Intensity correlation functions for the colored gain-noise model of dye lasers

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Predictions for dye-laser intensity correlation functions of a stochastic model incorporating colored gain-noise fluctuations are discussed and compared with the predictions of the model incorporating loss-noise fluctuations. It is shown that differences between the two models are generally masked by the finite correlation time of the noise. Important differences are shown to exist well above threshold when the correlation time of the colored noise is of the same order or smaller than the cavity decay time. A new algorithm for the simulation of stochastic equations with colored noise is used and discussed.

I. INTRODUCTION

The anomalous statistical properties of dye lasers are well known.[1] They have been used as a testing bench of various theories devised to deal with stochastic dynamical systems. In particular, theories and approximations to study external colored noise have been tested in this physical system. It is by now well established[1–5] that pump fluctuations with a finite correlation time (external colored noise) are responsible for the observed statistical properties which include[1] the intensity stationary distribution and associated intensity fluctuations, transient phenomena during laser switch-on, and intensity correlation functions. Besides direct experimental evidence of colored pump noise,[2] there are two important effects that have been proposed as a clear signature of the presence of colored noise. One is the existence of a first-order-like transition in the most probable intensity value.[6,7] This effect has been shown to be also possible within a white-noise model for pump fluctuations.[8] Second is the prediction,[9] and experimental corroboration,[2–4] of a rounding-off of the intensity correlation function in its initial decay. We are hence concerned with the theoretical description of such intensity correlation functions.

The standard theoretical model used in the study of the intensity correlation function is one that can be formally obtained replacing the loss parameter (cavity decay rate) by a fluctuating quantity in the equation for the electric field. We will refer to it as loss-noise model. A variety of calculations and simulations exist for this model in the literature. These include the limit in which spontaneous emission noise is neglected, either in the white-noise limit[10] for the fluctuating loss parameter or considering a finite correlation time.[9,11] Several analyses incorporating spontaneous emission noise, again in the white-noise limit[12] or for colored noise[2–4,8,9,13] have also been reported. However, there is clear experimental evidence that dye-laser fluctuations originate in noise associated with the gain parameter.[2] A more sensible physical model is one incorporating fluctuations of the gain parameter.[8,14,15] A detailed study of this alternative model in the white-noise limit[8] indicates that there are important differences between the statistical properties predicted by a loss- and a gain-noise model. Most of the qualitative predictions associated with colored noise in the loss-noise model can be also recovered within a white gain-noise model, except the initial rounding-off of the intensity correlation function.[8] This remains as a clear signature of colored noise. Nevertheless, no analysis of the predictions of a nonwhite gain-noise model for the intensity correlation function has yet been reported. The need of a detailed study of this sort becomes more urgent given the intriguing fact that experimental results for such intensity correlation functions seem to be well described by the colored loss-noise model.[2–4] Our purpose in this paper is to present such study and to elucidate when and why the colored gain-noise and colored loss-noise models are expected to give the same or different results for the intensity correlation function.

Our analysis in this paper is mostly based (in addition to a linear analysis used as a first guide to the problem) on numerical simulations of stochastic equations with colored noise. Numerical simulations[16] have a long tradition in this problem. A number of papers have recently discussed the algorithms involved in these simulations.[17,18] We have profited this opportunity to revisit the algorithm in Ref. 16 in the light of new contributions. Our conclusions on this point and the new algorithm proposed are discussed in an appendix.

Our findings can be summarized by saying that the presence of a correlation time of the noise which is large compared with the cavity decay time[2,3] hides the differences that appear between the gain- and loss-noise models in the white-noise limit. Differences in the power spectrum of the intensity fluctuations at large frequencies are cut off by the correlation time of the noise. Envisaging situations with other values of the noise parameters, the differences between the two models become explicit. Specifically, for situations well above threshold and with typical values of the gain and loss parameters of the same order of magnitude, large differences between the correlation functions predicted for the gain- and loss-noise models occur when the correlation time of the noise becomes of the same order or smaller than the cavity decay time. This situation can happen, depending on parameter values, in cases in which a linear theory is reliable and in

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cases in which a nonlinear description is needed.

We discuss in Sec. II the definition of the two models considered and the derivation of closed equations for the laser intensity. Section III studies the consequences of a linearized analysis. Section IV is devoted to situations in which a nonlinear description is needed. Technical aspects of numerical simulations are discussed in the Appendix.

II. GAIN-NOISE MODEL

Starting from the semiclassical Maxwell-Bloch equations for on-resonance single-mode operation and after adiabatic elimination, in the good cavity limit, of polarization and population inversion, the laser equation for the slowly varying amplitude of the complex electric field $E=E_1+iE_2$ is[19]

$$\frac{dE}{dt}=-\kappa E+\Gamma\frac{E}{1+\beta|E|^2}+(D/2)^{1/2}q(t)\ , \tag{1}$$

where κ is the loss parameter, Γ the gain parameter, and β a positive parameter involving the matter-radiation coupling constant and the polarization and population inversion decay rates. We have added a complex noise source term $q(t)=q_R(t)+iq_I(t)$ which models spontaneous emission fluctuations of intensity D. It is taken as Gaussian white noise of zero mean and correlation

$$\langle q(t)q^*(t')\rangle=\langle q_R(t)q_R(t')\rangle+\langle q_I(t)q_I(t')\rangle=4\delta(t-t')\ . \tag{2}$$

Gain noise is introduced in this model by replacing the parameter Γ by $\Gamma+(Q/2)^{1/2}\xi(t)$ where $\xi(t)$ models gain fluctuations of intensity Q. It is taken as a complex Gaussian noise $\xi(t)=\xi_R(t)+i\xi_I(t)$ statistically independent of $q(t)$ and with zero mean and a correlation time τ:

$$\langle\xi(t)\xi^*(t')\rangle=\langle\xi_R(t)\xi_R(t')\rangle+\langle\xi_I(t)\xi_I(t')\rangle=2\tau^{-1}e^{-|t-t'|/\tau}\ . \tag{3}$$

The imaginary part of $\xi(t)$ models fluctuations due to detuning effects. The gain-noise model for the complex electric field is then

$$\frac{dE}{dt}=-\kappa E+\Gamma\frac{E}{1+\beta|E|^2}+(Q/2)^{1/2}\frac{E}{1+\beta|E|^2}\xi(t)+(D/2)^{1/2}q(t)\ . \tag{4}$$

This model incorporates colored (i.e., $\tau\neq0$) fluctuations of the gain parameter as required by experimental evidence.[2] A more standard model used in the literature[1-9] to describe dye-laser fluctuations is written as

$$\frac{dE}{dt}=(\Gamma-\kappa)E-\Gamma\beta|E|^2E+E(Q/2)^{1/2}\xi(t)+(D/2)^{1/2}q(t)\ . \tag{5}$$

We will refer to (5) as the loss-noise model since it can be naturally interpreted as obtained from (1) by replacing κ by $\kappa+(Q/2)^{1/2}\xi(t)$ and also performing a third-order expansion of the saturation nonlinear term.

Equations (4) and (5) are for the complex electric field. We are here interested in intensity correlation functions. It is then useful to have a closed equation for the intensity $I=|E|^2$. To obtain it, the change of variables to intensity I and phase φ has to be introduced in (4), leading to coupled equations for I and φ:

$$\frac{dI}{dt}=-2\kappa I+\frac{2\Gamma I}{1+\beta I}+\frac{2I}{1+\beta I}(Q/2)^{1/2}\xi_R(t)+(2DI)^{1/2}\cos\varphi\, q_R(t)+(2DI)^{1/2}\sin\varphi\, q_I(t)\ , \tag{6a}$$

$$\frac{d\varphi}{dt}=(1+\beta I)^{-1}(Q/2)^{1/2}\xi_I(t)-(D/2I)^{1/2}\sin\varphi\, q_R(t)+(D/2I)^{1/2}\cos\varphi\, q_I(t)\ . \tag{6b}$$

In the Markovian case (white-noise limit), it is well known[7,8,20] that a statistically equivalent set of equations in which I is decoupled from φ exists. The statistical equivalence between both sets of equations is demonstrated in the case of white noise by showing that they lead to the same Fokker-Planck equation for the joint probability $P(I,\varphi;t)$. In the presence of colored noise, no exact Fokker-Planck equation exists for $P(I,\varphi;t)$, so that the question of statistical equivalence is not so obvious. The approach of Ref. 15 considers an approximate effective Fokker-Planck equation, and uses it to demonstrate the validity of the closed equation for I even in the case $\tau\neq0$ up to this level of approximation. We now demonstrate that the proposed closed equation for I is in fact exact. This can be seen by introducing the four-dimensional Markovian process $\mathbf{x}=(I,\varphi,\xi_R,\xi_I)$ by completing the coupled equations for I and φ with the equations for the Ornstein-Uhlenbeck noise:

$$\frac{d\xi_i}{dt}=-\frac{1}{\tau}\xi_i+\frac{1}{\tau}\eta_i,\quad i=R,I \tag{7}$$

$$\langle\eta_i(t)\eta_j(t')\rangle=2\delta_{ij}\delta(t-t')\ . \tag{8}$$

The exact Fokker-Planck equation for the joint density $P(I,\varphi,\xi_R,\xi_I;t)$ of this four-dimensional Markovian process can then be written out explicitly:

$$\frac{\partial P}{\partial t}=-\sum_i\frac{\partial}{\partial x_i}(\bar{D}_iP)+\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(\bar{\bar{D}}_{ij}P)\ , \tag{9}$$

where the vector $\bar{D}$ and the matrix $\bar{\bar{D}}$ are given by

$$\bar{D}=\begin{pmatrix}-2\kappa I+\dfrac{2\Gamma I}{1+\beta I}+\dfrac{2I}{1+\beta I}(Q/2)^{1/2}\xi_R(t)+2D\\ \dfrac{1}{1+\beta I}(Q/2)^{1/2}\xi_I\\ -\dfrac{1}{\tau}\xi_R\\ -\dfrac{1}{\tau}\xi_I\end{pmatrix} \tag{10a}$$

and

$$\bar{\bar{D}}=\begin{bmatrix} 2DI & 0 & & & & \\ 0 & \frac{D}{2I} & & 0 & & \\ & & & \frac{1}{\tau^2} & 0 & \\ & 0 & & & & \\ & & & 0 & \frac{1}{\tau^2} & \end{bmatrix} . \tag{10b}$$

From this Fokker-Planck equation it can be shown that a set of stochastic equations statistically equivalent to the pair of Eqs. (6) exist and in which the intensity I is decoupled from the other variables. The equations for the intensity I and the phase φ are

$$\frac{dI}{dt}=-2\kappa I+\frac{2\Gamma I}{1+\beta I}+D+\frac{I}{1+\beta I}(2Q)^{1/2}\xi_R(t)+(2DI)^{1/2}q_R(t) , \tag{11a}$$

$$\frac{d\varphi}{dt}=\frac{1}{1+\beta I}\left[\frac{Q}{2}\right]^{1/2}\xi_I+\left[\frac{D}{2I}\right]^{1/2}q_I . \tag{11b}$$

Likewise, for the loss-noise model we obtain a closed equation for the intensity

$$\frac{dI}{dt}=2(\Gamma-\kappa)I-2\Gamma\beta I^2+D+(2Q)^{1/2}I\xi_R(t)+(2DI)^{1/2}q_R(t) . \tag{12}$$

We will here analyze the gain-noise model (11a) and compare it with the results of the loss-noise model (12). In our analysis we will use values of the physical parameters in the range experimentally determined for their ring dye laser by Roy and co-workers.[2,3,5] We fix the parameters $\kappa=10^7$ sec^{-1} and $\beta=5\times10^{-3}$ and vary Γ to study different operating points of the laser. The value of β is consistent with the parameters used in Ref. 3. The noise parameters are also fixed: $D=4\times10^{-3}$ sec^{-1}, $Q=10^4$ sec^{-1}, and $\tau=2\times10^{-5}$ sec. The values of Q and τ will be eventually changed to evidentiate differences between the two models which critically depend on the parameters of the external noise.

III. LINEAR ANALYSIS

A first study of (11a) can be done by linearizing the equation around the deterministic steady-state intensity $I_0=(\Gamma-\kappa)/\beta\kappa$. We consider here an additive linearization in which the multiplicative noise terms take values at $I=I_0$. Writing $I=I_0+\delta$ we have

$$\frac{d\delta}{dt}=-2\frac{\kappa}{\Gamma}(\Gamma-\kappa)\delta+D+(2Q)^{1/2}\frac{I_0}{1+\beta I_0}\xi_R(t)+(2DI_0)^{1/2}q_R(t) . \tag{13}$$

The normalized correlation function defined[21] by

$$\lambda(t)=\lim_{t'\to\infty}\frac{\langle I(t+t')I(t')\rangle-\langle I(t')\rangle^2}{\langle I(t')\rangle^2} \tag{14}$$

is obtained by straightforward integration of (13) and making use of (2) and (3). Introducing $\gamma=2(\Gamma-\kappa)\kappa/\Gamma$ we find for $\gamma\neq\tau^{-1}$,

$$\lambda(t)=\frac{1}{I_0^2}\left[\frac{2I_0D}{\gamma}-\frac{Q\gamma\tau^{-2}}{2\beta^2\kappa^2}(\gamma^2-\tau^{-2})^{-1}\right]e^{-\gamma|t|}+\frac{e^{-|t|/\tau}}{I_0^2}\frac{Q\gamma^2\tau^{-1}}{2\beta^2\kappa^2}(\gamma^2-\tau^{-2})^{-1} \tag{15}$$

and, if $\gamma=\tau^{-1}$,

$$\lambda(t)=\frac{e^{-\gamma|t|}}{I_0^2}\left[\frac{2I_0D}{\gamma}+\frac{Q\gamma}{4\beta^2\kappa^2}(1+\gamma|t|)\right] . \tag{16}$$

Here only the lowest order in D has been retained. Higher-order corrections in D do appear due to the fact that the linearization has been performed at the determistic value I_0, which differs from the mean $\langle I\rangle$ in terms which are of order D.

Of particular interest is the result for the normalized intensity fluctuations,

$$\lambda(0)=\frac{2D}{I_0\gamma}+\frac{Q\gamma\tau^{-1}}{2(\Gamma-\kappa)^2(\gamma+\tau^{-1})} . \tag{17}$$

The power spectrum $S(\omega)$ of the intensity fluctuations defined as the Fourier transform of $\lambda(t)$ becomes in this approximation

$$S(\omega)=\frac{4I_0^{-1}D}{\omega^2+\gamma^2}+\frac{Q}{(\Gamma-\kappa)^2}\frac{\gamma^2\tau^{-2}}{(\omega^2+\tau^{-2})(\omega^2+\gamma^2)} . \tag{18}$$

The same analysis can be done[2] for the loss-noise model (12) linearizing around its deterministic steady-state intensity $I_0'=(\Gamma-\kappa)/\beta\Gamma$. It reproduces the same results (15)–(18), but with γ replaced by $\gamma'=\gamma\Gamma/\kappa$. For the typical values of Q and D considered, the contributions from spontaneous emission noise in (15)–(18) are negligible except in situations extremely close to threshold $(\Gamma=\kappa)$.

The above results give a useful guide on the validity of the linear approximation itself and also to know when the differences between the two models will be apparent. Generally speaking the linearization is valid when fluctuations are small, that is $\lambda(0)\ll1$. According to (17) the linear approximation will be safe for $\Gamma-\kappa\gg Q$. It is important to note that it also follows from (17) that fluctuations become smaller with increasing τ, so that the presence of nonwhite noise gives a wider range of validity to the linear approximation. It is also clear from (17) and (18) that the differences between the gain- and loss-noise model will be important for $\gamma\ll\tau^{-1}$. In this case, the ratio of the gain- to the loss-noise model values of $\lambda(0)$ is $\gamma/\gamma'=\kappa/\Gamma$, which gives a measure of the expected differences between the two models. The inequality $\gamma\ll\tau^{-1}$ will occur, for typical values in which Γ and κ are of the same order of magnitude, when τ is of the same order or smaller than κ^{-1}. In the opposite limit $\gamma\gg\tau^{-1}$, $\lambda(0)$ becomes independent of γ and the spectrum is essentially coincident with the noise spectrum. This implies that differences will be observed when the dye laser acts as a low-pass filter for the fluctuations of the pump laser.[2] We will see that situations in which this happens can be envisaged well above threshold $(\Gamma\gg\kappa)$. The result for the linearized spectrum also indicates that differences between the two models will occur, independently of the value of τ, when γ and γ' are grossly different, that is for $\Gamma \gg \kappa$. These differences would be reflected in $\lambda(0)$ if γ is not much larger than τ^{-1}. However, depending on the value of τ, such differences in the spectrum might occur only for too large frequencies with no practical significance.

The above discussion has been corroborated and made explicit by direct numerical simulations of (11a) and (12) for a variety of parameter values. We have integrated numerically Eqs. (11a) and (12) using the method described in the Appendix. Numerical results have focused mainly on the calculation of the stationary normalized correlation function $\lambda(t)$ defined in Eq. (14). In order to ensure that we are computing stationary values we need to run the stochastic equations up to a transient time t_0. t_0 was chosen to be 10 times the characteristic decay time of the initial intensity correlation functions. $I(0)$ was always given the deterministic stationary value (which depends on the model). Results for $\lambda(t)$ were averaged over a large number of realizations (typically of the order of 10 000). The time step h necessary for convergence was $h=10^{-3}$ or $h=10^{-4}$ depending on the particular values of the parameters and the model.

As a reference case we consider a situation well above threshold ($\Gamma=2\times10^7$ sec^{-1}) with values for the other parameters consistent with those in Refs. 2 and 3 given at the end of Sec. II [they correspond to $\eta\equiv(\Gamma/\kappa)-1=1$, that is, operation 100% above threshold]. In this reference situation, $\gamma=10^7$ sec^{-1} and $\gamma\tau=200$. For this case, and in agreement with our discussion, the linear theory reproduces within a 0.5% the exact simulation result for $\lambda(0)$ and both the gain- and the loss-noise models give the same normalized correlation function. It is however interesting to notice that even in this case very important differences[22] occur for the unnormalized correlation functions $\langle I(t+t')I(t')\rangle$ of both models because of the very different value for $\langle I\rangle$ in each case. Still in the domain of validity of the linear approximation and for the same operating point of the laser, differences between the two models become explicit when considering the possibility of a smaller correlation time of the noise (Fig. 1): Going in the direction of the white noise we evidentiate the differences that are expected between the two models well above threshold. These differences still exist closer to threshold but become less pronounced as seen in Fig. 2 when changing the operating point to $\Gamma=1.01\times10^7$ sec^{-1} (that is 1% above threshold and still in the linear regime). For this operating point, even if $\gamma\tau\ll1$, differences are small because κ/Γ is close to one. The possibility of differences between the two models arising just from different values of γ and γ' can be discussed imagining a situation still further above threshold than our reference case, taking for example $\Gamma=10\kappa=10^8$ sec^{-1}. For the reference value of $\tau=2\times10^{-5}$ sec the normalized correlation functions are coincident for both models. Differences do appear in the noise spectrum $S(\omega)$ but for frequencies for which $S(\omega)$ is zero for all practical purposes. However, if the correlation time is arbitrarily changed to smaller values (i.e., $\tau=10^{-7}$ sec) to meet the requirement of not large values of $\gamma\tau$, differences in the

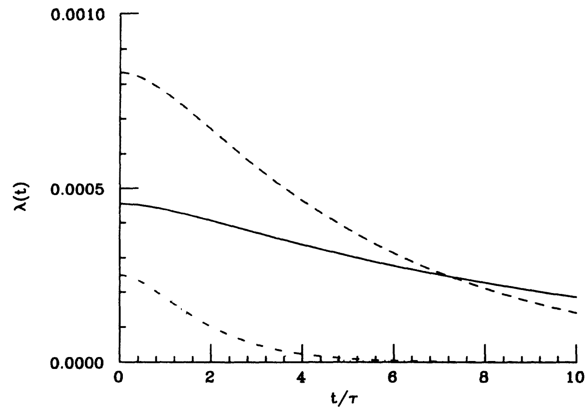

FIG. 1. Results for the normalized intensity-intensity correlation function $\lambda(t)$ from the linearized theory for the gain-noise model, Eq. (15), for $\tau=10^{-7}$ (dash-dotted line) and $\tau=10^{-8}$ (solid line), and from the corresponding linearized theory for the loss-noise model, Eq. (15) with γ substituted for $\gamma'=\gamma\Gamma/\kappa$, for $\tau=10^{-7}$ (dotted line), and $\tau=10^{-8}$ (dashed line). The rest of the parameters used in the equations are as follows: $\Gamma=2\times10^7$; $\kappa=10^7$; $Q=10^4$; $D=5\times10^{-3}$; $\beta=5\times10^{-3}$. Notice that going in the direction of white noise, differences between the two models become more apparent, as explained in the text.

normalized correlation functions become apparent again. The conclusion is that a large correlation time, as the one in our reference case, cuts off the differences in the spectrum that appear at large ω. By changing τ in the direction of white noise such differences are evidentiated.

The above linear analysis teaches us that having a rather large correlation time of the noise, compared with the inverse of the cavity decay rate, is a rather lucky situation in the sense that the validity of a linear analysis is larger than in situations with smaller τ and that differences between the loss- and gain-noise models that would appear for white noise at the same operating point

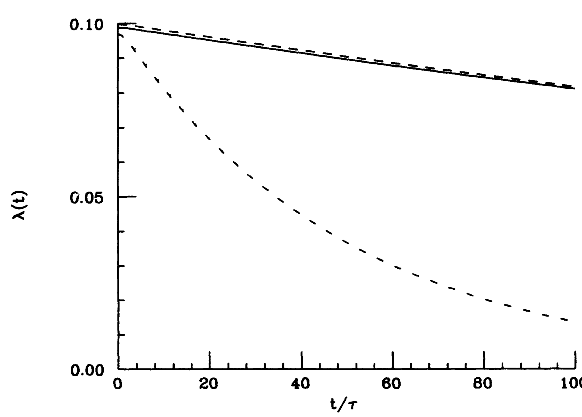

FIG. 2. As Fig. 1 but the value of Γ has been changed to 1.01×10^7, that is to a point where the laser operates 1% above threshold.

of the laser are hidden by the effect of τ. Envisaging possible experimental situations in which $\gamma\tau$ becomes of the order 1, practically κ^{-1} of the order of τ, differences between the gain- and loss-noise models are apparent in situations well above threshold and well described in a linear theory.

IV. CORRELATION FUNCTIONS IN THE NONLINEAR DOMAIN

The linear results are expected to be unreliable, at least quantitatively, when $\Gamma-\kappa$ is no longer much larger than Q. This can occur by simply changing the operating point of the laser and going closer to threshold or by envisaging a situation with a larger noise intensity and arbitrarily changing the value of Q. We have examined these possible situations by numerical simulations of (11a) and (12).

Starting from our reference case we move close to threshold setting $\Gamma=1.0001\times10^{7}$ $\sec^{-1}$, that is 0.01% above threshold. Results are shown in Fig. 3. For the reference value of $\tau=2\times10^{-5}$ sec the intensity correlation functions of the two models coincide and are grossly different from the one obtained in the linear approximation. Arbitrarily decreasing τ in the white-noise direction we find that for $\tau=10^{-7}$ sec the difference between the two models is still less than 0.1%. As shown in this figure, even for the smallest values of τ no significant differences are found. These findings are in agreement with the criteria discussed within the linear approximation: For the reference value of the noise parameters and for this operating point $\gamma\tau$ is already very small and it becomes smaller diminishing τ. However differences are not observed because Γ/κ is very close to one. In such operating point no significant differences occur even for white noise.

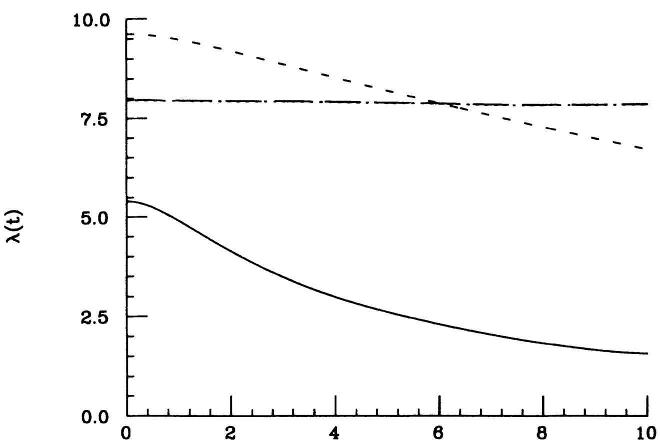

FIG. 3. Results for the normalized intensity-intensity correlation function $\lambda(t)$ from the results of the linearized theory for the gain-noise model, Eq. (15), for $\tau=2\times10^{-5}$ (dash-dotted line), compared with numerical simulations of the gain-noise model (solid line for $\tau=2\times10^{-5}$ and dotted line for $\tau=10^{-7}$) and the loss-noise models (short-dashed line for $\tau=2\times10^{-5}$ and long-dashed line for $\tau=10^{-7}$). Due to the fact that $\gamma'=\gamma\Gamma/\kappa\approx\gamma$, the linearized theory for the loss-noise model agrees almost exactly with the results of the linearized theory for the gain-noise model. The rest of the parameters are as in Fig. 1, except $\Gamma=1.0001\times10^{7}$. Note that the results for the numerical simulations for both models coincide for the same value of τ within the width of the lines drawn.

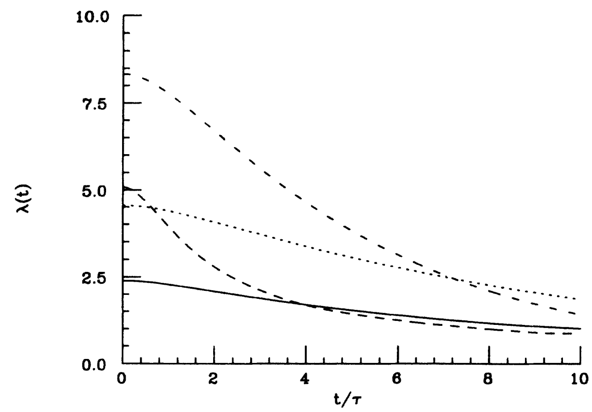

FIG. 4. Comparison of linearized theories and simulation results for the case $\tau=10^{-8}$, $\Gamma=2\times10^{7}$; $\kappa=10^{7}$; $Q=10^{8}$; $D=5\times10^{-3}$; $\beta=5\times10^{-3}$; as follows: linearized loss-noise model (dash-dotted line), linearized gain-noise model (dotted line), simulation of loss-noise model (short-dashed line), and simulation of gain-noise model (solid line).

Next we consider a possible situation of very strong external noise in which the linear theory is not reliable even well above threshold. Starting from our reference case we change arbitrarily the parameter Q to $Q=10^{8}$ $\sec^{-1}$. This value implies that $\Gamma-\kappa\ll Q$, but $\lambda(0)$ is still small due to the effect of setting $\tau=2\times10^{-5}$ sec, so that linear theory remains still valid. For these parameter values no differences exist between the two models. Linear theory breaks down if, in addition to taking $Q=10^{8}$ $\sec^{-1}$, we diminish τ in the direction of white noise. Significant differences between the two models occur for τ of the order of 10^{-7} sec and smaller (Fig. 4). These findings are again in agreement with the qualitative criteria discussed within the linear theory. Namely, when $\gamma\tau$ is of order 1 or smaller we observe differences between the two models because Γ/κ is not close to 1. These differences become more apparent moving to the white-noise limit and they are masked when τ becomes large. When Γ and κ are of the same order of magnitude, as they are here, differences between the two models occur for τ of the same order or smaller than the cavity decay time κ^{-1}. We remark that the situations considered in Fig. 4 show the possibility of differences between the two models at operating points well above threshold that, in addition, cannot be described quantitatively by a linearized theory.

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APPENDIX: NUMERICAL ALGORITHM

In this appendix, we develop a numerical scheme to solve a general stochastic differential equation of the form

$$\frac{dx(t)}{dt}=f(x)+\sigma(x)\xi(t)+g(x)q(t)\,,\quad x(0)=x_0\,, \tag{A1}$$

which is understood in the Stratonovich sense.[23] The random force $\xi(t)$ is a Gaussian-distributed Ornstein-Uhlenbeck process of zero mean and correlations given by

$$\langle \xi(t)\xi(t')\rangle=\frac{D}{\tau}e^{-|t-t'|/\tau} \tag{A2}$$

and $q(t)$ is a Gaussian-distributed white process of mean 0 and correlations

$$\langle q(t)q(t')\rangle=Q\delta(t-t')\,. \tag{A3}$$

The formal integration of Eq. (A1) between t and $t+h$ yields

$$x(t+h)=x(t)+\int_t^{t+h}f(x(s))ds+\int_t^{t+h}\sigma(x(s))\xi(s)ds+\int_t^{t+h}g(x(s))q(s)ds\,. \tag{A4}$$

We now expand the functions $f(x),\sigma(x),g(x)$ in a Taylor series around $x=x(t)$:

$$f(x(s))=f(x(t))+O(x(s)-x(t))\,, \tag{A5}$$

$$\sigma(x(s))=\sigma(x(t))+O(x(s)-x(t))\,, \tag{A6}$$

$$g(x(s))=g(x(t))+\left.\frac{dg}{dx}\right|_{x(t)}[x(s)-x(t)]+O([x(s)-x(t)]^2)\,. \tag{A7}$$

The orders of these expansions are enough to ensure that the error (taken in the square-mean quadratic sense) of the final formulas will be of the order $h^{3/2}$. Indeed, the three integrals appearing on the right-hand side of Eq. (A4) are, respectively, of orders h, h, and $h^{1/2}$. Using Eqs. (A4) and (A7) we have to the lowest order ($h^{1/2}$) the relation

$$x(s)-x(t)=g(x(t))\int_t^s q(u)du+O(h)\,. \tag{A8}$$

Substituting the expansions (A5), (A6), and (A7), into Eq. (A4) and making use of Eq. (A8) we obtain

$$x(t+h)=x(t)+f(x(t))h+\sigma(x(t))\int_t^{t+h}\xi(s)ds+g(x(t))\int_t^{t+h}q(s)ds+g(x(t))\left.\frac{dg}{dx}\right|_{x(t)}\int_t^{t+h}ds\int_t^s du\,q(u)q(s)+O(h^{3/2})\,. \tag{A9}$$

Since we have that

$$\int_t^{t+h}ds\int_t^s du\,q(u)q(s)=\tfrac{1}{2}\left[\int_t^{t+h}ds\,q(s)\right]^2\,, \tag{A10}$$

we get the recurrence relation

$$x(t+h)=x(t)+f(x(t))h+\sigma(x(t))\omega_h(t)+g(x(t))v_h(t)+\tfrac{1}{2}g(x(t))\left.\frac{dg}{dx}\right|_{x(t)}v_h(t)^2+O(h^{3/2})\,. \tag{A11}$$

Here we have introduced two stochastic processes $\omega_h(t)$ and $v_h(t)$ defined, respectively, by

$$\omega_h(t)=\int_t^{t+h}\xi(s)ds \tag{A12}$$

and

$$v_h(t)=\int_t^{t+h}q(s)ds\,. \tag{A13}$$

Both $\omega_h(t)$ and $v_h(t)$ are Gaussian distributed processes of mean zero. The correlations of $v_h(t)$ are readily computed as

$$\langle v_h(t_i)v_h(t_j)\rangle=Qh\delta_{ij}\,, \tag{A14}$$

where $t_i=ih, t_j=jh$ are the times that enter in the recurrence relation (A11). It is useful to redefine

$$v_h(t)=\sqrt{Qh}\,V(t)\,, \tag{A15}$$

so that $V(t)$ for different t represent independent Gaussian random variables of mean 0 and variance 1. On the other hand, $\omega_h(t)$ are correlated random variables for different times. More explicitly, we have

$$\langle \omega_h(t)^2\rangle=2D[h+\tau(p-1)] \tag{A16}$$

and for $i\neq j$

$$\langle \omega_h(t_i)\omega_h(t_j)\rangle=-\tau D\left[2-p-\frac{1}{p}\right]e^{-|t_j-t_i|/\tau}\,, \tag{A17}$$

where we have introduced $p=e^{-h/\tau}$. It is interesting to point out that to order h^3 we have

$$\langle \omega_h(t_i)\omega_h(t_j)\rangle = \frac{Dh^2}{\tau} e^{-|t_j-t_i|/\tau} + O(h^3) \tag{A18}$$

so that we can write

$$\omega_h(t) = h\xi_{\rm OU}(t) + O(h^{3/2}) \tag{A19}$$

with $\xi_{\rm OU}(t)$ a Ornstein-Uhlenbeck process characterized by (A2). This $\xi_{\rm OU}(t)$ could then be generated by the algorithm in Ref. 17, i.e., by the exact recursion relation

$$\xi_{\rm OU}(0) = \left[\frac{D}{\tau}\right]^{1/2} U(0) , \tag{A20}$$

$$\xi_{\rm OU}(t+h) = \xi_{\rm OU}(t)p + \left[\frac{D}{\tau}(1-p^2)\right]^{1/2} U(t) , \tag{A21}$$

where $U(t)$ are independent Gaussian random variables of mean 0 and variance 1. The method of Ref. 16 consists, essentially, in using (A19) above combined with an algorithm to generate $\xi_{\rm OU}(t+h)$ from $\xi_{\rm OU}(t)$ using the numerical solution of the differential equation satisfied by $\xi_{\rm OU}(t)$. As already pointed out in Ref. 16, this algorithm gives accurate results if a sufficiently small time step h is used.[17] We also note that on some occasions, such as the calculation of first-passage times,[24] a small time step is naturally demanded by the problem, and this algorithm is fully satisfactory. On the other hand, it seems that the possibility of exactly generating the process $\omega_h(t)$ would increase both the accuracy and the stability of the numerical solution by allowing the use of larger time steps when possible. Reference 18 has proposed a method to generate the process $\omega_h(t)$. Here, an alternative method will be derived. Let us introduce

$$W(t) = \int_0^t \xi(s)ds . \tag{A22}$$

In terms of $W(t)$, $\omega_h(t)$ is given by

$$\omega_h(t) = W(t+h) - W(t) . \tag{A23}$$

Given that the Ornstein-Unlenbeck process $\xi(t)$ satisfies a differential equation, namely,

$$\frac{d\xi(t)}{dt} = -\frac{1}{\tau}\xi(t) + \xi_W(t) , \tag{A24}$$

where $\xi_W(t)$ is a Gaussian white process of mean 0 and correlations

$$\langle \xi_W(t)\xi_W(t')\rangle = \frac{2D}{\tau^2}\delta(t-t') . \tag{A25}$$

$W(t)$ satisfies a second-order differential equation whose solution is

$$W(t) = \tau\xi_W(0)(1-e^{-t/\tau}) + \tau\int_0^t \xi_W(s)ds - \tau e^{-t/\tau}\int_0^t e^{s/\tau}\xi_W(s)ds ; \tag{A26}$$

here $\xi_W(0)$ is a Gaussian random variable of mean 0 and variance

$$\langle \xi_W(0)^2\rangle = \frac{D}{\tau} . \tag{A27}$$

By straightforward algebra, using (A23) and (A26) it is possible to get the following recurrence relation for $\omega_h(t)$:

$$\omega_h(0) = \tau(1-p)\xi_W(0) + f_1(0) + f_2(0) ,$$
$$\omega_h(t+h) = p\omega_h(t) - pf_1(t) + f_1(t+h) - f_2(t) + f_2(t+h) , \tag{A28}$$

where

$$f_1(t) = \tau\int_t^{t+h}\xi_W(s)ds \tag{A29}$$

and

$$f_2(t) = -\tau p e^{-t/\tau}\int_t^{t+h} e^{s/\tau}\xi_W(s)ds \tag{A30}$$

are two correlated Gaussian variables of zero mean and correlations (as before $t_i = ih, t_j = jh$):

$$\langle f_1(t_i)f_1(t_j)\rangle = 2Dh\delta_{ij} , \tag{A31}$$

$$\langle f_2(t_i)f_2(t_j)\rangle = D\tau(1-p^2)\delta_{ij} , \tag{A32}$$

$$\langle f_1(t_i)f_2(t_j)\rangle = -2D\tau(1-p)\delta_{ij} . \tag{A33}$$

The important point is that they are uncorrelated at different times $t_i \neq t_j$. $f_1(t), f_2(t)$ can now be easily generated by writing

$$f_1(t) = \alpha_1 U_1(t) , \tag{A34}$$

$$f_2(t) = \beta_1 U_1(t) + \beta_2 U_2(t) , \tag{A35}$$

where $U_1(t), U_2(t)$ are independent Gaussian variables of mean zero and variance unity. α_1, β_1, and β_2 are constants determined by the relations (A31), (A32), and (A33) as

$$\alpha_1 = \sqrt{2Dh} , \tag{A36}$$

$$\beta_1 = \left[\frac{2D}{h}\right]^{1/2} \tau(1-p) , \tag{A37}$$

$$\beta_2 = \left\{D\tau(1-p)\left[1-\frac{2\tau}{h} + \left[1+\frac{2\tau}{h}\right]p\right]\right\}^{1/2} . \tag{A38}$$

In summary, the numerical solution proceeds then using the recurrence relation given by (A11) with $v_h(t)$ given by Eq. (A15) and $\omega_h(t)$ by Eqs. (A28) and (A34)–(A38).

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