - E_1)F(\mathbf{p}) = \frac{e_1e_2}{(2\pi)^3}\int d^3q\frac{\omega_p+\omega_q}{2\sqrt{\omega_p\omega_q}}\frac{F(\mathbf{q})}{|\mathbf{p}-\mathbf{q}|^2} , \quad (3.17)$$

where $E_1 = E - M$. We note that this equation happens to be identical to the weak field limit[6] of the Klein-Gordon equation. The same equation is obtained in the purely Coulombic case, i.e., setting $G=0$ in (2.14) or, equivalently, $K_T=0$ in (3.12) for the $M\to\infty$ limit. This indicates that the transverse photon degrees of freedom do not manifest themselves in this limit.

IV. PERTURBATIVE SOLUTIONS

Equation (3.16) has the known eigenenergies ($\hbar = c = 1$)

$$E_{nlm_l} = m + M - \frac{1}{2}\frac{\mu\alpha^2}{n^2} , \quad (4.1)$$

where $\mu = mM/(M+m)$ is the reduced mass, $\alpha = e_1e_2/4\pi$ and n, l, m_l are the usual quantum numbers. The corresponding eigenfunctions[7,8] are

$$F_{nlm_l} = f_{nl}(p)Y_{lm_l}(\theta,\varphi) , \quad (4.2)$$

where

$$f_{nl}(p) = \left[\frac{2n(n-l-1)!}{\pi(n+l)!}\right]^{1/2} 2^{2l+2}l!\gamma^{(l+5/2)} \times \frac{p^l}{(p^2+\gamma^2)^{l+2}}C_{n-l-1}^{l+1}\left[\frac{p^2-\gamma^2}{p^2+\gamma^2}\right] . \quad (4.3)$$

In the above, $\gamma = \mu\alpha/n$, C_{n-l-1}^{l+1} are the Gegenbauer functions, and Y_{lm_l} are the spherical harmonics.

Using the zeroth-order approximate wave function and Eqs. (3.11) and (3.16), we obtain the lowest-order relativistic correction to the nonrelativistic energy (cf. Ref. 1), viz.,

$$\begin{aligned}\Delta E_{nlm_l} &= E - E_{nlm_l} \\ &= \int |F_{nlm_l}(\mathbf{p})|^2\left[\omega_p+\Omega_p-\left[m+M+\frac{1}{2\mu}\mathbf{p}^2\right]\right]d^3p - \frac{\alpha}{2\pi^2}\int F^*_{nlm_l}(\mathbf{p})F_{nlm_l}(\mathbf{q})\frac{K(\mathbf{p},\mathbf{q})-1}{|\mathbf{p}-\mathbf{q}|^2}d^3pd^3q \\ &\simeq -\tfrac{1}{2}\alpha^4\mu^4\left[\frac{1}{m^3}+\frac{1}{M^3}\right]\left[\frac{1}{n^3(l+\frac{1}{2})}-\frac{3}{4n^4}\right] \\ &\quad + \frac{\lambda_c'\alpha^3}{4\pi}\frac{\mu^3}{Mm}\frac{1}{n^3}\delta_{0l} - \frac{\alpha^4\mu^3}{Mm}\left[\left[\frac{3}{2l+1}-\delta_{0l}\right]\frac{1}{n^3}-\frac{1}{n^4}\right]+O(\alpha^6) . \end{aligned} \quad (4.4)$$

The three terms in the last two lines of Eq. (4.4) are, respectively, the α^4 contributions of the relativistic kinetic energy, the Coulomb interaction, and the transverse photon exchange interaction.

Rearranging (4.4), we obtain

$$E = m + M - \frac{\mu\alpha^2}{2n^2} + \frac{\alpha^4\mu}{8n^4}\left[3-\frac{\mu^2}{Mm}\right] - \frac{\alpha^4\mu}{n^3}\frac{1}{2l+1} + \frac{\alpha^4\mu^3}{Mm}\frac{1}{n^3}\left[\frac{\lambda_c'}{4\pi\alpha}+1\right]\delta_{0l}+O(\alpha^6) . \quad (4.5)$$

We note that, except for the last one, all terms in this expression are the same as those given by an empirical formula quoted by Itzykson and Zuber (Ref. 9, p. 84).

For the particle-antiparticle system, to order α^4 the perturbative energies are exactly the same as Eq. (4.5) with λ_c' replaced by $4\lambda_c$. This occurs because the second term on the right-hand side of Eq. (3.15) affects only p states and contributes at order α^6 and higher. This is not difficult to understand, as this term represents particle-antiparticle annihilation into a Coulomb photon.

V. NUMERICAL SOLUTIONS

Letting $F(\mathbf{p}) = f(p)Y_{lm_l}(\theta,\varphi)$ and carrying out the angular integration in (3.11) or (3.15), we obtain the radial integral equation

$$f(p)(\omega_p+\Omega_p-E) = \frac{\alpha}{\pi}\int_0^\infty f(q)\frac{q}{p}K(p,q)dq , \quad (5.1)$$

where the radial kernels $K(p,q)$ for s and p states are given in the Appendix. We note that the term with coupling constant λ_c' or λ_c in $K(\mathbf{p},\mathbf{q})$ only contributes to the s-state radial kernel $K(p,q)$. The second term in (3.15)

only affects the p-state radial kernel. In fact, these characteristics are already evident from the perturbative energies.

Before solving (5.1) approximately, we note that for the s-state case in the massless limit,[5] viz, $m=M=0$, Eq. (5.1) has the exact solution $f(p)=1/p^2$, $E=0$ at $\alpha_0=8\pi/(\pi^2+12)=1.149\ldots$ [or $\alpha_0=8\pi/(\pi^2+4)=1.812\ldots$ for the purely Coulombic case as pointed out in paper I]. This is true only for $\lambda_c'=4\pi\alpha$ (or $\lambda_c=\pi\alpha$ for the particle-antiparticle system), while for other values of λ_c' (or λ_c) and for $l\neq 0$ no analytic solutions of Eq. (5.1) have been obtained.

Equation (5.1) can be solved approximately in different ways.[1-3,10] Using the variational method we have obtained eigenvalues and eigenfunctions as functions of $\alpha=e_1e_2/4\pi$ for $1s$ and $2p$ states in various cases. For comparison purposes we have also calculated some results for the radial excitations using the basis-state expansion method, which has been discussed in paper I.

In the variational approach, Eq. (5.1) is replaced by

$$E(p_0,N)=\left[\int_0^\infty dp(\omega_p+\Omega_p)p^2f^2(p,p_0,N) - \frac{\alpha}{\pi}\int_0^\infty\int_0^\infty pdp\,qdq\,f(p,p_0,N) \times f(q,p_0,N)K(p,q)\right] \times\left[\int_0^\infty f^2(p,p_0,N)p^2dp\right]^{-1}, \tag{5.2}$$

where p_0,N are variational parameters and $f(p,p_0,N)$ are generalizations of (4.3); for example,

$$f(p,p_0,N)=\begin{cases}\dfrac{1}{(p^2+p_0^2)^N} & \text{for the } 1s \text{ state}, \quad (5.3)\\[2ex] \dfrac{p}{(p^2+p_0^2)^N} & \text{for the } 2p \text{ state}. \quad (5.4)\end{cases}$$

The variational wave-function parameters are determined from

$$\frac{\partial E}{\partial N}=0 \quad \text{and} \quad \frac{\partial E}{\partial p_0}=0 \tag{5.5}$$

and the corresponding minimal value of $E(p_0,N)$ is taken as the approximate eigenenergy for each α.

Figure 1 is a plot of the eigenenergies as functions of α for the $1s$, $2s$, $2p$, and $3s$ states of the system of two distinct particles when $\lambda_c'=4\pi\alpha$. Two mass ratios are considered: the lower four curves are for $M=m$, while the upper ones are for $M=(m_K/m_\pi)m$, where $m_K/m_\pi=3.536\,97$, i.e., the mass ratio of K to π mesons. The $1s$ and $2p$ results were obtained using both the variational and basis-state expansion (BSE) methods, but only the variational results are shown on the graph for these states. The BSE results agree to within plotting accuracy with the variational ones, except near the critical coupling, where they are slightly higher and extrapolation to an infinite number of basis states is required (see Ref. 1).

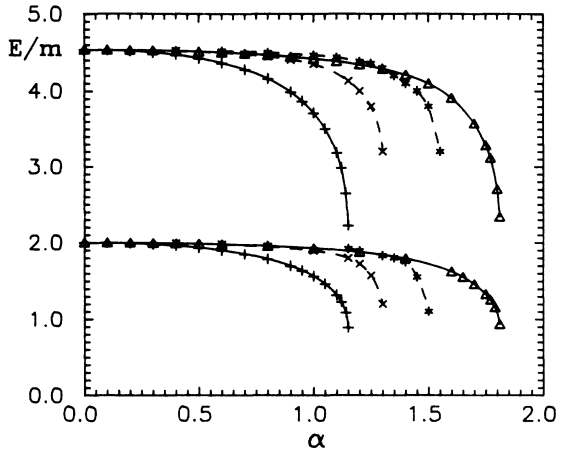

FIG. 1. Two-particle bound state energy E/m as a function of the coupling constant α for the case $M=m$ (lower set of curves) and $M\approx 3.537m$ (kaon-to-pion mass ratio) when $\lambda_c'=4\pi\alpha$. +, $1s$; ×, $2s$; *, $3s$, △, $2p$.

One can observe that the energies for all states start out at weak coupling from the nonrelativistic values and exhibit the corresponding angular-momentum degeneracies. As the coupling increases, however, the energy curves decrease rapidly and this degeneracy is broken. The energies for the $1s$ states decrease rapidly towards zero at a critical coupling $\alpha_0\approx 1.15$, independently of the mass ratio. Note, however, that another critical coupling value, namely the one at which the system becomes unstable with respect to decay into a particle-antiparticle pair of the lighter species (mass m), decreases slowly as the mass ratio increases. This effect was studied in I for the purely Coulombic case. The value of α_0 for the $1s$ state, found here for the massive case, is quite close to the analytic massless result of $8\pi/(\pi^2+12)$ discussed above. The energies for the $2s$ and $3s$ states drop rapidly towards zero at about $\alpha\approx 1.3$ and 1.5, respectively, while for the $2p$ states this behavior sets in at around $\alpha=1.8$.

The variational parameters p_0 and N for the wave functions (5.3) and (5.4) are plotted in Figs. 2 and 3, respectively for the $1s$ and $2p$ states. We note that the momentum scale parameter p_0, for each state, increases rapidly from the nonrelativistic result $\mu\alpha/n$, as α increases towards α_0. Both power parameters N decrease monotonically from the nonrelativistic values $2+l$ down to $1+l/2$ as $\alpha\to\alpha_0$. (Note that the wave functions are not normalizable for $N\leq 1+l/2$.) The variation of the power parameter as a function of α is almost independent of the mass ratio.

In Fig. 4 we compare the results for the equal mass, $M=m$, $1s$ state in both the pure Coulomb and Coulomb+transverse photon approximations. These results are the same for the system of two distinct particles, as well as for the particle-antiparticle system due to our choice of $\lambda_c=\pi\alpha$ and $\lambda_c'=4\pi\alpha$. The energies corresponding to the two cases begin to differ significantly beyond $\alpha=0.5$. The wave-function parameters, however, and, in particular, the power parameter N, begin to

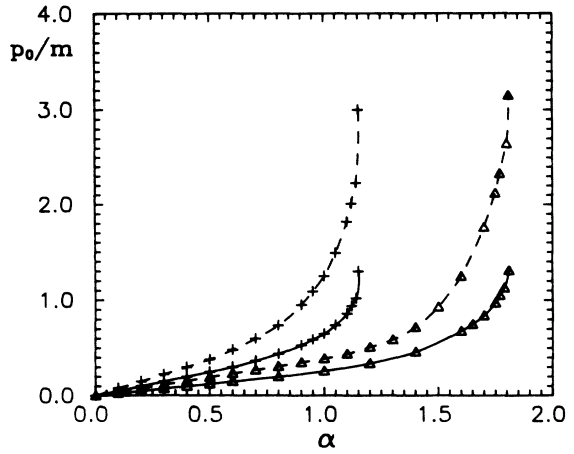

FIG. 2. Variational parameter p_0/m as a function of the coupling constant α for the $1s$-states (+) and for the $2p$ states (△) shown in Fig. 1. The $M=m$ results are connected with solid lines while the M_K/m_π values are shown as dashed lines.

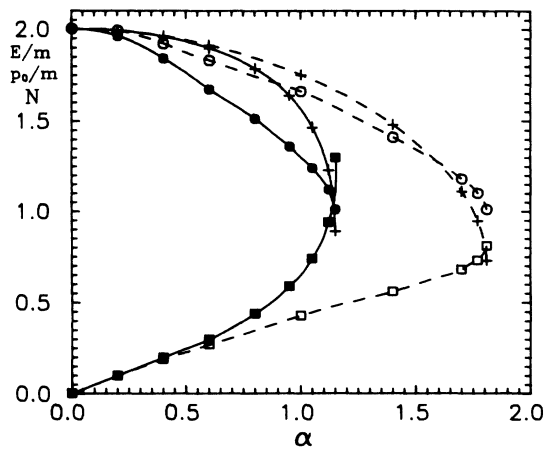

FIG. 4. Energy and variational wave-function parameters for the $1s$ state in the $M=m$ case. Solid line and solid symbols, Coulomb+transverse photon exchange; dashed line and open symbols, pure Coulomb case; crosses, E/m; circles, N; squares, p_0/m.

diverge at weaker coupling. This behavior might be expected, as first-order deviations in the wave function appear only in second order in the energy. Thus one should expect that at weak coupling, effects of transverse photon exchange will be more easily detected in quantities that depend sensitively on the wave function (e.g., decay widths).

Figure 5 shows similar results as Fig. 4, but for the $2p$ state of the particle-antiparticle system. The $2p$ state is less relativistic than the $1s$ state and differences between the pure Coulomb and Coulomb+transverse photon approximations set in at stronger coupling: the energies begin to deviate from each other beyond $\alpha=1$, but again, as in the $1s$ case, the power parameter N behaves differently in the two approximations for couplings as low as $\alpha=0.2$. Unlike for the other cases, the power parameter N does not decrease down to $1+l/2=\frac{3}{2}$ as $\alpha\to\alpha_0$ due to the extra annihilation contribution [see Eqs. (3.15) and (A4)].

Figures 4 and 5 show that the inclusion of transverse photon exchange reduces significantly the values of the coupling constant at which the energies turn towards zero. This is not unexpected, as the transverse photon exchange contribution is attractive and a similar effect has been observed in the case of bound pairs of fermions.[2] A note of warning is, however, in order: the pure Coulomb results are obtained in a variational approximation to the field theory without further approximations. The results including transverse photon exchange, on the other hand, were obtained by approximately decoupling the variational equations. This way of decoupling the equations suggests that the α_0 values derived from the resulting single integral equation may be too low and that a proper solution of the coupled equations (2.14) and (2.15) will lead to higher α_0 values.

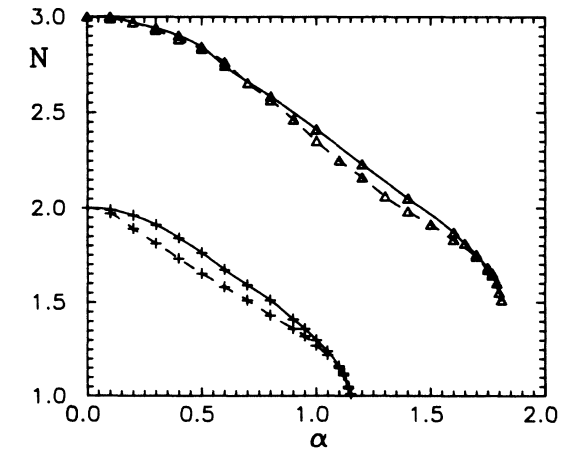

FIG. 3. Same as in Fig. 2 but for the power parameter N.

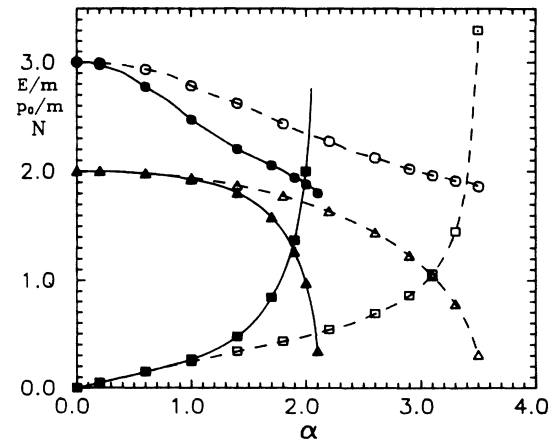

FIG. 5. Same as in Fig. 4 but for the $2p$ state of the particle-antiparticle system. The energy eigenvalues are marked by triangles.

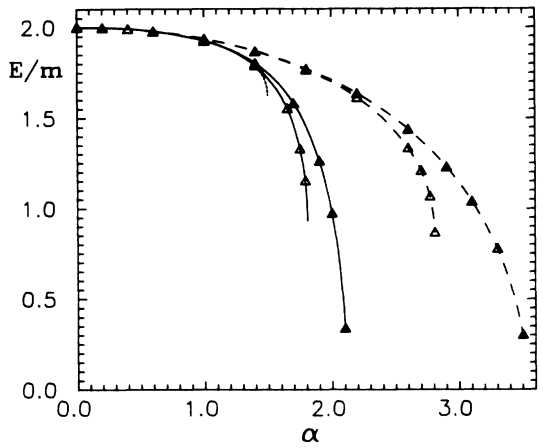

FIG. 6. Two-particle bound state energies E/m for the $2p$ states of a pair of distinct particles of equal mass (open triangles) and a pair of identical particles (solid triangles) in the pure Coulomb case (dashed lines) and the Coulomb+transverse photon case (solid lines). Dotted line, Eq. (5.6).

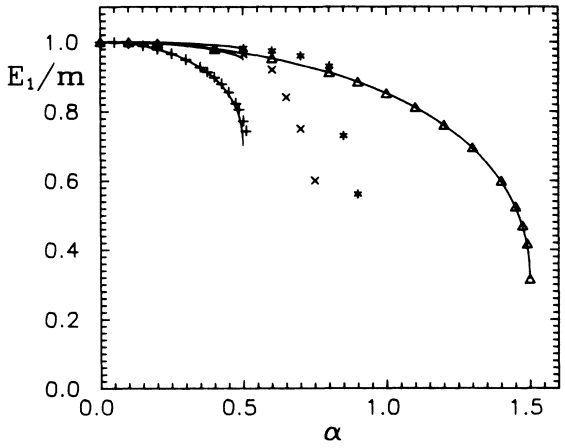

FIG. 7. Single-particle energy E_1/m ($E_1=E-M$) from the $M\to\infty$ limit. +, $1s$; ×, $2s$; ∗, $3s$; △, $2p$. Shown as solid lines are the Klein-Gordon-Coulomb results for $1s$, $2s$, $3s$, and $2p$.

In Fig. 6 we compare the $2p$-state energies for pairs of identical and distinct particles of equal mass and opposite charge for $K_T=0$ and $K_T\neq 0$. Also provided is a plot of an empirical formula given in the book of Itzykson and Zuber.[9] It can be seen that the energies for the cases of identical and distinct particles disagree in the strong relativistic limit and that the discrepancy is evident already in the pure Coulomb case. The formula provided by Itzykson and Zuber,

$$E^2=m^2+M^2+\frac{2mM}{(1+\alpha^2/\lambda_l^2)^{1/2}}\ , \tag{5.6}$$

where

$$\lambda_l=n-(l+\tfrac{1}{2})+[(l+\tfrac{1}{2})^2-\alpha^2]^{1/2}\ , \tag{5.7}$$

is based on the Klein-Gordon result, taking into account the relative motion of the particles. In analogy with the Klein-Gordon spectrum, the eigenvalues are real for a given angular momentum value only up to some maximum coupling constant. This maximum coupling constant is even smaller than the critical coupling α_0 for our $K_T\neq 0$ case. The situation is very similar in the case of s states. We note that, in contrast to the empirical formula (5.6), our critical coupling values α_0 are n-dependent.

We have also worked out the solutions of the integral equations in the limit $M\to\infty$. In this limit the transverse photon contribution disappears, as can be seen from Eq. (3.14). It turns out that our integral equation becomes identical, in this limit, to the approximately decoupled Klein-Gordon equation in momentum space [Eq. (2.58) in Ref. 6]. The variational solutions to the equation for the $1s$ and $2p$ states follow the Klein-Gordon results quite closely, as can be seen in Fig. 7. For $\alpha>0.5$, the Klein-Gordon energy E_{1s} becomes complex.[11] Our variational results indicate that at $\alpha\approx 0.515$ the variational power parameter N reaches the boundary value at which the wave function is no longer normalizable, while the energy value for the $1s$ state does not go to zero. The BSE results, however, do extend towards zero, but show a very slow convergence with respect to the number of basis states. It is very difficult to obtain numerically accurate results from either method in this regime.

For the $2p$ state our energies, in the $M\to\infty$ limit, agree closely with the Klein-Gordon results up to $\alpha=1.5$, which corresponds to a maximum value of the coupling constant for all p states in the Klein-Gordon equation. The situation is different for the radially excited s states. The Klein-Gordon eigenvalues become complex beyond $\alpha=0.5$ independently of n. For $n>2$ the imaginary part is initially small and the real part of the eigenvalue continues to fall in the vicinity of $\alpha=0.5$. By contrast, the BSE results for our integral equation show a distinct n dependence in the variation of the eigenvalues beyond $\alpha=0.5$.

VI. CONCLUSIONS

We have used a limited Fock-space *ansatz*, in a variational formulation of scalar QED, to derive wave equations for systems of two bound scalars of arbitrary mass. The coupled equations that are obtained by using a pure two-scalar and two-scalar+transverse photon Fock state ansatz were decoupled approximately resulting in a single integral equation. This wave equation is shown to contain all energy contributions up to order α^4, when compared to the standard perturbative approach, as happens also for the system of two bound fermions.[2,3]

In contrast to the two-fermion case, however, it is possible and indeed necessary[12,13] to include a scalar interaction term $[\lambda_c(\phi^*\phi)]^2$ (or $\lambda_c'\phi^*\phi\psi^*\psi$ for the system of two distinct scalars) in the Lagrangian. This then raises the question of choices for the coupling constants λ_c and λ_c'. Although for s states it is possible to choose them in such a way that the resulting kernels become the same for pairs of identical and distinct scalar particles, the choice of proper values for λ_c and λ_c' eventually will be dictated by experiment. In the present work we have made a

choice of λ_c (or λ_c'), which ensures that the s-state integral equation possesses an analytic solution in the massless limit. For states that carry angular momentum, this scalar contact interaction term does not contribute. There is, however, a difference in the p-state kernel for pairs of identical versus distinct scalars of the same mass, which is shown to be important in the strongly relativistic regime.

Numerical solutions were provided for the $1s$ and $2p$ states using a variational method (thereby giving insight into the form of the wave functions) and a basis-state expansion method for radial s-state excitations. It is shown that the inclusion of the transverse photon degrees of freedom reduces the values of the critical coupling constant α_0 near which the bound-state energy drops rapidly. Differences between the wave functions for the pure Coulomb and Coulomb+transverse photon cases show up even for intermediate coupling.

Future work should address the question of solving the coupled integral equations without approximately decoupling and of further improving the Fock-space *ansatz*. Future experimental work on bound-pion and -kaon spectroscopy will eventually provide a test of the present results.

ACKNOWLEDGMENTS

The financial support by the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

APPENDIX

Carrying out the angular integration in Eqs. (3.11) or (3.15) we obtain a radial equation of the following form:

$$f(p)(\omega_p+\Omega_p-E)=\frac{\alpha}{\pi}\int_0^\infty f(q)\frac{q}{p}K(p,q)dq\ , \quad \text{(A1)}$$

where the radial kernels $K(p,q)$ are different for states of different angular-momentum quantum number l.

The radial kernels for s and p states are given as follows:

(1) For s-states of two distinct scalars,

$$K(p,q)=\frac{pq}{2J_1(p,q)}\left[\frac{J_2(p,q)+2(p^2+q^2)}{2pq}\ln\left|\frac{p+q}{p-q}\right|-\left[\frac{\lambda_c'}{4\pi\alpha}+2\right]\right], \quad \text{(A2a)}$$

where

$$J_1(p,q)=(\Omega_p\Omega_q\omega_p+\omega_q)^{1/2} \quad \text{(A2b)}$$

and

$$J_2(p,q)=(\Omega_p+\Omega_q)(\omega_p+\omega_q)\ . \quad \text{(A2c)}$$

For the particle-antiparticle system, the s-state radial kernel takes the same form but with λ_c' replaced by $4\lambda_c$.

(2) For p states of two distinct scalars,

$$K(p,q)=\left[\frac{J_2(p,q)(p^2+q^2)+3(p^4+q^4)+2p^2q^2}{2pq}\times\ln\left|\frac{p+q}{p-q}\right|-J_2(p,q)-3(p^2+q^2)\right]\Big/4J_1(p,q) \quad \text{(A3)}$$

and for the particle-antiparticle system, the p-state kernel is given by (A3) plus the additional term

$$-\frac{1}{18}\frac{p^2q^2}{\Omega_p^2\Omega_q^2}\ . \quad \text{(A4)}$$

In addition, for the purely Coulombic case, i.e., setting $G=0$ in (2.12) or, equivalently, setting $K_T=0$ in (3.11) or (3.15), we have the following:

(1) For s states of two distinct scalars,

$$K(p,q)=\frac{pq}{2J_1(p,q)}\left[\frac{J_2(p,q)}{2pq}\ln\left|\frac{p+q}{p-q}\right|-\frac{\lambda_c'}{4\pi\alpha}\right], \quad \text{(A5)}$$

where $J_1(p,q)$ and $J_2(p,q)$ are the same as above, and the s-state kernel of the particle-antiparticle system is given by (A5) with λ_c' replaced by $4\lambda_c$.

(2) For p states of two distinct scalars,

$$K(p,q)=\frac{J_2(p,q)}{4J_1(p,q)}\left[\frac{p^2+q^2}{2pq}\ln\left|\frac{p+q}{p-q}\right|-1\right], \quad \text{(A6)}$$

and the p-state kernel of the particle-antiparticle system is given by (A6) plus (A4).

[1]J. W. Darewych and M. Horbatsch, J. Phys. B **22**, 973 (1989).
[2]J. W. Darewych and M. Horbatsch, J. Phys. B **23**, 337 (1990).
[3]W. Dykshoorn and R. Koniuk, Phys. Rev. A **41**, 64 (1990).
[4]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), p. 90.
[5]J. Finger, D. Horn, and J. E. Mandula, Phys. Rev. D **20**, 3253 (1979).
[6]H. Feshbach and F. Villars, Rev. Mod. Phys. **30**, 24 (1958).
[7]H. Bethe and E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Plenum, New York, 1977).
[8]B. Podolsky and L. Pauling, Phys. Rev. **34**, 109 (1929).
[9]C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
[10]G. Hardekopf and J. Sucher, Phys. Rev. A **30**, 703 (1984); **31**, 2020 (1985).
[11]W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer, Berlin, 1985).
[12]S. D. Joglekar and A. Misra, Phys. Rev. D **38**, 2546 (1988); **40**, 4111 (1989).
[13]F. Rohrlich, Phys. Rev. **80**, 666 (1950).