Q-function approach to a two-photon laser

Ning Lu*
Department of Chemistry, University of Rochester, Rochester, New York 14627
(Received 7 February 1990)

A quantum theory of a two-photon laser is developed by making use of the antinormal-ordering Q function. Starting from the microscopic atom-field interaction Hamiltonian for cascade three-level atoms and including saturable absorbers, we first present a master equation for the field-density operator, and then transform the master equation into a Fokker-Planck equation for the Q function. The Q-function approach enables us to study the two-photon laser analytically, obtaining simple expressions for nonlinear gain, mean photon number, frequency pulling, natural linewidth, and photon-number variance in a unified method. We find that the field in a two-photon laser will build up from a vacuum without triggering if its linear gain is larger than the cavity loss. Also hysteresis can occur in the two-photon laser even without triggering. With an overall two-photon resonance, the normalized photon-number variance approaches a common value $\frac{11}{8}$ well above "threshold," independent of the one-photon detuning. We compare these results with previous results obtained from an effective interaction Hamiltonian. We find that the domain of validity of the effective Hamiltonian for the photon-number variance is much smaller than that for the mean photon number. Well above "threshold" the effective Hamiltonian overestimates both the natural linewidth and the photon-number variance.

I. INTRODUCTION

The quantum theory of a (single-mode) two-photon laser was studied theoretically for many years,[1–15] but with few experimental realizations.[16–18] In recent years there has been new interest in this novel system. First, Brune *et al.* have studied a two-photon micromaser both theoretically[19,20] and experimentally.[21] In the experiment of Brune *et al.*[21] the cavity frequency is turned to half the frequency of the $40S \rightarrow 39S$ transition of rubidium Rydberg atoms, so that the two-photon emission rate is enhanced whereas the one-photon emission rate is suppressed. They observed two-photon oscillation in the microwave region. Second, Scully *et al.*[22] have shown that two-photon correlated-spontaneous-emission lasers (two-photon CEL's, which are coherently pumped, single-mode two-photon lasers) can generate bright light with phase squeezing.[14,23] By assuming large one-photon detunings, most of the theoretical investigations on the two-photon laser used an effective atom-field interaction Hamiltonian which has a single two-photon exchange term. The use of the effective Hamiltonian gives two-photon gain but no one-photon gain, and also does not predict any dynamic Stark shift. Brune *et al.*[19] derived an effective two-photon Rabi frequency starting from a microscopic atom-field interaction Hamiltonian that includes explicitly a middle level. By assuming large one-photon detunings (compared with one-photon Rabi frequencies), they first treated the middle level to first order in one-photon Rabi frequencies and then eliminated the middle level. The application of the effective Rabi frequency gives only two-photon gain (as that of the effective Hamiltonian does), but it does predict a dynamic Stark shift and thus a frequency pulling at two-photon resonance.[20,24] Zhu and Li[12] studied the photon-number distribution of the two-photon laser by using directly the microscopic Hamiltonian (i.e., no elimination of the middle level). They found both one- and two-photon gains, and argued the difficulty in defining a criterion for the threshold of the two-photon laser. The use of the microscopic Hamiltonian allows the one-photon detunings to vary from very large to very small, and represents a more realistic model compared with the use of the effective Hamiltonian as is evident in the experiment of Brune *et al.*[21] Very recently, Boone and Swain[15] have further studied the photon-number distribution by also using the microscopic Hamiltonian. However, none of the two papers gave any analytic expression for photon-number variance, which is an important quantity in photon statistics.

Comparing the current understanding of the two-photon laser from the microscopic Hamiltonian with that from the effective Hamiltonian, and with our understanding of a one-photon laser,[25] which is well known and has played an important role in laser physics, we still do not know something and need to answer some questions. One is the threshold and the buildup processes in the two-photon laser. Is a triggering required to start up the laser operation? The answer from the effective Hamiltonian is always yes.[14] Another is the general expression for the photon-number variance. Well above threshold, does the normalized photon-number variance approach unity (the value of the Poisson distribution) as in the one-photon laser? The effective Hamiltonian predicts[4,14] it to be $\frac{3}{2}$, and thus the answer is no. To answer these questions based on the results of the microscopic atom-field interaction Hamiltonian, we develop in this paper a quantum theory of a two-photon laser by using the Fokker-Planck equation for the antinormal-ordering Q function.[26,27] We include the effects of saturable absorbers[28,29] in our study. We find that, when the linear gain of the two-photon laser is larger than the cavity loss, the laser field will build up from a vacuum without triggering. Well above "threshold" the normalized photon-number variance is found to be $\frac{11}{8}$, independent of one-photon detuning (overall two-photon resonance is assumed). We also compare these results with those in Ref. 14 obtained by using the effective Hamiltonian. Recently, Lu and Zhu[23] have made such a comparison for the two-photon CEL's, and Boone and Swain[15] have done so for a single-mode two-photon laser. In this paper we find that the domain of validity of the effective Hamiltonian is much smaller than that for the mean photon number, and the effective Hamiltonian overestimates the natural linewidth well above "threshold," in disagreement with the conclusion of Ref. 15.

The layout of the paper is as follows. In Sec. II we present the master equation for the reduced field density operator. In Sec. III we transform the master equation into a Fokker-Planck equation for the Q function. In Sec. IV we discuss the operation of the laser and give general expressions for laser intensity and frequency pulling. In Sec. V we study the natural linewidth and the photon-number variance by using the Q function. In Sec. VI we compare in detail these results with those in Ref. 14 and discuss the domain of validity of the effective Hamiltonian. Finally, we summarize our results in Sec. VII.

II. MASTER EQUATION

We consider cascade three-level atoms interacting with a single mode of cavity field (see Fig. 1). The top level $|a\rangle$ and the bottom level $|c\rangle$ are of the same parity, which is opposite to that of the middle level $|b\rangle$. The energy of level $|A\rangle$ ($A=a,b,c$) is $\hbar\omega_A$. The active three-level atoms are *incoherently* pumped to the lasing levels $|a\rangle$, $|b\rangle$, and $|c\rangle$. The corresponding initial atomic populations are ρ_{aa}, ρ_{bb}, and ρ_{cc}, respectively. Population ρ_{cc} (also ρ_{bb} to the upper transition) plays the role of saturable absorbers.[28,29] To ensure the single-mode operation of the laser we assume that the cavity frequency has been turned close to half the frequency of the a-c transition but away from the frequencies of a-b and b-c transitions, so that single-photon emission channels are closed. We study such a two-photon laser by using the microscopic atom-field interaction Hamiltonian and the Scully-Lamb model for lasers.

The Scully-Lamb theory[25] of lasers can be reformulated, yielding two basic equations of motion.[30] One deals with the reduced field density operator ρ for the laser field in the interaction picture

$$\dot{\rho}=-i(\Omega-\nu)[a^{\dagger}a,\rho]-i\sum_j \Theta(t-t_j)\mathrm{Tr}_{A^j}[\tilde{V}_j,\tilde{\rho}_j^f] + \tfrac{1}{2}\gamma(2a\rho a^{\dagger}-a^{\dagger}a\rho-\rho a^{\dagger}a) . \tag{2.1}$$

The other treats the reduced density operator $\tilde{\rho}_j^f$ for the

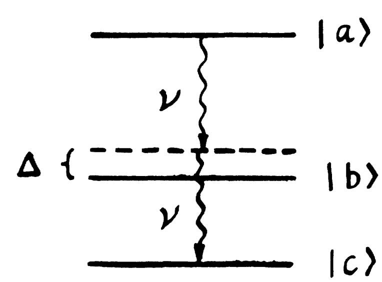

FIG. 1. Energy-level diagram for a single-mode two-photon laser.

jth atom and the field in the interaction picture

$$\dot{\tilde{\rho}}_j^f=-i\Theta(t-t_j)[\tilde{V}_j,\tilde{\rho}_j^f]-\Gamma\tilde{\rho}_j^f . \tag{2.2}$$

In Eqs. (2.1) and (2.2), Ω is the cavity-mode frequency, ν the actual laser frequency, a ($a^{\dagger}$) the field annihilation (creation) operator, γ the cavity-loss rate, t_j the injection time of the jth atom (assumed to obey a Poissonian distribution[31]), $\Theta(t-t_j)$ the unit step function $[\Theta(t-t_j)=1$ for $t\geq t_j$ and $\Theta(t-t_j)=0$ for $t<t_j]$, and Γ the decay rate of the lasing atoms (assumed to be the same for all three atomic levels). Also $\hbar\tilde{V}_j$ is the interaction Hamiltonian of the jth atom with the laser field in the interaction picture. Summation over atoms in Eq. (2.1) can be replaced by an integral over the injection time t_j,

$$\sum_j \Theta(t-t_j)\rightarrow r_a\int_{-\infty}^{t} dt_j ,$$

where r_a is the total pumping rate to all three levels.

For the two-photon laser (see Fig. 1) the microscopic interaction Hamiltonian $\hbar V_j$ in the Schrödinger picture is (under the rotating-wave approximation)

$$V_j=(g_1|a^j\rangle\langle b^j|+g_2|b^j\rangle\langle c^j|)a+\mathrm{H.c.} , \tag{2.3}$$

where g_1 and g_2 are the atom-field coupling constants for the a-b and b-c transitions, respectively, and are chosen to be real. For simplicity we consider the case of the actual two-photon resonance $\omega_{ac}=2\nu$ and the equal coupling constants $g_1=g_2\equiv g$, where $\omega_{AA'}=\omega_A-\omega_{A'}$. We denote one-photon detunings by

$$\Delta=\omega_{ab}-\nu=-(\omega_{bc}-\nu) . \tag{2.4}$$

In the good-cavity limit $\gamma\ll\Gamma$, where the laser field does not change appreciably on a time scale of atomic lifetimes, one can obtain the coarse-grained time rate of change for the field operator ρ from Eqs. (2.1)–(2.4). The master equation for the laser field has thus been derived in Ref. 23 for more general initial atomic conditions. For the *incoherently* pumped two-photon laser studied in this paper, the master equation is simplified to

$$
\begin{aligned}
\dot{\rho}_{nm} = {} & \alpha\rho_{aa}\{ -\tfrac{1}{2}\rho_{nm}[(n+2)(m+1)\mu_{nm}^{-1}+(n+1)(m+2)\epsilon_{nm}^{-1}+2(n+1)(m+1)\kappa_{nm}\sigma_{nm}^{-1}]+\rho_{n-1,m-1}\sqrt{nm}\;\xi_{n-1,m-1} \\
& +\tfrac{1}{2}\rho_{n-2,m-2}[n(n-1)m(m-1)]^{1/2}(\mu_{n-2,m-2}^{-1}+\epsilon_{n-2,m-2}^{-1}-2\kappa_{n-2,m-2}\sigma_{n-2,m-2}^{-1})\} \\
& +\alpha\rho_{bb}[\rho_{n+1,m+1}(n+1)^{1/2}(m+1)^{1/2}\xi_{nm}^{-1}-\rho_{nm}\kappa_{m-1,n-1}\xi_{n-1,m-1}^{-1}+\rho_{n-1,m-1}\sqrt{nm}\;\xi_{n-2,m-2}^{-1}] \\
& +\rho_{cc}\{\tfrac{1}{2}\rho_{n+2,m+2}[(n+1)(n+2)(m+1)(m+2)]^{1/2}(\mu_{nm}^{-1}+\epsilon_{nm}^{-1}-2\kappa_{nm}\sigma_{nm}^{-1}) \\
& \quad +\rho_{n+1,m+1}(n+1)^{1/2}(m+1)^{1/2}\xi_{n-1,m-1}^{-1} \\
& \quad -\tfrac{1}{2}\rho_{nm}[(n-1)m\mu_{n-2,m-2}^{-1}+n(m-1)\epsilon_{n-2,m-2}^{-1}+2nm\kappa_{n-2,m-2}\sigma_{n-2,m-2}^{-1}]\} \\
& -i(\Omega-\nu)(n-m)\rho_{nm}+\gamma(n+1)^{1/2}(m+1)^{1/2}\rho_{n+1,m+1}-\tfrac{1}{2}\gamma(n+m)\rho_{nm} \;, \qquad (2.5)
\end{aligned}
$$

with

$$\xi_{nm}=1+\delta^2+(n+m+3)\beta/\alpha+(n-m)^2\beta^2/4\alpha^2 \;, \qquad (2.6a)$$

$$\sigma_{nm}=(2n+3)(2m+3)\xi_{nm} \;, \qquad (2.6b)$$

$$\mu_{nm}=(2n+3)[1+i\delta+(2m+3)\beta/4\alpha] \;, \qquad (2.6c)$$

$$\epsilon_{nm}=(2m+3)[1-i\delta+(2n+3)\beta/4\alpha] \;, \qquad (2.6d)$$

$$\kappa_{nm}=n+m+3+(n-m)^2\beta/2\alpha+i\delta(n-m) \;, \qquad (2.6e)$$

where

$$\alpha=2r_a(g/\Gamma)^2, \quad \beta=8r_a(g/\Gamma)^4, \quad \delta=\Delta/\Gamma \qquad (2.7)$$

are a linear-gain coefficient, a saturation parameter, and a normalized one-photon detuning, respectively. In the absence of saturable absorbers, the master equation is a special case of Eq. (2.5) with $\rho_{bb}=\rho_{cc}=0$, in which case we can set $\rho_{aa}=1$.

The equations of motion for the diagonal elements $\rho_{nn}=p_n$ of the field density operator are readily obtained from Eqs. (2.5) and (2.6) by setting $m=n$,

$$
\begin{aligned}
\dot{p}_n = {} & T_{n-2}(\rho_{aa}p_{n-2}-\rho_{cc}p_n)-T_n(\rho_{aa}p_n-\rho_{cc}p_{n+2})+\rho_{aa}(A_{n-1}p_{n-1}-A_np_n+\tilde{A}_{n-1}p_{n-1}-\tilde{A}_{n-2}p_{n-2}) \\
& +\rho_{bb}(B_{n-2}p_{n-1}-B_{n-1}p_n+\tilde{B}_np_{n+1}-\tilde{B}_{n-1}p_n)+\rho_{cc}(C_{n-1}p_{n+1}-C_{n-2}p_n+\tilde{C}_{n-1}p_{n+1}-\tilde{C}_np_{n+2}) \\
& +\gamma(n+1)p_{n+1}-\gamma np_n \;, \qquad (2.8)
\end{aligned}
$$

where

$$T_n=\frac{\alpha(n+1)(n+2)[1+(2n+3)\beta/4\alpha]}{(2n+3)\{[1+(2n+3)\beta/4\alpha]^2+\delta^2\}} \;, \qquad (2.9a)$$

$$A_n=\frac{\alpha(n+1)^2}{(2n+3)\xi_{nn}} \;, \qquad (2.9b)$$

$$\tilde{A}_n=\tilde{C}_n=\frac{n+2}{n+1}A_n \;, \qquad (2.9c)$$

$$B_n=\alpha(n+2)/\xi_{nn} \;, \qquad (2.9d)$$

$$\tilde{B}_n=\alpha(n+1)/\xi_{nn} \;, \qquad (2.9e)$$

$$C_n=\frac{\alpha(n+2)^2}{(2n+3)\xi_{nn}} \;. \qquad (2.9f)$$

In Eq. (2.8) the terms containing T's represent direct two-photon processes in the flow of probability for finding n photons, those containing $\tilde{A}$'s and $\tilde{C}$'s represent indirect two-photon processes, and the rest terms give one-photon processes. In steady state we have $\dot{p}_n=0$. By adding and subtracting same terms $T_{n-1}(\rho_{aa}p_{n-1}-\rho_{cc}p_{n+1})$ in Eq. (2.8) we have detailed balance. For example, when $\rho_{bb}=\rho_{cc}=0$ and $\rho_{aa}=1$, detailed balance implies that

$$\gamma np_n=(T_{n-1}+A_{n-1})p_{n-1}+(T_{n-2}-\tilde{A}_{n-2})p_{n-2} \;. \qquad (2.10)$$

For small n such that $n\beta/\alpha \ll 1$, Eq. (2.10) becomes approximately

$$\gamma np_n=\alpha np_{n-1}/(1+\delta^2) \;, \qquad (2.11)$$

which is similar to the situation in a one-photon laser. However, in contrast to the one-photon laser,[25] the recurrence relation (2.10) does not admit any simple analytic solutions for, say, photon-number variance. Zhu and Li[12] and Boone and Swain[15] calculated numerically the photon-number distribution of the two-photon laser. To better understand the properties of the two-photon laser, however, analytic expressions for the photon-number variance, etc., are desired. In this paper we will use the approach of the Q function to find analytic expressions for mean photon number and photon-number variance (which determine the first two moments of the photon-number distribution), etc. The advantage of using the Q function is that it can determine the mean photon number, frequency pulling, natural linewidth, and photon-number variance in a unified approach.

III. FOKKER-PLANCK EQUATION FOR THE Q FUNCTION

In this section we convert the master equation (2.5) into a Fokker-Planck equation for the Q function, which is an antinormal-ordering quasiprobability function.[26,27] The Q function can be expressed in terms of the field density-matrix elements ρ_{nm}:

$$Q(\mathcal{E},\mathcal{E}^*)=\pi^{-1}\sum_{n,m=0}^{\infty} e^{-|\mathcal{E}|^2}\frac{(\mathcal{E}^*)^n\mathcal{E}^m}{\sqrt{n!m!}}\rho_{nm} . \tag{3.1}$$

The expectation value of an antinormally ordered function $F_{\rm anti}(a,a^\dagger)$ can be calculated by using the Q function,[26,27]

$$\langle F_{\rm anti}(a,a^\dagger)\rangle=\int F_{\rm anti}(\mathcal{E},\mathcal{E}^*)Q(\mathcal{E},\mathcal{E}^*)d^2\mathcal{E} . \tag{3.2}$$

The derivation of the Fokker-Planck equation for the Q function in this work is very similar to that in Ref. 14, and we will only outline the derivation here. Taking the time derivative on both sides of Eq. (3.1) and substituting the master equation (2.5) into Eq. (3.1) we obtain an equation of motion for the Q function. This equation contains derivatives with respect to $\mathcal{E}$ and $\mathcal{E}^*$ in both numerators and denominators of its various terms.[14,30,32] The Fokker-Planck equation for the Q function is obtained by expanding the equation in terms of the derivatives and keeping terms up to second order in the derivatives. Assuming that the average photon number of the two-photon laser is much larger than unity, we can safely neglect 1 compared to $|\mathcal{E}|^2$ in the Q's equation of motion. We find the Fokker-Planck equation for the Q function after some lengthy calculations:

$$\frac{\partial}{\partial t}Q(\mathcal{E},\mathcal{E}^*,t)=\left[-\frac{\partial}{\partial\mathcal{E}}d_{\mathcal{E}}+\frac{\partial^2}{\partial\mathcal{E}\partial\mathcal{E}^*}D_{\mathcal{E}\mathcal{E}^*}+\frac{\partial^2}{\partial\mathcal{E}^2}D_{\mathcal{E}\mathcal{E}}+\text{c.c.}\right]Q(\mathcal{E},\mathcal{E}^*,t) , \tag{3.3}$$

where the drift coefficient is

$$d_{\mathcal{E}}=\frac{\alpha\mathcal{E}}{2}\left[\frac{(\rho_{aa}-\rho_{cc})(1+|\mathcal{E}|^2\beta/2\alpha)}{(1+|\mathcal{E}|^2\beta/2\alpha)^2+\delta^2}+i\delta\frac{2\rho_{bb}-\rho_{aa}-\rho_{cc}}{1+\delta^2+2|\mathcal{E}|^2\beta/\alpha}\right]+[i(\nu-\Omega)-\tfrac{1}{2}\gamma]\mathcal{E} , \tag{3.4}$$

and the diffusion coefficients are

$$D_{\mathcal{E}\mathcal{E}^*}=\frac{\alpha}{4}\left[\frac{2\rho_{cc}(1+|\mathcal{E}|^2\beta/2\alpha)}{(1+|\mathcal{E}|^2\beta/2\alpha)^2+\delta^2}+\frac{|\mathcal{E}|^2\beta}{2\alpha}\frac{(\rho_{aa}-\rho_{cc})[(1+|\mathcal{E}|^2\beta/2\alpha)^2-\delta^2]}{[(1+|\mathcal{E}|^2\beta/2\alpha)^2+\delta^2]^2}+\frac{2\rho_{bb}+(\rho_{aa}+2\rho_{bb}+\rho_{cc})|\mathcal{E}|^2\beta/\alpha}{1+\delta^2+2|\mathcal{E}|^2\beta/\alpha}\right]+\frac{\gamma}{2} , \tag{3.5a}$$

$$D_{\mathcal{E}\mathcal{E}}=\frac{\alpha\mathcal{E}}{4\mathcal{E}^*}\left[\frac{(\rho_{aa}+\rho_{cc})(1+|\mathcal{E}|^2\beta/2\alpha)+i\delta(\rho_{aa}-\rho_{cc})}{(1+|\mathcal{E}|^2\beta/2\alpha)^2+\delta^2}-\frac{(1+i\delta)\rho_{aa}+(1-i\delta)\rho_{cc}+(\rho_{aa}+2\rho_{bb}+\rho_{cc})|\mathcal{E}|^2\beta/\alpha}{1+\delta^2+2|\mathcal{E}|^2\beta/\alpha}+\frac{(\rho_{aa}-\rho_{cc})|\mathcal{E}|^2\beta/2\alpha}{(1+i\delta+|\mathcal{E}|^2\beta/2\alpha)^2}-\frac{2i|\mathcal{E}|^2\beta\delta(\rho_{aa}-2\rho_{bb}+\rho_{cc})}{\alpha(1+\delta^2+2|\mathcal{E}|^2\beta/\alpha)^2}\right] . \tag{3.5b}$$

The cavity-loss rate γ enters into the diffusion coefficient $D_{\mathcal{E}\mathcal{E}^*}$ when the antinormal-ordering Q function is used. Without saturable absorbers we have $\rho_{aa}=1$ and $\rho_{bb}=\rho_{cc}=0$ in Eqs. (3.4) and (3.5).

In order to study the properties of the intensity and phase of the two-photon laser, we rewrite the Fokker-Planck equation (3.3) in terms of intensity and phase variables I and ϕ through the relation $\mathcal{E}=\sqrt{I}\,e^{i\phi}$,

$$\frac{\partial}{\partial t}Q_2(I,\phi,t)=\left[-\frac{\partial}{\partial I}d_I-\frac{\partial}{\partial\phi}d_\phi+\frac{\partial^2}{\partial I^2}D_{II}+\frac{\partial^2}{\partial\phi^2}D_{\phi\phi}+2\frac{\partial^2}{\partial I\partial\phi}D_{I\phi}\right]Q_2(I,\phi,t) , \tag{3.6}$$

where

$$d_I=2\sqrt{I}\ \text{Re}(d_{\mathcal{E}}e^{-i\phi}) , \tag{3.7a}$$

$$d_\phi=\text{Im}(d_{\mathcal{E}}e^{-i\phi})/\sqrt{I} \tag{3.7b}$$

are the intensity- and phase-drift coefficients, respectively, and

$$D_{II}=2I[D_{\mathcal{E}\mathcal{E}^*}+\text{Re}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi})] , \tag{3.8a}$$

$$D_{\phi\phi}=[D_{\mathcal{E}\mathcal{E}^*}-\text{Re}(D_{\mathcal{E}\mathcal{E}}e^{-2i\phi})]/2I , \tag{3.8b}$$

$$D_{I\phi}=\text{Im}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi}) \tag{3.8c}$$

are, respectively, the intensity-, phase-, and cross-diffusion coefficients for the Q function. In arriving at Eq. (3.7) we have neglected diffusion-induced drift terms[14] by assuming that the mean photon number is much larger than 1. The new Q function is related to the old one by

$$2Q_2(I,\phi,t)=Q(\mathcal{E},\mathcal{E}^*,t) , \tag{3.9a}$$

such that

$$\int Q_2(I,\phi,t)dI\,d\phi=\int Q(\mathcal{E},\mathcal{E}^*,t)d^2\mathcal{E}=1 . \tag{3.9b}$$

Substituting Eq. (3.4) into Eqs. (3.7) we obtain the drift coefficients for the intensity and phase, respectively,

$$d_I=(G-\gamma)I\ , \tag{3.10a}$$

$$d_\phi=\nu-\Omega-\frac{\alpha(\rho_{aa}-2\rho_{bb}+\rho_{cc})\mathrm{sgn}\delta}{2(|\delta|+|\delta|^{-1}+4N)}\ , \tag{3.10b}$$

where

$$G(N)=\frac{\alpha(\rho_{aa}-\rho_{cc})(N+|\delta|^{-1})}{|\delta|[1+(N+|\delta|^{-1})^2]} \tag{3.10c}$$

is the nonlinear gain of the two-photon laser, and $N=I\beta/2\alpha|\delta|$ represents a normalized photon number. One needs population inversion (i.e., $\rho_{aa}>\rho_{cc}$) to achieve positive gain. The role of the middle-level population ρ_{bb} (it is a loss for the upper transition but a gain for the lower transition) in the gain G has been canceled, since we have taken $g_1=g_2$. When $\rho_{bb}=0$, the linear gain $G(0)=\alpha/(1+\delta^2)$ equals the linear gain for an off-resonant one-photon laser. Substituting Eqs. (3.5) into Eqs. (3.8) we have the diffusion coefficients for the intensity and phase:

$$\begin{aligned}D_{II}=\gamma I&+\tfrac{1}{4}\alpha I|\delta|^{-1}(2\rho_{bb}-\rho_{aa}-\rho_{cc})(|\delta|+|\delta|^{-1}+4N)^{-1}\\&+\tfrac{1}{2}\alpha I|\delta|^{-1}\{(N+|\delta|^{-1})^2[\rho_{aa}(3N+|\delta|^{-1})+\rho_{cc}(N+3|\delta|^{-1})]+\rho_{aa}(|\delta|^{-1}-N)\\&+\rho_{cc}(3|\delta|^{-1}+5N)\}[1+(N+|\delta|^{-1})^2]^{-2}\ ,\end{aligned} \tag{3.11a}$$

$$D_{\phi\phi}=\frac{\gamma}{4I}+\frac{\alpha}{8I|\delta|}\left[\frac{(\rho_{cc}-\rho_{aa})(N+|\delta|^{-1})}{1+(N+|\delta|^{-1})^2}+\frac{(\rho_{aa}+2\rho_{bb}+\rho_{cc})(N+|\delta|^{-1})}{|\delta|+|\delta|^{-1}+4N}\right]\ , \tag{3.11b}$$

$$D_{I\phi}=0\ . \tag{3.11c}$$

IV. OPERATION, INTENSITY AND FREQUENCY PULLING

In this section we examine the operation of the two-photon laser. Since the Q function is an antinormal-ordering function, we find, by using Eq. (3.2), the expectation value of the photon-number operator $\hat{n}=a^\dagger a$,

$$\langle\hat{n}\rangle=\langle\,\vdots\hat{n}\vdots\,\rangle-1=\langle I\rangle-1\ , \tag{4.1}$$

and the photon-number variance

$$\begin{aligned}\langle(\Delta\hat{n})^2\rangle&=\langle\,\vdots(\Delta\hat{n})^2\vdots\,\rangle-\langle\,\vdots\hat{n}\vdots\,\rangle\\&=\langle(\delta I)^2\rangle-\langle I\rangle\ ,\end{aligned} \tag{4.2}$$

where $\vdots\ \vdots$ denotes antinormal ordering of a and $a^\dagger$, and $\delta I=I-\langle I\rangle$. Making use of the Fokker-Planck equation (3.6) we find the equations of motion for the intensity and phase of the field:

$$\frac{d}{dt}\langle I\rangle=\langle d_I\rangle\ , \tag{4.3a}$$

$$\frac{d}{dt}\langle\phi\rangle=\langle d_\phi\rangle\ , \tag{4.3b}$$

and that for the antinormally ordered photon-number variance:

$$\frac{d}{dt}\langle(\delta I)^2\rangle=2\langle d_I\delta I\rangle+2\langle D_{II}\rangle\ . \tag{4.4}$$

When the linear gain is larger than the cavity loss, $G(0)>\gamma$, we see from Eq. (4.3a) that the laser intensity $\langle I\rangle$ will start to increase; i.e., the laser field will build up from a vacuum via spontaneous emission. On the other hand, when $G(0)<\gamma$, the laser field cannot build up from the vacuum through spontaneous emission. Consequently, $G(0)=\gamma$ is a "triggering-free" threshold of the two-photon laser, as in the one-photon laser. This result can be regarded as a semiclassical result. Note that, by setting $n=1$ in Eq. (2.11), we have $p_1/p_0=G(0)/\gamma$. Thus, $G(0)=\gamma$ means $p_1=p_0$, and $G(0)>\gamma$ is equivalent to $p_1>p_0$. Davidovich *et al.*[20] also found the condition $p_1>p_0$ for the self-starting in a two-photon micromaser.

In the steady state we have $d/dt=0$. It follows from Eqs. (4.3) that the position n_0 of a peak (or a valley) of the steady-state Q function, which is the steady-state mean photon number $\langle\hat{n}\rangle_{ss}$ when there is only one peak, satisfies the following deterministic equations:

$$d_I(n_0)=0\ , \tag{4.5a}$$

$$d_\phi(n_0)=0\ . \tag{4.5b}$$

Equation (4.5a) determines the laser intensity. Substitution of Eq. (3.10a) into Eq. (4.5a) gives

$$G(N_0)=\gamma \tag{4.6}$$

and/or

$$N_0=0\ . \tag{4.7}$$

Physically, Eq. (4.6) means that the nonlinear gain equals the cavity loss in steady state. We plot the gain G as a function of the intensity N in Fig. 2. The gain G behaves differently depending on whether $|\delta|\le 1$ or $|\delta|>1$. In the former case, the gain G decreases to zero monotonically with increasing N, and there exists one (none) positive solution of N_0 to Eq. (4.6) if $G(0)>\gamma$ ($<\gamma$). These are similar to a one-photon laser. In the latter case, as N increases, G first increases to its maximum value

$$\begin{aligned}G_{\max}&=\alpha(\rho_{aa}-\rho_{cc})/2|\delta|\\&=G(0)(1+\delta^2)/2|\delta|\end{aligned} \tag{4.8}$$

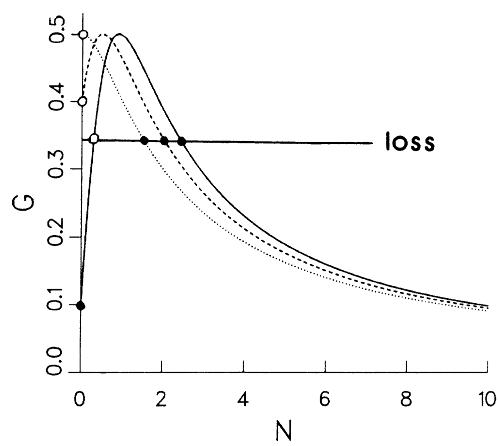

FIG. 2. Gain curve G [in units of $\alpha(\rho_{aa}-\rho_{cc})|\delta|^{-1}$] of the two-photon laser as a function of a normalized laser intensity $N=I\beta/2\alpha|\delta|$ for $|\delta|=10$ (solid line), $|\delta|=2$ (dashed line), and $|\delta|=1$ (dotted line). The closed (open) circles denote stable (unstable) solutions to Eq. (4.5a).

at $N=N_m=1-|\delta|^{-1}$ and then decreases to zero. Consequently, there exists (i) none, (ii) two, and (iii) one positive solution of N_0 to Eq. (4.6) for (i) $\gamma>G_{\max}$, (ii) $G(0)<\gamma<G_{\max}$, and (iii) $\gamma<G(0)$, respectively. To start laser operation one needs triggering in case (ii) since the linear gain $G(0)<\gamma$, whereas triggering is no longer needed in case (iii). Namely, $G_{\max}=\gamma$ is a "triggering-needed" threshold. Overall, it follows from Eqs. (4.6), (3.10c), and (4.8) that the normalized mean photon number N_0 are

$$N_0=G_{\max}\gamma^{-1}-|\delta|^{-1}\pm(G_{\max}^2\gamma^{-2}-1)^{1/2}\ , \tag{4.9}$$

provided $G_{\max}\geq\gamma$ when $|\delta|>1$, or $G(0)\geq\gamma$ when $|\delta|\leq 1$. The smaller N_0 (with minus sign), and even the larger N_0 (with plus sign) may be negative. The negative N_0 should be dropped. For example, in the case of one-photon resonance $\delta=0$ and without saturable absorbers, Eq. (4.9) gives

$$n_0=2\frac{\alpha}{\beta}\left[\frac{\alpha}{\gamma}-1\right]\ , \tag{4.10}$$

which is twice the mean photon number in a (resonant) one-photon laser with the same parameters α, β, and γ [cf. Eq. (5.9)]. Physically we may interpret this result by saying that each atom emits two photons in the resonant two-photon laser instead of one photon in the one-photon laser.

We now examine the stability of the solutions N_0's found above. The condition for a stable intensity n_0 (i.e., for a peak in the steady-state Q function) is

$$A_{II}\equiv\frac{\partial d_I(n_0)}{\partial I}=N_0\frac{\partial G(N_0)}{\partial N}<0\ , \tag{4.11}$$

where the second equality is obtained after using Eqs. (3.10a) and (4.6). Using Eq. (3.10c) we find the "locking strength"

$$A_{II}=N_0\frac{\alpha(\rho_{aa}-\rho_{cc})[1-(N_0+|\delta|^{-1})^2]}{|\delta|[1+(N_0+|\delta|^{-1})^2]^2}\ . \tag{4.12}$$

In the case of small detuning $|\delta|\leq 1$, the solution N_0 is always stable, again similar to the one-photon laser. In the case of large detuning $|\delta|>1$, a solution N_0 is stable only if

$$N_0>N_m=1-|\delta|^{-1}\ . \tag{4.13}$$

Physically we can simply determine the stability of a solution N_0 by looking at Fig. 2 and using inequality (4.11). For the case $G(0)>\gamma$ (with arbitrary $|\delta|$), in which there is only one positive N_0, the intensity N_0 is always stable, in agreement with the above discussions. For case (ii), i.e., $|\delta|>1$ and $G(0)<\gamma<G_{\max}$, the larger N_0 is stable whereas the smaller N_0 is unstable, in agreement with (4.13). The unstable N_0 corresponds to a valley in the Q function. Thus, we conclude that only the plus sign in Eq. (4.9) should be retained in any circumstance.

For the solution at origin, $N_0=0$, we find from Eqs. (3.10a) that

$$\frac{\partial d_I(0)}{\partial I}=G(0)-\gamma\ . \tag{4.14}$$

Consequently, the solution at the origin is stable (i.e., there exists a peak at $I=0$) only if the linear gain $G(0)<\gamma$, as expected physically. The peak at the origin is just a thermal field, arising from the below-threshold operation of the two-photon laser. For case (ii) discussed above, there exist two stable solutions n_0 to Eq. (4.5a) (see Fig. 2); namely, there exists bistability in the two-photon laser in the range of $2|\delta|<\alpha(\rho_{aa}-\rho_{cc})/\gamma<1+\delta^2$ when $|\delta|>1$.

Let us summarize here the behavior of the two-photon laser in the case of $|\delta>1$ and without triggering. When the linear gain $G(0)$ is less than the cavity loss γ, the laser field is a thermal field at $N_0=0$. As we continuously increase the pumping of the active atoms so that $\alpha(\rho_{aa}-\rho_{cc})/\gamma$ becomes slightly larger than $1+\delta^2$ [i.e., $G(0)$ is slightly larger than γ], there will be a discontinuous jump in the laser intensity from $N_0=0$ to $N_0=N_m(|\delta|+1)$, giving rise to a first-order phase transition. After this, the laser intensity will increase continuously when α/γ does so. At this time, if we decrease the pumping of the active atoms continuously, then the laser intensity will also decrease continuously until $N_0=N_m$, which corresponds to $\alpha(\rho_{aa}-\rho_{cc})/\gamma=2|\delta|$ or $G_{\max}=\gamma$. After this, the laser intensity will jump discontinuously from $N_0=N_m$ to $N_0=0$ as the pumping is further continuously reduced. In other words, we have hysteresis in the two-photon laser even without triggering (see Fig. 3). If the quantum fluctuations are included, then we expect that the discontinuous jumps will become less dramatic.

Having found N_0 we obtain the actual laser frequency from Eqs. (4.5b) and (3.10b),

$$\nu=\frac{\Gamma\Omega+\tilde{\gamma}\omega_{ab}}{\Gamma+\tilde{\gamma}}\ , \tag{4.15a}$$

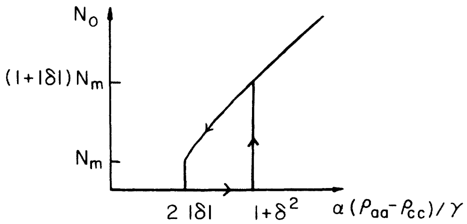

FIG. 3. Hysteresis in the two-photon laser. That $\alpha(\rho_{aa}-\rho_{cc})/\gamma=2|\delta|$ corresponds to $G_{\max}=\gamma$ and $\alpha(\rho_{aa}-\rho_{cc})/\gamma=1+\delta^2$ to $G(0)=\gamma$.

where

$$\tilde{\gamma}=\frac{\alpha(\rho_{aa}-2\rho_{bb}+\rho_{cc})}{2|\delta|(|\delta|+|\delta|^{-1}+4N_0)} \tag{4.15b}$$

plays the role of the cavity-loss rate γ for determining the actual laser frequency in a one-photon laser. Note that $\tilde{\gamma}$ depends on the laser intensity N_0, which in turn depends on the ratio $G_{\max}/\gamma$ and $|\delta|$. Equation (4.15a) shows that, even in the case of the actual two-photon resonance $\omega_{ac}=2\nu$, there still exists frequency pulling in the off-resonant case[20] $\delta\neq 0$. This frequency pulling originates from a dynamic Stark shift between levels $|a\rangle$ and $|c\rangle$ induced by the cavity field through the middle level $|b\rangle$. The direction of the mode pulling follows that of one-photon detuning for the upper transition $a-b$. When $\Omega=\omega_{ab}$, Eq. (4.15a) predicts no frequency pulling, $\nu=\Omega=\omega_{ab}=\omega_{bc}$. This is expected since it is just the resonant case $\delta=0$. In order to achieve the actual two-photon resonance for the usual off-resonant case, it is of practical importance to know how to adjust the cavity-mode frequency according to the atomic transition frequencies. The rule follows from Eq. (4.15a) as

$$2\Omega=\omega_{ac}+(\omega_{bc}-\omega_{ab})\tilde{\gamma}/\Gamma \ . \tag{4.16}$$

$2\Omega-\omega_{ac}$ has the same sign as $\omega_{bc}-\omega_{ab}$.

V. NATURAL LINEWIDTH AND PHOTON-NUMBER VARIANCE

In this section we investigate the quantum fluctuations in the two-photon laser peaked at $I=n_0\gg 1$. In steady state the diffusion coefficients take their values at $I=n_0$. Making use of Eq. (4.9) we find from Eqs. (3.11) the steady-state diffusion coefficients

$$D_{II}(n_0)=\frac{n_0}{2}\left[\frac{\gamma(3\rho_{aa}+\rho_{cc})}{\rho_{aa}-\rho_{cc}}+\frac{\alpha(2\rho_{bb}-\rho_{aa}-\rho_{cc})}{|\delta|(|\delta|+|\delta|^{-1}+4N_0)}\right]-n_0A_{II} \ , \tag{5.1a}$$

$$D_{\phi\phi}(n_0)=\frac{1}{8n_0}\left[\gamma+\frac{\alpha(\rho_{aa}+2\rho_{bb}+\rho_{cc})(|\delta|^{-1}+4N_0)}{|\delta|+|\delta|^{-1}+4N_0}\right] . \tag{5.1b}$$

It is important to note that $D_{\phi\phi}(n_0)$ is half the natural linewidth[33] of the two-photon laser. The first term $\gamma/8n_0$ of it comes from the vacuum fluctuations associated with the cavity loss, which can be reduced and even eliminated[34] if we can replace the ordinary vacuum by a squeezed vacuum (e.g., by shining a broadband squeezed vacuum into the laser cavity).[35] The second term of $D_{\phi\phi}(n_0)$ is due to the spontaneous emission. Comparing the effects of different levels on the natural linewidth, we see that the contribution from ρ_{bb} is twice (gain for the lower transition and loss for the upper transition) as much as that from ρ_{aa} (gain only) or from ρ_{cc} (loss only). Only in the resonant case $\delta=0$ will all the gain and loss mechanisms contribute on the equal position to the natural linewidth.

Since the drift coefficients (3.10) and the diffusion coefficients (3.11) are independent of the phase variable ϕ, the steady-state solution $Q_2(I)$ of the Fokker-Planck equation (3.6) must be ϕ independent too. Consequently, it satisfies the equation

$$\frac{\partial}{\partial I}\left[d_I-\frac{\partial}{\partial I}D_{II}\right]Q_2(I)=0 \ . \tag{5.2}$$

The detailed-balance solution of Eq. (5.2) is

$$Q_2(I)=\frac{C}{D_{II}(I)}\exp\left[\int_0^I\frac{d_I(x)}{D_{II}(x)}dx\right] , \tag{5.3}$$

where C is a normalization constant to be determined from Eq. (3.9b). Using Eqs. (3.10a), (3.10c), and (3.11a) we can obtain an expression for $Q_2(I)$. As an approximation, we expand d_I and D_{II} around $I=n_0$ up to first and zeroth order in $I-n_0$, respectively, and obtain

$$Q_2(I)=\left[\frac{|A_{II}|}{2\pi D_{II}(n_0)}\right]^{1/2}\exp\left[-\frac{|A_{II}|(I-n_0)^2}{2D_{II}(n_0)}\right] , \tag{5.4}$$

which, as a function of I, is a Gaussian distribution centered at $I=n_0$ with a "variance" $\langle(\delta I)^2\rangle=D_{II}(n_0)/|A_{II}|$.

Since $Q_2(I)$ is well peaked at $I=n_0$, we can expand d_I and D_{II} in Eqs. (4.4) around $I=n_0$ up to first order in δI, and set $d/dt=0$ (steady state). Substituting the resulting expression for $\langle(\delta I)^2\rangle$ into Eq. (4.2) and using Eqs. (5.1a), (4.12), (4.9), and (4.8) we arrive at

$$\langle(\Delta\hat{n})^2\rangle=\frac{D_{II}(n_0)}{|A_{II}|}-n_0$$
$$=n_0\frac{3\rho_{aa}+\rho_{cc}+(\rho_{aa}-\rho_{cc})(2\rho_{bb}-\rho_{aa}-\rho_{cc})\alpha\gamma^{-1}|\delta|^{-1}(|\delta|+|\delta|^{-1}+4N_0)^{-1}}{\rho_{aa}-\rho_{cc}-2\gamma|\delta|\alpha^{-1}(|\delta|^{-1}+N_0)^{-1}}\frac{|\delta|^{-1}+N_0}{2N_0} \ . \tag{5.5}$$

Alternatively we can obtain Eq. (5.5) by using directly the Q function in Eq. (5.4). In the absence of saturable absorbers (i.e., $\rho_{bb}=\rho_{cc}=0$, $\rho_{aa}=1$) we can express the normalized photon-number variance $\langle(\Delta\hat{n})^2\rangle/n_0$ in terms of N_0 and δ^2 only by eliminating α/γ:

$$\frac{\langle(\Delta\hat{n})^2\rangle}{n_0}=\frac{2(1+\delta^{-2})+N_0(13|\delta|^{-1}+3|\delta|+11N_0)}{2N_0(|\delta|+|\delta|^{-1}+4N_0)}\frac{(N_0+|\delta|^{-1})^2+1}{(N_0+|\delta|^{-1})^2-1} . \tag{5.6}$$

For the resonant case $\delta=0$, in which there is no frequency pulling, half the natural linewidth [Eq. (5.1b)] and the photon-number variance [Eq. (5.6)] reduce to

$$D_{\phi\phi}(n_0)=\frac{\alpha+\gamma}{8n_0} , \tag{5.7}$$

$$\langle(\Delta\hat{n})^2\rangle=\frac{11\alpha-9\gamma}{8\alpha-6\gamma}\left[\frac{\alpha n_0}{\alpha-\gamma}\right] . \tag{5.8}$$

In Fig. 4 we plot the normalized photon-number variance $\langle(\Delta\hat{n})^2\rangle/n_0$ as a function of the relative pumping level α/γ. One sees that $\langle(\Delta\hat{n})^2\rangle/n_0$ decreases monotonically from much larger than unity near threshold $\alpha\gtrsim\gamma$ to the value $\frac{11}{8}$ far above threshold $\alpha\gg\gamma$. It is interesting to compare these results with those of a (resonant) one-photon laser having the same linear gain (coefficient) α and the saturation parameter β. It is well known that the normalized mean photon number, half the natural linewidth, and the photon-number variance in such a one-photon laser are[25]

$$n_1=\frac{\alpha}{\beta}\left[\frac{\alpha}{\gamma_1}-1\right] , \tag{5.9}$$

$$D^1_{\phi\phi}(n_0)=\frac{\alpha+\gamma_1}{8n_1} , \tag{5.10}$$

$$\langle(\Delta\hat{n})^2\rangle_1=\frac{\alpha n_1}{\alpha-\gamma_1} , \tag{5.11}$$

respectively, where γ_1 is the cavity-loss rate for the one-photon laser. In the case of the equal cavity-loss rates $\gamma=\gamma_1$, (1) Eqs. (5.9) and (4.10) give $n_0=2n_1$, (2) consequently, Eqs. (5.10) and (5.7) lead to $D_{\phi\phi}(n_0)=\frac{1}{2}D^1_{\phi\phi}(n_1)$, i.e., the natural linewidth of the two-photon laser is only half that of the one-photon laser, and (3) it is easy to show from Eqs. (5.8) and (5.11) that the normalized photon-number variances obey the inequality $\langle(\Delta\hat{n})^2\rangle/n_0>\langle(\Delta\hat{n})^2\rangle_1/n_1$ (see Fig. 4), which implies that $\langle(\Delta\hat{n})^2\rangle>\langle(\Delta\hat{n})^2\rangle_1$. In the case of the equal mean photon numbers $n_0=n_1$, which requires $\gamma_1=\gamma/(2-\gamma\alpha^{-1})<\gamma$, we find that $D_{\phi\phi}(n_0)>D^1_{\phi\phi}(n_1)$ and $\langle(\Delta\hat{n})^2\rangle>\langle(\Delta\hat{n})^2\rangle_1$. All these results show that the degree of the photon-number fluctuations in the resonant two-photon laser is always larger than that in the one-photon laser, but the natural linewidth in the two-photon laser can be either smaller or larger than that in the one-photon laser.

For the (one-photon) off-resonant case $\delta\neq0$, it is straightforward to show that the normalized photon-number variance decreases monotonically with increasing intensity N_0, and approaches the value $\frac{11}{8}$ again, which is independent of δ. In Fig. 5 we plot the normalized photon-number variance as a function of the ratio G_{max}/γ for $|\delta|\geq1$. Far above the "triggering-needed" threshold $G_{max}\gg\gamma$, Eq. (5.6) reduces to

$$\frac{\langle(\Delta\hat{n})^2\rangle}{n_0}=\frac{3|\delta|+11N_0}{2|\delta|+8N_0} , \tag{5.12}$$

whose value is between $\frac{11}{8}$ and $\frac{3}{2}$.

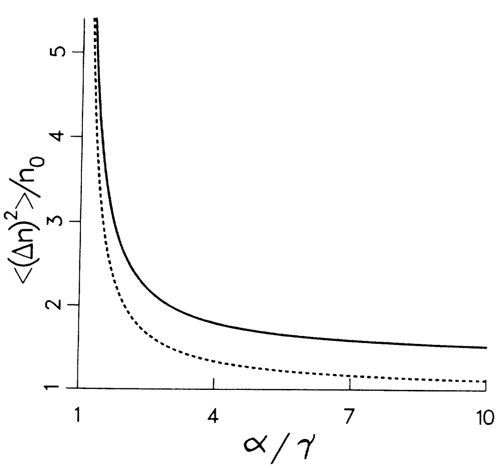

FIG. 4. Normalized photon-number variance $\langle(\Delta\hat{n})^2\rangle/n_0$ as a function of the relative pumping level α/γ for the resonant case $\delta=0$ [Eq. (5.8)]. The dashed curve is that for a (resonant) one-photon laser with the same parameters [Eq. (5.11)].

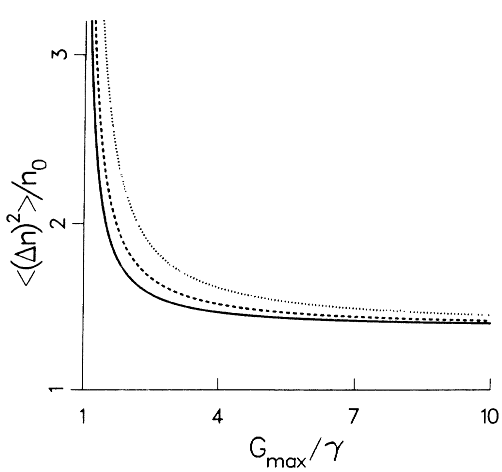

FIG. 5. Normalized photon-number variance as a function of the ratio G_{max}/γ for the initial atomic condition $\rho_{aa}=1$, $\rho_{bb}=\rho_{cc}=0$. Solid line, $|\delta|=10$; dashed line, $|\delta|=2$; and dotted line, $|\delta|=1$.

VI. COMPARISON WITH THE EFFECTIVE HAMILTONIAN MODEL

In preceding sections we have analyzed the two-photon laser with an arbitrary one-photon detuning δ, starting from the microscopic atom-field interaction Hamiltonian V_j in Eq. (2.3). On the other hand, the two-level–two-photon laser has also been studied in Ref. 14, starting from an effective atom-field interaction Hamiltonian for a two-photon transition, which is usually regarded to be valid in the off-resonant case with large one-photon detunings. In this section we compare the predictions for the two-photon laser obtained in this paper with those obtained in Ref. 14, and discuss the domain of validity of the effective interaction Hamiltonian. The comparison is made in the case of the actual two-photon resonance $\omega_{ac}=2\nu$ ($\Delta=0$ in Ref. 14) and the equal coupling constants $g_1=g_2\equiv g$. Also we have to set $\rho_{bb}=0$ in the following comparison.

In order to distinguish notations and equation numbers we denote the notations and equation numbers of Ref. 14 by primes. The effective atom-field interaction Hamiltonian $\hbar V_j$ used in Ref. 14 becomes

$$V_j'=g'|a^j\rangle\langle c^j|a^2+\text{H.c.} , \tag{6.1}$$

whose counterpart is V_j in Eq. (2.3). Here g' is an effective atom-field coupling constant for the two-photon transition between levels $|a\rangle$ and $|c\rangle$. The linear gain coefficient α and the saturation parameter β in Eq. (2.19′) become $\alpha'=2r_a(g'/\Gamma)^2$ and $\beta'=8r_a(g'/\Gamma)^4$, respectively.

A. Master equation

The master equation obtained from the microscopic Hamiltonian V_j is Eq. (2.5), and that obtained from the effective Hamiltonian V_j' is Eq. (2.17′) (with $\bar{\rho}_{ac}=\bar{\rho}_{ca}=0$). The loss parts of the two master equations are identical. The gain part of Eq. (2.17′) consists of terms containing ρ_{nm}, $\rho_{n-2,m-2}$, and $\rho_{n+2,m+2}$ only, which represent the direct two-photon processes. Namely, the indirect two-photon processes and the one-photon processes are missing if we use the effective Hamiltonian. To see this point more clearly we should look at the equations of motion for the diagonal elements p_n: Eq. (2.8) for the microscopic Hamiltonian; and Eq. (2.20′) for the effective Hamiltonian, i.e.,

$$\dot{p}_n=T_{n-2}'(\rho_{aa}p_{n-2}-\rho_{cc}p_n)-T_n'(\rho_{aa}p_n-\rho_{cc}p_{n+2}) +\gamma(n+1)p_{n+1}-\gamma n p_n , \tag{6.2}$$

where

$$T_n'=\frac{\alpha'(n+1)(n+2)}{1+(n+1)(n+2)\beta'/\alpha'} . \tag{6.3}$$

To have the same direct two-photon–process terms, i.e., $T_n=T_n'$, we only need the condition

$$n\beta/2\alpha \gg 1 , \tag{6.4}$$

provided that we identify

$$g'=g^2/\Delta . \tag{6.5}$$

To have the direct two-photon process dominate over the one-photon and indirect two-photon processes, i.e., $T_n \gg A_n, \tilde{A}_n, C_n, \tilde{C}_n$, however, we need the conditions

$$1 \ll n\beta/2\alpha \ll \delta^2 . \tag{6.6}$$

Equation (2.8) reduces to Eq. (6.2) only under conditions (6.6). On the other hand, it would be inaccurate to say that the mean photon number n_0' obtained from the effective Hamiltonian is valid *only* under the conditions $1 \ll n_0'\beta/2\alpha \ll \delta^2$, as will be seen in Sec. VI B.

B. Operation and intensity

The operation and the mean photon number of the two-photon laser are determined by the nonlinear gains G in Eqs. (3.10c) and G' in (4.8a′) and the cavity loss γ. Two nonlinear gains obey a simple relation

$$G(N-|\delta|^{-1})=G'(N) , \tag{6.7}$$

where $N=I\beta/2\alpha|\delta|=I\sqrt{\beta'/\alpha'}$ [by Eq. (6.5)]. The two gains agree with each other when inequality (6.4) is satisfied ($I\approx n$). Since $G'(0)=0<\gamma$ always, however, the effective Hamiltonian incorrectly predicts that the solution at the origin [cf. Eq. (4.7)] is always stable and triggering is also always needed. Relation (6.7) implies that the normalized mean photon number $N_0'=n_0'\sqrt{\beta'/\alpha'}$ is

$$N_0'=N_0+|\delta|^{-1} . \tag{6.8}$$

Consequently, the effective Hamiltonian is valid for the mean photon number under the condition

$$N_0 \gg |\delta|^{-1} , \tag{6.9}$$

which corresponds to (6.4). When $|\delta|\geq 1$, $G_{\max}\gg\gamma$ is a sufficient condition but not a necessary one for satisfying (6.9). Also, the intensity locking strength A_{II} derived from the microscopic Hamiltonian reduces to that derived from the effective one under condition (6.9), as can be seen from Eqs. (4.11) and (4.11′).

C. Frequency pulling

Under no circumstance [including the region of (6.6)] will d_ϕ in Eq. (3.10b) reduce to $d_\phi'=\nu-\Omega$ in Eq. (4.8b′). One finds no frequency pulling (i.e., $\nu-\Omega=0$) from the effective Hamiltonian, which is different from the frequency-pulling prediction, $\nu-\Omega=\frac{1}{2}\omega_{ac}-\Omega=(\omega_{ab}-\omega_{bc})\tilde{\gamma}/2\Gamma$ [see Eq. (4.16)], from the microscopic Hamiltonian. Thus, the effective Hamiltonian fails to predict the frequency pulling $\nu-\Omega$ at the actual two-photon resonance $\omega_{ac}=2\nu$, as pointed out in Ref. 14. The physical reason for this is that the effective Hamiltonian neglects the dynamic Stark shift.

D. Natural linewidth

Half the natural linewidth obtained from the effective Hamiltonian was given in Eq. (4.13b′), which can be rewritten as

$$D'_{\phi\phi}(n'_0)=\frac{\gamma+\alpha(\rho_{aa}+\rho_{cc})2N'_0|\delta|^{-1}}{8n'_0} \quad (6.10)$$

by using Eq. (6.5). Comparing this with $D_{\phi\phi}(n_0)$ [Eq. (5.1b)] in the region of (6.9), where $n_0 \approx n'_0$, we notice that, while the contributions from the vacuum fluctuations (represented by γ) are the same, the contributions from the spontaneous emission are different. (1) Under the conditions

$$|\delta|^{-1} \ll N_0 \ll |\delta| \quad (6.11)$$

[which corresponds to (6.6)], the spontaneous-emission part of $D_{\phi\phi}(n_0)$ is twice that of $D'_{\phi\phi}(n'_0)$. To satisfy conditions (6.11) we must have $|\delta| \geq 10$. The effective Hamiltonian underestimates the natural linewidth in the region of (6.11). For example, when $\rho_{aa}=1$, $\rho_{cc}=0$, and $|\delta| \gg 1$ with $G_{\max}$ being slightly larger than γ, we have $N_0 \approx N'_0 = 1$ and, consequently, $D_{\phi\phi}(n_0) \approx 1.8 D'_{\phi\phi}(n'_0)$. (2) On the other hand, well above "threshold" such that

$$4N_0 \gg |\delta|, |\delta|^{-1} , \quad (6.12)$$

half the linewidth takes the form in the resonant case, $D_{\phi\phi}(n_0)=[\gamma+\alpha(\rho_{aa}+\rho_{cc})]/8n_0$, and (the spontaneous-emission part of) $D_{\phi\phi}(n_0)$ is a small fraction $|\delta|/2N_0$ of (that of) $D'_{\phi\phi}(n'_0)$. Thus, the effective Hamiltonian overestimates the natural linewidth well above "threshold" [entering the region of (6.12)]. This conclusion is somewhat in disagreement with the corresponding conclusion of Ref. 15. (3) Between the above two regions we find a small region $N_0 \approx |\delta|/4 \gg |\delta|^{-1}$ in which the two natural linewidths are approximately equal. Last, we notice that $D_{I\phi}(n_0) \neq 0$ in general, in contrast to $D'_{I\phi}=0$ in Eq. (4.13c′).

E. Photon-number variance

In the region of (6.11) the photon-number variance in Eq. (5.5) reduces to

$$\langle(\Delta\hat{n})^2\rangle = n_0 \frac{3\rho_{aa}+\rho_{cc}}{2(\rho_{aa}-\rho_{cc}-\gamma/\alpha' n_0)} , \quad (6.13)$$

which is exactly Eq. (4.17′). For the simple case of $\rho_{aa}=1$ and $\rho_{cc}=0$, both normalized photon-number variances are $\frac{3}{2}$ when $1 \ll N_0 \ll |\delta|$ [see also Eq. (5.12)]. When the laser intensity further increases to the region of (6.12), the normalized photon-number variance obtained from the microscopic Hamiltonian changes from $\frac{3}{2}$ to $\frac{11}{8}$ [see (5.12)], whereas that obtained from the effective one remains at $\frac{3}{2}$. This difference is completely due to the difference in the steady-state intensity diffusion coefficients (D_{II}), since the locking strengths (A_{II}) from both Hamiltonians are the same.

VII. CONCLUSION

We have developed a quantum theory of the two-photon laser by using the microscopic atom-field interaction Hamiltonian and the Q function. Including saturable absorbers and assuming an actual two-photon resonance $2\nu=\omega_{ac}$ (but allowing one-photon detuning δ), we first present the master equation for the reduced field density operator and then transform it into a Fokker-Planck equation for the antinormal-ordering Q function. The Fokker-Planck equation, which is represented by its drift and diffusion coefficients, takes a relatively simple form and makes an analytic study on the two-photon laser possible. We discover that, when $|\delta|>1$, there exists a "triggering-needed" threshold $G_{\max}=\gamma$ and a "triggering-free" threshold $G(0)=\gamma$ (i.e., the linear gain equals the cavity loss). If $G(0)>\gamma$, then the laser field will build up from a vacuum without triggering. We also stress the physical meaning of the stability of the laser intensity. We find the simple expressions for mean photon number, frequency pulling, natural linewidth, and photon-number variance, etc. Well above "threshold" satisfying (6.12), the normalized photon-number variance is $\frac{11}{8}$.

The results for the off-resonant case $\delta \neq 0$ have been compared in detail with those obtained from the effective Hamiltonian.[14] We find that the effective Hamiltonian V'_j fails to predict the possible self-buildup of the laser intensity and the frequency pulling in the two-photon laser. For the wide range of (6.9), the effective Hamiltonian is accurate for the mean photon number. For a smaller range (6.11), which is within the range of (6.9), the effective Hamiltonian is accurate for the photon-number variance but underestimate the natural linewidth. For another smaller range (6.12), which is also within the range of (6.9), the effective Hamiltonian overestimates both the natural linewidth and the photon-number variance.

---

*Present address: Program Development Corporation, 300 Hamilton Avenue, Suite 409, White Plains, NY 10601.

[1]K. J. McNeil and D. F. Walls, J. Phys. A **8**, 104 (1975); **8**, 111 (1975).
[2]N. Nayak and B. K. Mohanty, Phys. Rev. A **19**, 1204 (1979).
[3]A. R. Bulsara and W. C. Schieve, Phys. Rev. A **19**, 2046 (1979).
[4]M. S. Zubairy, Phys. Lett. **80A**, 225 (1980).
[5]M. Reid, K. J. McNeil, and D. F. Walls, Phys. Rev. A **24**, 2029 (1981).
[6]L. Sczaniecki, Opt. Acta **27**, 251 (1980); **29**, 69 (1982).
[7]L. A. Lugiato and G. Strini, Opt. Commun. **41**, 374 (1982).
[8]M. D. Reid and D. F. Walls, Phys. Rev. A **28**, 332 (1983).
[9]U. Herzog, Opt. Acta **30**, 639 (1983).
[10]Z. C. Wang and H. Haken, Z. Phys. B **55**, 361 (1984); **56**, 77 (1984); **56**, 83 (1984).
[11]P. Garcia Fernandez and L. Sainz De Los Terreros, Opt. Commun. **61**, 151 (1987).
[12]S. Y. Zhu and X. S. Li, Phys. Rev. A **36**, 3889 (1987).
[13]I. Ashraf and M. S. Zubairy, Opt. Commun. **71**, 92 (1989).
[14]N. Lu, F. X. Zhao, and J. Bergou, Phys. Rev. A **39**, 5189 (1989).
[15]A. W. Boone and S. Swain, Phys. Rev. A **41**, 343 (1990).
[16]B. Nikolaus, D. Z. Zhang, and P. E. Toschek, Phys. Rev. Lett. **47**, 171 (1981).

[17]G. Grynberg, E. Giacobino, and F. Biraben, Opt. Commun. **36**, 403 (1981).

[18]J. Y. Gao, W. W. Eidson, M. Squicciarini, and L. M. Narducci, J. Opt. Soc. Am. B **1**, 606 (1984).

[19]M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. A **35**, 154 (1987).

[20]L. Davidovich, J. M. Raimond, M. Brune, and S. Haroche, Phys. Rev. A **36**, 3771 (1987).

[21]M. Brune, J. M. Raimond, P. Goy, L. Davidovich, and S. Haroche, Phys. Rev. Lett. **59**, 1899 (1987).

[22]M. O. Scully, K. Wódkiewicz, M. S. Zubairy, J. Bergou, N. Lu, and J. Meyer ter Vehn, Phys. Rev. Lett. **60**, 1832 (1988).

[23]N. Lu and S. Y. Zhu, Phys. Rev. A **40**, 5735 (1989).

[24]The approach of Refs. 19 and 20 is equivalent to an improved effective two-photon model, which is often used in quantum optics involving two-photon transitions. The model is derived by assuming that a set of intermediate levels are sufficiently far from one-photon resonances; The intermediate levels are first treated to first order in one-photon Rabi frequencies and then eliminated from the model. This model includes the dynamic Stark shifts and is sometimes called the two-photon two-level model; see D. A. Holm and M. Sargent III, Phys. Rev. A **33**, 1073 (1986); P. Alsing and M. S. Zubairy, J. Opt. Soc. Am. B **4**, 177 (1987); P. Meystre and M. Sargent III, *Elements of Quantum Optics* (Springer, Berlin, 1990), Sec. 5-4 and references therein.

[25]M. O. Scully and W. E. Lamb, Jr., Phys. Rev. **159**, 208 (1967); M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974), Chap. 17.

[26]Y. Kano, J. Math. Phys. **6**, 1913 (1965).

[27]C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965).

[28]R. Roy, Opt. Commun. **30**, 90 (1979); Phys. Rev. A **20**, 2093 (1979).

[29]I. Ashraf and M. S. Zubairy, Phys. Rev. A **38**, 854 (1988).

[30]N. Lu and J. A. Bergou, Phys. Rev. A **40**, 237 (1989).

[31]For a Poissonian pumping of active atoms, the Scully-Lamb theory of laser contains the influence of pump noise. For discussions about the effect of pumping statistics (including Piossonian pumping and regular pumping) on photon statistics in a one-photon laser, see J. Bergou, L. Davidovich, M. Orszag, C. Benkert, M. Hillery, and M. O. Scully, Phys. Rev. A **40**, 5037 (1989); M. A. M. Marte and P. Zoller, *ibid.* **40**, 5774 (1989).

[32]F. Casagrande and L. A. Lugiato, Phys. Rev. A **14**, 778 (1976).

[33]M. Lax and W. H. Louisell, Phys. Rev. **185**, 568 (1969); W. H Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973), p. 489.

[34]N. Lu (unpublished).

[35]J. Gea-Banacloche, Phys. Rev. Lett. **59**, 543 (1987).