Quantum theory of optical phase correlations

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(Received 19 April 1990)

We extend the theory of the Hermitian optical phase operator to analyze the quantum phase properties of pairs of electromagnetic field modes. The operators representing the sum and difference of the two single-mode phases are simply the sum and difference of the two single-mode phase operators. The eigenvalue spectra of the sum and difference operators have widths of 4π, but phases differing by 2π are physically indistinguishable. This means that the phase sum and difference probability distributions must be cast into a 2π range. We obtain mod(2π) probability distributions for the phase sum and difference that unambiguously reveal the signatures of randomness, phase correlations, and phase locking. We use our approach to investigate the phase sum and difference properties for uncorrelated modes in random and partial phase states and the phase-locked properties of the two-mode squeezed vacuum states. We reveal the fundamental property of two-mode squeezed states that the phase sum is locked to the argument of the squeezing parameter. The variance of the phase sum depends dilogarithmically on $1+\tanh r$, where r is the magnitude of the squeezing parameter, vanishing in the large squeezing limit.

I. INTRODUCTION

The quantum nature of the phase of the electromagnetic field is an old[1] and much studied problem.[2] Recently, we introduced a new approach to the analysis of the quantum optical phase that utilizes the Hermitian optical phase operator.[3-5] This operator does not exist in the conventional harmonic-oscillator infinite Hilbert space, but rather in a state space Ψ of finite but arbitrarily large dimension $(s+1)$. The procedure involves calculating c numbers such as expectation values, variances, and other properties of the field as functions of s *before* allowing s to tend to infinity. Our approach avoids indeterminacies which are inherent in the conventional approach employing an infinite Hilbert space from the outset. We also have shown that this limiting procedure solves many of the problems associated with a quantum formulation of angle variables for rotating systems.[6]

The Hermitian optical phase operator has been applied to a calculate the phase properties of a number of single-mode field states. These include number states and mixed-thermal states,[4,5] coherent states,[4,5] and squeezed states.[7,8] Number-phase minimum uncertainty states[9] and minimum phase-noise states[10] have also been constructed. In addition, the formalism has been used to study the phenomenon of phase diffusion in optical amplifiers[11] and in an analysis of phase dynamics in some nonlinear optical systems.[12]

In this paper we extend our formalism to pairs of field modes and apply it to investigate both uncorrelated and correlated phase behavior. We demonstrate the correlated, phase-locked property of two-mode squeezed states.

II. HERMITIAN OPTICAL PHASE OPERATOR

We have described the properties of the Hermitian optical phase operator ϕ_θ in detail in Refs. 4 and 5. The operator ϕ_θ exists in an $(s+1)$-dimensional states space Ψ spanned by the $s+1$ number states and the $s+1$ orthornormal phase states:

$$|\theta_m\rangle=(s+1)^{-1/2}\sum_{n=0}^{s}\exp(in\theta_m)|n\rangle\ , \quad (2.1)$$

where the $s+1$ phase values θ_m are equally spaced between θ_0 and $\theta_0+2\pi$:

$$\theta_m=\theta_0+\frac{2\pi m}{s+1}\ , \quad (2.2)$$

and m takes integer values from 0 to s. We are free to choose any value for θ_0, giving an uncountable infinity of orthonormal phase-state bases. The Hermitian optical phase operator is defined as

$$\hat{\phi}_\theta\equiv\sum_{m=0}^{s}\theta_m|\theta_m\rangle\langle\theta_m|\ , \quad (2.3)$$

and consequently has phase states as eigenstates with the eigenvalues θ_m. These eigenvalues are restricted to a 2π range and therefore the phase operator is single valued. Different phase-state bases correspond to distinct phase operators with different 2π ranges of eigenvalues.

The restricted range of the phase eigenvalues means that we must be careful when interpreting the calculated moments of the phase operator. The results obtained will depend on the range of eigenvalues and therefore on θ_0.

For example, the expectation value of the phase in a (randomly phased) number or thermal state is $\theta_0+\pi$.[4] Further complications arise when interpreting the sum and difference of the phases associated with two modes.

It is not just the formulation of the phase operator that is important[13] but the limiting procedure is also a crucial part of our approach. We do not take the limit of the states or the operators as s tends to infinity as these are only defined in the state space Ψ. The limit as s tends to infinity is only taken after c-number expressions, such as the moments of operators, are obtained. These limits are well behaved for physical (that is, experimentally accessible) systems.

III. PHASE SUMS AND DIFFERENCES

It is natural to define the phase sum and difference operators for two modes (a and b) to be the sum and difference of the single-mode operators ($\phi_{\theta a}$ and $\phi_{\theta b}$).[5] However, the 4π eigenvalue ranges of these two-mode operators adds further subtlety to the interpretation of the phase probability distributions. If we wish to describe the sum or difference of two single-mode phases, each of which is determined relative to third (reference) phase, then it is natural that this sum or difference will be expressed in a 4π range. This is because each single-mode phase will be expressed in a 2π range. However, if we are interested in only the sum or difference of the two single-mode phases and not the individual phases, then it is more meaningful to restrict the sum or a difference to a single 2π range. This restriction to a 2π range makes it easier to interpret two-mode phase correlations. We will illustrate the use of these two ranges by some examples.

A. Uncorrelated fields of random phase

We begin by examining the simplest possible case of two uncorrelated modes each in a state of random phase (for example, any number or thermal state[4]). The expectation values of the sum and difference of the phase eigenvalues are simply the sum and difference of the individual mean values. Moreover, the variances of the sum and difference are the equal to the sum of individual variances:

$$\Delta(\phi_{\theta a}\pm\phi_{\theta b})^2=\frac{2\pi^2}{3}=\Delta\phi_{\theta a}^2+\Delta\phi_{\theta b}^2 \ . \tag{3.1}$$

This property follows directly from the uncorrelated nature of the two-mode state in question. The compound probability distribution for the sum of the phases is readily derived from the single-mode distributions as follows.

Although it is possible to work with single-mode state spaces of different dimensionality,[5] it is sufficient (and simpler) to let each of these spaces to have dimension $(s+1)$. Then the single-mode phase operators have eigenvalues

$$\theta_{ma}=\theta_{0a}+\frac{2\pi m_a}{s+1} \ , \tag{3.2a}$$

$$\theta_{mb}=\theta_{0b}+\frac{2\pi m_b}{s+1} \ , \tag{3.2b}$$

where m_a and m_b both range from 0 to s. The possible values of the phase eigenvalue sum are

$$\theta_M=\theta_{0a}+\theta_{0b}+\frac{2\pi M}{s+1} \ , \tag{3.3}$$

where M has integer values between 0 and $2s$ inclusive. The probability for finding the two-mode field with a phase sum θ_M is

$$P(\theta_M)=\begin{cases}\displaystyle\sum_{m_a=0}^{M} P(\theta_{ma})P(\theta_M-\theta_{ma}) & (M\le s)\\ \displaystyle\sum_{m_a=M-s}^{s} P(\theta_{ma})P(\theta_M-\theta_{ma}) & (M\ge s) \ .\end{cases} \tag{3.4}$$

The phase distributions for the individual modes are uniform and therefore all the $P(\theta_{ma})$ and $P(\theta_{mb})$ are equal to $1/(s+1)$. Thus we find that the phase eigenvalue sum probability distribution is

$$P(\theta_M)=\begin{cases}\dfrac{M+1}{(s+1)^2} & (M\le s)\\ \dfrac{2s-M+1}{(s+1)^2} & (M\ge s) \ ,\end{cases} \tag{3.5}$$

which gives a triangular probability distribution as shown in Fig. 1. This is analogous to the probability distribution for the sum of the numbers shown on two dice. The probability for scoring any number between 1 and 6 with a single die is uniformly $\frac{1}{6}$, but the probability for the total score has a triangular distribution: $P(2)=\frac{1}{36}$, $P(3)=\frac{2}{36},\ldots,P(7)=\frac{6}{36},\ldots P(12)=\frac{1}{36}$. The difference between the two single-mode phase eigenvalues $\phi_{\theta a}-\phi_{\theta b}$ will range from $\theta_{0a}-\theta_{0b}-2\pi$ to $\theta_{0a}-\theta_{0b}+2\pi$ and has a similar triangular probability distribution.

While it is perfectly legitimate to calculate the distributions of the phase sums and differences in these 4π ranges, there is a redundancy implicit in using 4π radians to express the result. For example, a phase difference of α radians should be physically indistinguishable from a

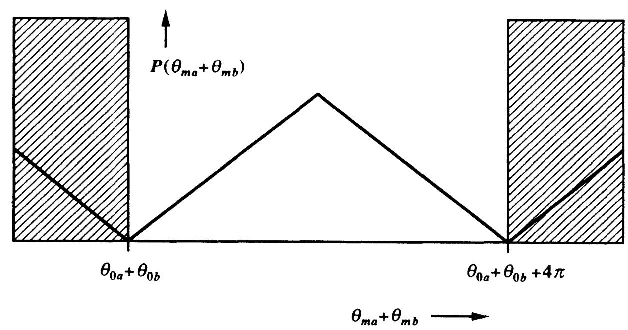

FIG. 1. Probability distribution for sum of the single-mode phase eigenvalues for two uncorrelated modes in states of random phase. The sum has values within a 4π range and the randomness of the phase sum is not immediately evident.

phase difference of $\alpha+2\pi$ radians, but both these values can occur in the same 4π range. It is therefore desirable to reduce the possible values of phase sums and differences to a 2π interval. We can achieve this by selecting a 2π interval within the 4π range and adding or subtracting 2π (as necessary) to or from values outside the selected interval in order to shift these values into the interval. As a result of this procedure we are left with a 2π range for the phase sum and difference probability distributions. Clearly, the moments of the phase sum and differences will, in general, be different when we use a 4π or a 2π range. While we can select any 2π subinterval, we specialize to a simple example by choosing the first half of the range. This reduced-range probability distribution is

$$P_{2\pi}(\theta_M)=P(\theta_M)+P(\theta_M+2\pi) \quad (M\le s)$$
$$=\frac{1}{s+1}\ , \tag{3.6}$$

which is a uniform distribution characteristic of a phase randomly distributed over a 2π interval. The reduced-range probability distribution is shown in Fig. 2 and can also be obtained from Fig. 1 by adding the second 2π section of Fig. 1 to the first. Naturally, the phase difference distribution will also be uniform when expressed mod(2π). This procedure also has an analogy in the two-dice system. The sum shown on the dice may be cast into the range 1–6 by representing the sum as mod(6).[14] This means that there are two ways of scoring 2, 3, 4, 5, and 6 and that the probability for registering any score between 1 and 6 is uniformly $\frac{1}{6}$.

We now have two methods for expressing the probability distribution of the phase sum or difference and we can calculate phase sum and difference moments for each distribution. Naturally, these will be different and some care is required when interpreting these moments. For the random phases discussed in this section we have already calculated the means and variances of the sum and difference for the 4π distribution. In the mod(2π) distribution for the phase sum, ranging from $\theta_{0a}+\theta_{0b}$ to $\theta_{0a}+\theta_{0b}+2\pi$, the mean and variance are

$$\langle \hat{\phi}_{\theta a}+\hat{\phi}_{\theta b}\rangle_{2\pi}=\theta_{0a}+\theta_{0b}+\pi\ , \tag{3.7a}$$

$$\Delta_{2\pi}(\phi_{\theta a}+\phi_{\theta b})^2=\frac{\pi^2}{3}\ , \tag{3.7b}$$

where the 2π subscript denotes the use of the mod(2π) distribution. We note that with this distribution the sum of the means of the phases is *not* the mean of the sum of the phases. This is a consequence of the periodicity of the phase variable and is not of quantum-mechanical origin. Two uncorrelated classical fields with random phases will admit precisely the same choice of 2π or 4π ranges in which to express the sum or difference probability distributions. Moreover the results from an analysis of such a classical system are readily shown to be identical to those obtained above for the quantum phases, when the large-s limit is finally taken.

The 4π and 2π distributions are both valid and useful. The former explicitly reveals the existence of correlations between the single-mode phases; in particular, if the variance in the phase sum or difference is not equal to the sum of the individual variances, then the phases are correlated. The latter is easier to interpret as a phase probability distribution because in it the phase sum or difference is a single-valued variable; for example, the sum and difference distributions for two uncorrelated fields in states of random phase are uniform. This is important because in the 4π distribution there is no unique shape signifying a randomness of the phase sum or difference. There are many distributions in the 4π range that become a uniform 2π distribution.

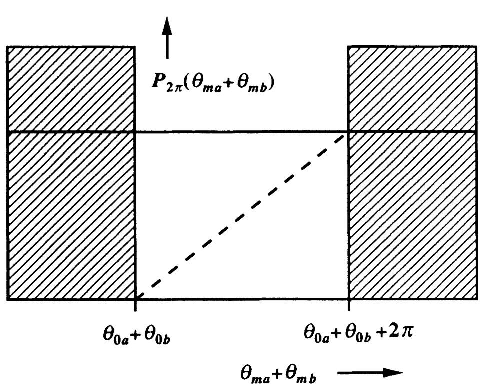

FIG. 2. Phase-sum probability distribution for two uncorrelated modes in states of random phase. This probability distribution for the sum of the phase eigenvalues has been cast into a 2π range of physically distinct values to form this distribution. The randomness of the phase sum is shown by the uniformity of the distribution.

B. Uncorrelated fields in partial phase states

Consider the two modes to be prepared in uncorrelated partial phase states:[5]

$$|a\rangle=\sum_{n=0}^{s} a_n \exp(in\beta_a)|n\rangle\ , \tag{3.8a}$$

$$|b\rangle=\sum_{n=0}^{s} b_n \exp(in\beta_b)|n\rangle\ , \tag{3.8b}$$

where a_n and b_n are real and positive. These states are important in the discussion of phase dependence and include coherent states, squeezed states, phase states, and number states as special cases. We have previously shown[5] that for a suitable choice of eigenvalue ranges with

$$\theta_{0a}=\beta_a-\frac{\pi s}{s+1} \tag{3.9}$$

(and a similar choice for mode b), the mean phase values are β_a and β_b. Moreover, the resulting phase probability distributions are symmetric about these mean values. For reasonably well-defined phases, the distributions have a pronounced central peak. When we find the phase eigenvalue sum distribution in the 4π range, we shall obtain another symmetric distribution with a central peak about a mean of $\beta_a+\beta_b$. The variance of the phase eigenvalue sum will be the sum of the individual single-mode variances, showing the lack of phase correlation between the modes. A corresponding result holds for the phase eigenvalue difference where the mean will be $\beta_a-\beta_b$.

The question now arises—what is the best way to cast the phase-sum distribution into a 2π range? Of course, all such choices are valid but the most convenient (and most easily interpreted) is that which reproduces the mean value as $\beta_a+\beta_b$. We realize this distribution by selecting the *central* 2π interval from the full 4π range. If the 4π distribution is sharply peaked (as, for example, when both field modes are prepared in intense coherent states[4]), then inserting the outside parts will have little effect on the moments of the phase sum. If instead we choose to cast the 4π distribution into the *first* 2π interval, then the original single peak is split into two parts, one at each end of the 2π interval. The mean values is shifted by π and the variance is markedly increased. Thus a poor choice of the 2π interval leads to the same interpretational problems encountered for a poor choice of θ_0 in the single-mode case.[4]

A simple, but important, example of a partial phase state is the coherent state. For suitable choices of the phase eigenvalue ranges, the expectation values of the phase sum and difference will simply be the sum and difference of the arguments the two coherent-state amplitudes. For intense coherent states, which have small phase variances,[4] the variances in both the sum and difference will be the sum of the two single-mode variances.

C. Phase locking and two-mode squeezing

Phase sum and differences are most interesting for states in which the two modes exhibit quantum phase correlations. We apply our two-mode phase formalism to analyze the phase sum and difference properties in the two-mode squeezed vacuum state.[15–18] The work of Reid and Drummond[19] suggests that these states might exhibit quantum phase correlations as a result of the correlations between the field quadratures.

The two-mode squeezed state is generalized from the two-mode vacuum by the action of a unitary Bogoliubov transformation and has the form[15–17]

$$|\zeta\rangle=\sum_n \frac{(\tanh r)^n}{\cosh r}\exp(in\xi)|n,n\rangle \;, \tag{3.10}$$

where the squeezing parameter is $\zeta=r\exp(i\xi)$. Expanding the number states in the single-mode phase state bases[4,5] gives

$$|\zeta\rangle=\frac{1}{(s+1)\cosh r}\times\sum_n\sum_{m_a}\sum_{m_b}\{\tanh r\exp[i(\xi-\theta_{ma}-\theta_{mb})]\}^n \times|\theta_{ma}\rangle|\theta_{mb}\rangle \;. \tag{3.11}$$

The compound probability $P(\theta_{ma},\theta_{mb})$ of finding the modes with phases θ_{ma} and θ_{mb} is obtained from the square modulus of the projection of (3.11) onto $|\theta_{ma}\rangle|\theta_{mb}\rangle$. Evaluating the geometric progression involved, we find for large s that

$$P(\theta_{ma},\theta_{mb})=\frac{1}{4\pi^2\cosh^2 r}\left[\frac{1}{1+\tanh^2 r-2\cos(\xi-\theta_{ma}-\theta_{mb})\tanh r}\right]\delta\theta_a\delta\theta_b \;, \tag{3.12}$$

where $\delta\theta_a$ and $\delta\theta_b$ are the spacings between successive phase eigenvalues for the two modes and have the value $2\pi/(s+1)$. In the limit as s tends to infinity we obtain from this the continuous probability density for the phases θ_a and θ_b:[20]

$$\mathcal{P}(\theta_a,\theta_b)=\frac{1}{4\pi^2\cosh^2 r}\left[\frac{1}{1+\tanh^2 r-2\cos(\xi-\theta_a-\theta_b)\tanh r}\right] . \tag{3.13}$$

This distribution is an explicit function of the phase sum, but not of the phase difference, and suggests that there will be a preferred value for the phase sum corresponding to the maximum of the cosine.

We can now calculate the single-mode, and phase-sum, and difference properties of the two-mode squeezed vacuum state. The phase probability densities for one of the modes is obtained by integrating the joint distribution with respect to the phase of the other mode:

$$\mathcal{P}(\theta_a)=\int_{\theta_{0b}}^{\theta_{0b}+2\pi}\mathcal{P}(\theta_a,\theta_b)d\theta_b \;. \tag{3.14}$$

The integrand is a periodic function of θ_b and therefore the integral will have the same value when evaluated over any 2π range. Reassigning the limits to be $\xi-\theta_a-\pi$ and $\xi-\theta_a+\pi$, we obtain eventually

$$\mathcal{P}(\theta_a)=\frac{1}{2\pi^2\cosh^2 r}\int_0^{\pi}\frac{1}{1+\tanh^2 r-2\cos\theta_b\tanh r}d\theta_b =\frac{1}{2\pi} \;. \tag{3.15}$$

This uniform probability density shows that the phase of mode a is random, which is consistent with the well-known thermal properties of single modes in a two-mode squeezed state.[16] Clearly, the b mode will also have random phase.

In order to investigate the phase sum and difference properties we introduce the new variables

$$\theta_+ \equiv \theta_a + \theta_b \ , \tag{3.16a}$$

$$\theta_- \equiv \theta_a - \theta_b \ . \tag{3.16b}$$

The Jacobean for this transformation is 2 and therefore the new probability density is

$$\mathcal{P}(\theta_+,\theta_-) = \frac{1}{8\pi^2\cosh^2 r} \times \left[\frac{1}{1+\tanh^2 r - 2\cos(\theta_+ - \xi)\tanh r} \right] . \tag{3.17}$$

The probability densities for the phase sum or difference are obtained by integrating with respect to θ_- or θ_+, respectively. Taking care with the appropriate limits of integration we find that the eigenvalue sum probability density, in the 4π range, is

$$\mathcal{P}(\theta_+) = \begin{cases} \displaystyle\int_{-\theta_+ + 2\theta_{0a}}^{\theta_+ - 2\theta_{0b}} \mathcal{P}(\theta_+,\theta_-) d\theta_- & (\theta_+ \le \theta_{0a} + \theta_{0b} + 2\pi) \\ \displaystyle\int_{\theta_+ - 2\theta_{0b} - 4\pi}^{-\theta_+ + 2\theta_{0a} + 4\pi} \mathcal{P}(\theta_+,\theta_-) d\theta_- & (\theta_+ \ge \theta_{0a} + \theta_{0b} + 2\pi) \ . \end{cases} \tag{3.18}$$

The integrals are straightforward and yield a single expression

$$\mathcal{P}(\theta_+) = \frac{1}{4\pi^2\cosh^2 r} \times \left[\frac{2\pi - |\theta_+ - \theta_{0a} - \theta_{0b} - 2\pi|}{1+\tanh^2 r - 2\cos(\theta_+ - \xi)\tanh r} \right] . \tag{3.19}$$

We are free to choose θ_{0a} and θ_{0b} to have any value and for simplicity we set

$$\theta_{0a} = \theta_{0b} = \frac{\xi}{2} - \pi \ , \tag{3.20}$$

giving a symmetrical distribution for the sum of the phase eigenvalues which is centered on a mean of ξ:

$$\mathcal{P}(\theta_+) = \frac{1}{4\pi^2\cosh^2 r} \times \left[\frac{2\pi - |\theta_+ - \xi|}{1+\tanh^2 r - 2\cos(\theta_+ - \xi)\tanh r} \right] . \tag{3.21}$$

We see that use of the 4π range has led to a distribution with a discontinuity of slope at $\theta_+ = \xi$. There is no physical significance in this discontinuity as we will demonstrate when we turn our attention to the corresponding mod(2π) distribution. This shows the problems that are associated with interpreting the 4π distribution. The phase eigenvalue difference probability density, in the 4π range, is

$$\mathcal{P}(\theta_-) = \begin{cases} \displaystyle\int_{-\theta_- + 2\theta_{0a}}^{\theta_- + 2\theta_{0b} + 4\pi} \mathcal{P}(\theta_+,\theta_0) d\theta_+ & (\theta_- \le \theta_{0a} - \theta_{0b}) \\ \displaystyle\int_{\theta_- + 2\theta_{0b}}^{-\theta_- + 2\theta_{0a} + 4\pi} \mathcal{P}(\theta_+,\theta_-) d\theta_+ & (\theta_- \ge \theta_{0a} - \theta_{0b}) \ . \end{cases} \tag{3.22}$$

The evaluation of these integrals is more complicated, but we find eventually the analytic result

$$\mathcal{P}(\theta_-) = \frac{1}{4\pi^2} \arctan \times \left[\exp(-2r)\tan\left[\frac{\theta_+ - \xi}{2}\right] \right] \Bigg|_{|\theta_-| - 2\pi + \xi}^{-|\theta_-| + 2\pi + \xi} , \tag{3.23}$$

where we have chosen θ_{0a} and θ_{0b} as in (3.20). Again we see the problems associated with using the 4π range. For the case $r = 0$, the two-mode squeezed state becomes the double vacuum and both of the phase eigenvalue sum and difference probability densities become

$$\mathcal{P}(\theta_+) = \frac{1}{4\pi^2}(2\pi - |\theta_+ - \xi|) \quad (\xi - 2\pi \le \theta_+ \le \xi + 2\pi) \tag{3.24}$$

$$\mathcal{P}(\theta_-) = \frac{1}{4\pi^2}(2\pi - |\theta_-|) \quad (-2\pi \le \theta_- \le 2\pi) \ . \tag{3.25}$$

These are precisely the triangular distributions discussed for uncorrelated fields with random phases in Sec. III A. However, as we have previously stated, a triangular distribution does not uniquely signify a random phase sum or difference. We can only determine the true nature of the phase sum and difference from the mod(2π) distribution.

We can obtain the mod(2π) probability densities for the phase sum and difference in two ways. We can either start with the 4π distributions derived above and shift the appropriate parts of this distribution into the chosen 2π interval, or we can return to $\mathcal{P}(\theta_+,\theta_-)$ and shift appropriate parts of this joint distribution into the chosen ranges prior to integration. These methods produce identical distributions, but the former is more complicated and is carried out in the Appendix. From (3.18) and (3.22) the ranges of values that θ_+ and θ_- can take for the joint distribution are

$$|\theta_-| + \xi - 2\pi \le \theta_+ \le -|\theta_-| + \xi + 2\pi \ , \tag{3.26a}$$

$$|\theta_+ - \xi| - 2\pi \le \theta_- \le -|\theta_+ - \xi| + 2\pi \ , \tag{3.26b}$$

where the absolute bounds on θ_+ and θ_- are $\xi - 2\pi$ to $\xi + 2\pi$ and -2π to 2π, respectively. We now cast the joint distribution into the central 2π ranges $\xi - \pi$ to $\xi + \pi$ and $-\pi$ to π for θ_+ and θ_-, respectively by adding or subtracting 2π as necessary to values of θ_+ and θ_- outside these 2π ranges. This gives the joint mod(2π) probability density

$$\mathcal{P}_{2\pi}(\theta_+,\theta_-)=\frac{1}{2\pi}\frac{1}{2\pi\cosh^2 r}\times\left[\frac{1}{1+\tanh^2 r-2\cos(\theta_+-\xi)\tanh r}\right], \tag{3.27}$$

where now

$$\xi-\pi\le\theta_+\le\xi+\pi, \tag{3.28a}$$

$$-\pi\le\theta_-\le\pi. \tag{3.28b}$$

The first point to notice is that both $\mathcal{P}_{2\pi}(\theta_+,\theta_-)$ and the range of θ_+ are independent of θ_-, so the integral of the distribution over θ_+ will also be independent of θ_-. Indeed, evaluating this integral we find that the mod(2π) probability density for the phase difference is

$$\mathcal{P}_{2\pi}(\theta_-)=\frac{1}{2\pi}. \tag{3.29}$$

This is the uniform distribution characteristic of a random phase. Working with the mod(2π) distribution has made it clear that the correlation between the modes is not manifest in the phase difference and that the complicated 4π distribution (3.23) is in fact a random phase distribution. The mod(2π) distribution for the phase sum is obtained by integrating the mod(2π) joint distribution over θ_-:

$$\mathcal{P}_{2\pi}(\theta_+)=\frac{1}{2\pi\cosh^2 r}\times\left[\frac{1}{1+\tanh^2 r-2\cos(\theta_+-\xi)\tanh r}\right]. \tag{3.30}$$

This function is the Airy pattern familiar from the theory of the Fabry-Pérot étalon and has a single central peak, within the defined range $\xi-\pi$ to $\xi+\pi$, at $\theta_+=\xi$. The correlation between the modes is manifest in the preferred value of the phase sum

$$\langle\hat{\phi}_{\theta a}+\hat{\phi}_{\theta b}\rangle_{2\pi}=\xi \tag{3.31}$$

and in the variance of the phase sum

$$\Delta_{2\pi}(\phi_{\theta a}+\phi_{\theta b})^2=\frac{1}{2\pi\cosh^2 r}\times\int_{\xi-\pi}^{\xi+\pi}\frac{(\theta_+-\xi)^2 d\theta_+}{1+\tanh^2 r-2\cos(\theta_+-\xi)\tanh r}. \tag{3.32}$$

We can evaluate this integral by first expanding $(\theta_+-\xi)^2$ as a Fourier series in the range of $-\pi$ to π and obtain eventually

$$\Delta_{2\pi}(\phi_{\theta a}+\phi_{\theta b})^2=\frac{\pi^2}{3}+4\sum_{k=1}\frac{(-1)^k}{k^2}\tanh^k r=\frac{\pi^2}{3}+4\,\mathrm{dilog}\,(1+\tanh r), \tag{3.33}$$

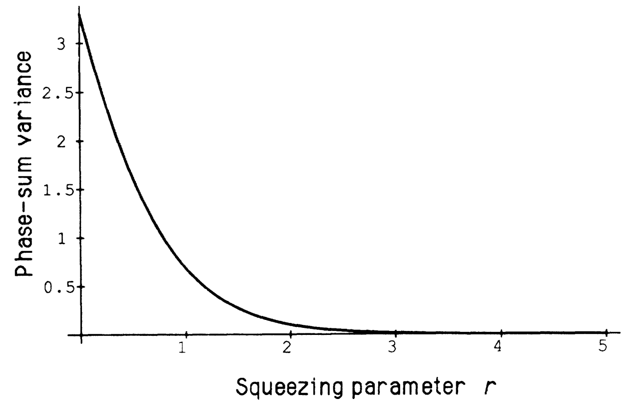

FIG. 3. Phase-sum variance $[\Delta_{2\pi}(\phi_{\theta a}+\phi_{\theta b})^2]$ for a two-mode squeezed vacuum state as a function of r, the modulus of the squeezing parameter. For zero squeezing the phase-sum variance has the value $\pi^2/3$, characteristic of random phase. In the large squeezing limit this variance vanishes.

where dilog( ) is the dilogarithm function.[21] The dilogarithm varies from 0 to $-\pi^2/12$ monotonically as r varies from 0 to ∞. The corresponding variation of the phase-sum variance is from $\pi^2/3$, corresponding to randomness, down to 0. This zero variance denotes the fact that the phase sum (in the 2π range) becomes perfectly locked to the value ξ in the large squeezing limit. This dilogarithmic variation is shown in Fig. 3.

Thus the two-mode squeezed state has simple phase properties that are accessible by exact analytic methods. The progressively stronger locking of the phase sum to the argument of the squeezing parameter with increasing squeezing brings to light another fundamental property of these states.

IV. CONCLUSION

The formulation of the Hermitian optical phase operator has made it possible to examine rigorously the quantum phase properties of light. In this paper we have extended the formalism to discuss two-mode phase properties as revealed in the sum and difference between the single-mode phase operators. This extension involves the added subtlety that the sum and difference operators have a 4π rather than a 2π range of eigenvalues. However, physically distinct phases exist only in a 2π range and therefore the unambiguous interpretation of phase correlations requires us to cast the phase sum and difference probability distributions into a 2π interval. When this is done the physical consequences of phase correlations (or the lack of them) become evident. We have illustrated this procedure with the examples of two uncorrelated modes of random phase, two uncorrelated modes in partial phase states, and the correlated two-mode squeezed state.

It is beyond the scope of this paper to address the experimental implications of our results. Indeed, no precise experimental procedure has yet been suggested for measuring the quantum phase. However, the "measured phase" observables,[22] which are accessible in homodyne detection and other coherent detection techniques, provide a good approximation to the corresponding Hermitian combinations of the unitary phase operators for fields of moderate to high intensity.[9] This strong-field correspondence can also be seen between the phase probability distribution and the angular properties of the Wigner function.[23] The intermode phase correlations exhibited by the two-mode squeezed states increase in strength with the squeezing parameter and thus with the field intensity. Therefore we expect that the effects of these phase correlations will be observable in suitable experiments involving homodyne detection of the individual modes comprising the two-mode squeezed state. In addition, phase-*difference* correlations, similar to the phase-sum correlations studied here, may be present in the output from a correlated emission laser.[24]

Two-mode squeezed states have received a great deal of attention in quantum optics and indeed the two-mode squeezed vacuum was the first squeezed state to be prepared in an experiment.[25] They are of fundamental interest because of the strong quantum correlations between the modes that are responsible for intensity correlations and for the squeezing property itself.[17] The individual modes display random thermal fluctuations and this has led to their application in finite-temperature field theory[26] and in thermodynamic problems in quantum optics.[16] Moreover they are more strongly correlated than any other two-mode state of light.[18] To this impressive list of fundamental properties of this important class of states we can now add their elegant phase properties. They have random single-mode phases and the phase difference is also random, but the phases lock so that their sum has a preferred value. With increasing squeezing this phase locking becomes more rigid until ultimately, in the limit of perfect squeezing, the phase sum can have only one value, which is the argument of the squeezing parameter.

ACKNOWLEDGMENTS

One of us (S.M.B.) thanks the General Electric Company and the United Kingdom Fellowship of Engineering for support.

APPENDIX

Here we derive the mod(2π) distributions $\mathcal{P}_{2\pi}(\theta_+)$ and $\mathcal{P}_{2\pi}(\theta_-)$ for the phase sum and differences from the 4π distributions. In both cases we choose to cast the distribution into the 2π center portion. For the sum distribution this means

$$\mathcal{P}_{2\pi}(\theta_+)=\begin{cases}\mathcal{P}(\theta_+)+\mathcal{P}(\theta_+ +2\pi) & (\xi-\pi\le\theta_+\le\xi)\\ \mathcal{P}(\theta_+)+\mathcal{P}(\theta_+ -2\pi) & (\xi\le\theta_+\le\xi+\pi)\ .\end{cases} \tag{A1}$$

This gives, from (3.21),

$$\mathcal{P}_{2\pi}(\theta_+)=\frac{1}{4\pi^2\cosh^2 r}\left[\frac{2\pi-|\theta_+-\xi|}{1+\tanh^2 r-2\cos(\theta_+-\xi)\tanh r}\right]+\frac{1}{4\pi^2\cosh^2 r}\left[\frac{2\pi-|\theta_+\pm 2\pi-\xi|}{1+\tanh^2 r-2\cos(\theta_+-\xi)\tanh r}\right], \tag{A2}$$

where the + and − signs refer to the two ranges in (A1). The final form of the mod(2π) distribution is

$$\mathcal{P}_{2\pi}(\theta_+)=\frac{1}{2\pi\cosh^2 r}\left[\frac{1}{1+\tanh^2 r-2\cos(\theta_+-\xi)\tanh r}\right], \tag{A3}$$

in agreement with the result derived in III C.

For the mod(2π) phase difference distribution we have

$$\mathcal{P}_{2\pi}(\theta_-)=\begin{cases}\mathcal{P}(\theta_-)+\mathcal{P}(\theta_- +2\pi) & (-\pi\le\theta_-\le 0)\\ \mathcal{P}(\theta_-)+\mathcal{P}(\theta_- -2\pi) & (0\le\theta_-\le\pi)\ .\end{cases} \tag{A4}$$

This gives, from (3.23),

$$\begin{aligned}\mathcal{P}_{2\pi}(\theta_-)=&\frac{1}{4\pi^2}\arctan\left[\exp(-2r)\tan\left[\frac{\theta_+-\xi}{2}\right]\right]\Bigg|_{|\theta_-|-2\pi+\xi}^{-|\theta_-|+2\pi+\xi}\\ &+\frac{1}{4\pi^2}\arctan\left[\exp(-2r)\tan\left[\frac{\theta_+-\xi}{2}\right]\right]\Bigg|_{|\theta_-\pm 2\pi|-2\pi+\xi}^{-|\theta_-\pm 2\pi|+2\pi+\xi},\end{aligned} \tag{A5}$$

where the + and − signs refer to the two ranges in (A4). It is important to note that the definition of the arctangent function employed here is such that

$$\arctan[\tan(\alpha)]=\alpha \quad \text{for } -\pi\le\alpha\le\pi \tag{A6}$$

but if α is *not* in the range $-\pi$ to π, then it must be shifted by an integer multiple of 2π to bring it into this range. Thus $\arctan[\tan(\alpha+\pi)]$ will be $\alpha+\pi$ if α is between $-\pi$ and 0, but will be $\alpha-\pi$ if α is between 0 and π. Moreover, by writing $\exp(-2r)\tan(\alpha)$ as $\tan(\beta)$ and requiring continuity of the arctangent function with variance or r, we obtain

$$\arctan[\exp(-2r)\tan(\alpha+\pi)]=\arctan[\exp(-2r)\tan(\alpha)]\pm\pi\ . \tag{A7}$$

Clearly, extreme care must be used in evaluating (A5). The result is

$$\mathcal{P}_{2\pi}(\theta_-)=\frac{1}{4\pi^2}\left\{\arctan\left[\exp(-2r)\tan\left[\frac{-|\theta_-|+2\pi}{2}\right]\right]-\arctan\left[\exp(-2r)\tan\left[\frac{|\theta_-|-2\pi}{2}\right]\right]\right\}$$
$$+\frac{1}{4\pi^2}\left\{\arctan\left[\exp(-2r)\tan\left[\frac{-|\theta_-\pm 2\pi|+2\pi}{2}\right]\right]-\arctan\left[\exp(-2r)\tan\left[\frac{|\theta_-\pm 2\pi|-2\pi}{2}\right]\right]\right\} . \quad \text{(A8)}$$

The first and fourth terms differ by π as do the second and third terms. Thus this complicated looking expression reduces to the uniform distribution characteristic of random phase:

$$\mathcal{P}_{2\pi}(\theta_-)=\frac{1}{2\pi} . \quad \text{(A9)}$$

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