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Multifractal features of random walks on random fractals

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We find that the fluctuations of probability density $P(r,t)$ of random walks on random fractals, for fixed distance r and time t, have a broad logarithmic distribution. The average moments $\langle P^q\rangle$ scale in a multifractal way, $\langle P^q\rangle \sim \langle P\rangle^{\tau(q)}$, where $\tau(q)\sim q^\gamma$, $\gamma<1$. The exponent γ characterizes the structural disorder in the fractal. In chemical l space, the distribution of $P(l,t)$ is narrow and $\langle P^q\rangle \sim \langle P\rangle^q$.

Several physical properties of random fractal structures show multifractal behavior. Examples are the growth probabilities in diffusion-limited aggregates and the voltage drops in percolation clusters (for recent reviews see, e.g., Refs. 1–3). Both quantities have a broad logarithmic distribution and their moments are described by an infinite hierarchy of exponents. The question how multifractality in random fractals occurs has been discussed extensively,[1–3] but the origin of this intriguing behavior is still unclear.[4] In this Rapid Communication we present an alternative example of multifractality in random fractals, where the multifractality can be studied analytically and therefore its origin can be understood.

We study the fluctuations of the probability density $P(r,t)$ of random walks on random fractals, for fixed distance r and time t. The fluctuations of $P(r,t)$ can be described by the histogram $N(\log P)$ giving the number of sites with values P between $\log P$ and $\log P + d\log P$ and by the moments $\langle P^q(r,t)\rangle$, $q>0$.[5] Specifically, we consider percolation clusters at criticality and linear fractals such as self-avoiding random walks[6] (SAW's).

We find that in both types of fractal structures the histogram $N(\log P)$ is broad and distributed algebraically as

$$N(\log P)\sim |\log(P/P_0)|^{-\alpha}\exp[-b/|\log(P/P_0)|^{\beta}]\,, \qquad P_0\equiv P(0,t)\,. \tag{1}$$

Accordingly, the average moments cannot be described by a single exponent but show multifractal features,[7] i.e.,

$$\langle P^q\rangle \sim \langle P\rangle^{\tau(q)},\ \tau(q)\sim q^\gamma,\ \gamma<1\ \ (q>0)\,. \tag{2}$$

The exponents α, β, and γ are related to the standard exponents characterizing the fractal,

$$\alpha=[(g+d)\nu(d_w^l-1)+1]/d_w^l\,, \tag{3a}$$

$$\beta=(d_w^l-1)/(d_w-d_w^l)\,, \tag{3b}$$

and

$$\gamma=(d_w^l-1)/(d_w-1)\,. \tag{3c}$$

Here d_w and d_w^l characterize how the mean distance $R(t)$ and the mean chemical distance[8] $L(t)$ travelled by a random-walker scale with time t, $R(t)\sim t^{1/d_w}$ and $L(t)\sim t^{1/d_w^l}$; $\nu=d_l/d_f$ denotes the ratio between chemical dimension d_l (Ref. 8) and fractal dimension d_f, and g characterizes the probability of having two sites on the fractal separated by large chemical distance l and small Euclidean distance r.

First we discuss linear fractal structures where (1)–(3) can be derived from rigorous arguments. Along the chain (which is one dimensional in l space, i.e., $d_l=1$) the probability $P(l,t)$ to find the walker at time t at a chemical distance l from its starting point is a Gaussian,

$$P(l,t)=P(0,t)\exp\{-c[l/L(t)]^2\}\,,$$

where $P(0,t)\sim t^{-1/2}$ and $L(t)\sim t^{1/2}$ is the mean chemical distance travelled by the walker, i.e., $d_w^l=2$. For topologically linear random fractals like the SAW, $P(l,t)$ and $L(t)$ are the *same* for every random configuration, and the moments of the distribution function in r space can be written as the convolution integral between $P^q(l,t)$ and the probability $\langle P(r|l)\rangle$ to find two sites at Euclidean distance r separated by the chemical distance l,

$$\langle P^q(r,t)\rangle=\int_0^\infty \langle P(r|l)\rangle P^q(l,t)\,dl\,. \tag{4}$$

It is known (see, e.g., Refs. 9 and 10) that asymptotically $\langle P(r|l)\rangle$ has the exponential form

$$\langle P(r|l)\rangle \sim l^{-\nu d}(r/l^\nu)^g\exp[-(r/l^\nu)^{1/(1-\nu)}]\,, \tag{5}$$

where $\nu\equiv d_l/d_f=1/d_f$ in this case. The exponent g characterizes the probability of having two sites separated by large l and small r. For the SAW, for example, we have $d_f\cong(2+d)/3$ and $g=\frac{4}{9}$, $\frac{5}{18}$, and 0 for $d=2$, 3, and 4, respectively. For linear fractals generated by random walks we have $\nu=\frac{1}{2}$ and $g=0$. By substituting (5) into (4) we obtain the general scaling relation

$$\langle P^q(r,t)\rangle \sim \langle P(rq^{\nu/2},t)\rangle\,. \tag{6}$$

Using the method of steepest descent to evaluate the integral one finds[10]

$$\ln\frac{\langle P(r,t)\rangle}{\langle P(0,t)\rangle}\sim-\left[\frac{r}{R(t)}\right]^{1/(1-\nu/2)}. \tag{7}$$

By combining (6) and (7) we obtain (2) with $\gamma=\nu/(2-\nu)=1/(2d_f-1)$. Since $d_\omega=2d_f=2/\nu$ for linear fractals, the exponent in the right-hand side of (7) can be written as $d_w/(d_w-1)$, and γ becomes identical to (3c).

For calculating the histogram $N(\log P)$ we use the identity

$$\langle P^q\rangle=\int_0^\infty P^q N(\log P)\,d\log P. \tag{8}$$

When comparing Eqs. (4) and (8) we change the variables in (4) from l to P. From

$$P(l,t)=P(0,t)\exp\{-c[l/L(t)]^2\}$$

follows $l\sim|\log P/P_0|^{1/2}$ and

$$dl\sim|\log(P/P_0)|^{-1/2}(d\log P)/P.$$

Comparing the resulting integrand in (4) with the integrand in (8), we obtain our basic result Eq. (1), with $\alpha=[(g+d)\nu+1]/2$ and $\beta=\nu/[2(1-\nu)]$ from (3a) and (3b).

The origin of the multifractality can be understood as follows. According to our derivation, the broad logarithmic distribution (1) leading to the multifractal behavior of the moments originates from the convolution of two comparatively narrow distributions in l space, $\langle P(r|l)\rangle$ and $P^q(l,t)$. Although the distribution $\langle P(r|l)\rangle$ is rather narrow in l, the resulting distribution for $P(r,t)$ is logarithmically broad, since l scales in a *logarithmic* way with P.

In order to show that the multifractal behavior is more general and not only restricted to linear fractals, we have considered also percolation clusters at criticality. In general, $P^q(l,t)$ depends on the configuration considered and the integrand in (4) is substituted by $\langle P(r|l)P^q(l,t)\rangle$. We can extend the above considerations to general random fractals, if the following two assumptions for the fluctuations of $P(l,t)$ (which are trivially valid for SAW's) are satisfied: (i) the moments $\langle P^q(l,t)\rangle$ scale as $\langle P(l,t)\rangle^q$, and (ii) $\langle P(l,t)\rangle$ has the form

$$\langle P(l,t)\rangle\sim\langle P(0,t)\rangle\exp\{-c[l/L(t)]^{d_w^l/(d_w^l-1)}\}. \tag{9}$$

From (i) follows that the histogram $N(\log P)$ in l space is narrow, and hence $\langle P(r|l)P^q(l,t)\rangle$ can be decoupled into $\langle P(r|l)\rangle\langle P^q(l,t)\rangle$. We show below that both assumptions are justified numerically for percolation clusters.

In percolation clusters, the form of $\langle P(r|l)\rangle$ is the same[10] as for SAW's, Eq. (5), with $\nu\equiv d_l/d_f\cong0.88$ in $d=2$ and $\cong0.75$ in $d=3$ (Ref. 11) and $g\cong2.5$ in $d=2$ and $\cong4$ in $d=3$. Using the above assumptions and the modified form of $\langle P(r|l)\rangle$, the convolutional integral (4) can be calculated as above and we obtain

$$\langle P^q(r,t)\rangle\sim\langle P(rq^{\nu(d_w^l-1)/d_w^l},t)\rangle \tag{10}$$

and

$$\ln\frac{\langle P(r,t)\rangle}{\langle P(0,t)\rangle}\sim-\left[\frac{r}{R(t)}\right]^{d_w/(d_w-1)}. \tag{11}$$

By combining (10) and (11) we obtain Eq. (2), and by identifying (4) and (8) we recover Eq. (1), with α, β, and γ from (3).

To test these predictions and the underlying assumptions numerically, we generated clusters of 300 shells at criticality, using the Leath growth method.[12] On each configuration we determined the probability $P(\mathbf{r},t)$ to find the random walker on site $\mathbf{r}$ at time t. From $P(\mathbf{r},t)$ we calculated $P^q(\mathbf{r},t)$ and, by summing $P^q(\mathbf{r},t)$ over all sites $\mathbf{r}$ on the same chemical shell l, we obtained $P^q(l,t)$. Similarly, $P^q(r,t)$ was obtained by summing over all cluster sites $\mathbf{r}$ within $r-\frac{1}{2}$ and $r+\frac{1}{2}$. To obtain the configurational averages $\langle P^q(r,t)\rangle$ and $\langle P^q(l,t)\rangle$ we averaged over 1400 configurations.

Figure 1 shows $-\log[\langle P^q(l,t)\rangle/\langle P^q(0,t)\rangle]^{1/q}$ as a function of $l/L(t)$ for several values of q at $t=1000$ time steps. All moments collapse asymptotically to a single straight line, supporting assumption (i). From the mean chemical distance $L(t)$ the walker travelled at time $t=1000$, we deduced the exponent $d_w^l=2.40\pm0.05$. The asymptotic slope of the curves is 1.71 ± 0.03, in agreement with assumption (ii), Eq. (9).

Figure 2(a) shows $-\log[\langle P^q(r,t)\rangle/\langle P^q(0,t)\rangle]^{1/q}$ as a function of $r/R(t)$ for representative values of q, at time $t=1000$, where $d_w=2.81\pm0.05$. In contrast to the situation in l space (Fig. 1), the moments do not collapse to a single line. The asymptotic slope of the lines is 1.54 ± 0.03, in agreement with the prediction of Eq. (11), $d_w/(d_w-1)\approx1.55$.[13] To test the scaling result, Eq. (10), we have plotted $-\log[\langle P^q(r,t)\rangle/\langle P^q(0,t)\rangle]$ as a function of $rq^{\nu(d_w^l-1)/d_w^l}/R(t)$, in Fig. 2(b). The data collapse supports (10).

From Fig. 2(a) we deduced the exponent $\tau(q)$ defined in Eq. (2). The result is shown in Fig. 3. The slope of $\tau(q)$ in the log-log plot is $\gamma=0.81\pm0.03$, in excellent

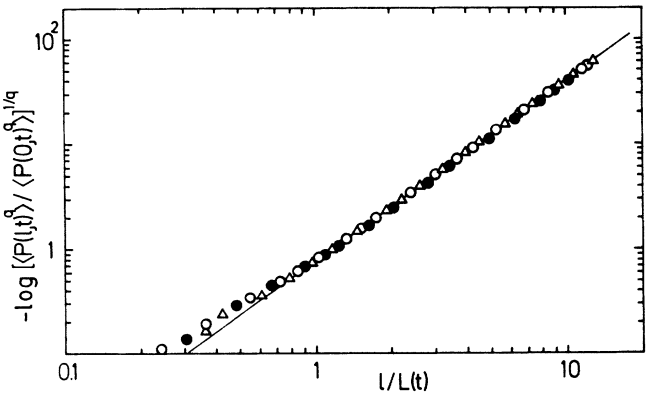

FIG. 1. Random walks on percolation clusters in $d=2$ at criticality, in l space: Plot of $-\log[\langle P^q(l,t)\rangle/\langle P^q(0,t)\rangle]^{1/q}$ as a function of $l/L(t)$ for several values of q and $t=1000$. Different symbols represent different values of q: $q=0.3$(△), $q=1$(○), $q=4$(●). All data collapse to a single line asymptotically.

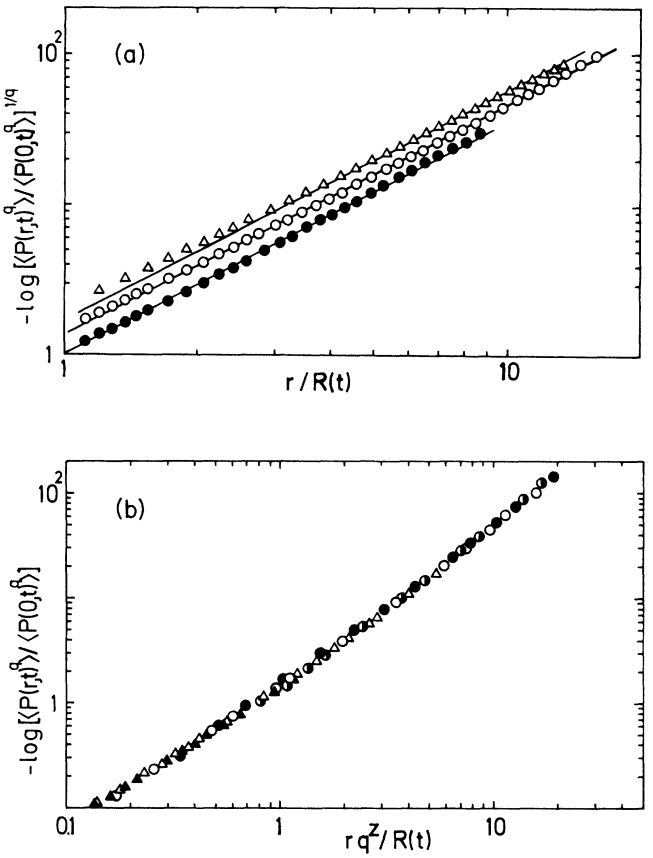

FIG. 2. Random walks on percolation clusters in $d=2$ at criticality, in r space: (a) Plot of $-\log[\langle P^q(r,t)\rangle/\langle P^q(0,t)\rangle]^{1/q}$ as a function of $r/R(t)$ for $t=1000$ and representative values of q: $q=0.3$(△), $q=1$(○), and $q=4$(●). (b) Plot of $-\log[\langle P^q(r,t)\rangle/\langle P^q(0,t)\rangle]^{1/q}$ vs $rq^z/R(t)$, $z=\nu(d_w^l-1)/d_w^l$, for $t=1000$ and $q=0.1$(▲), 0.3(△), 1(○), 4(●), and 10(◑).

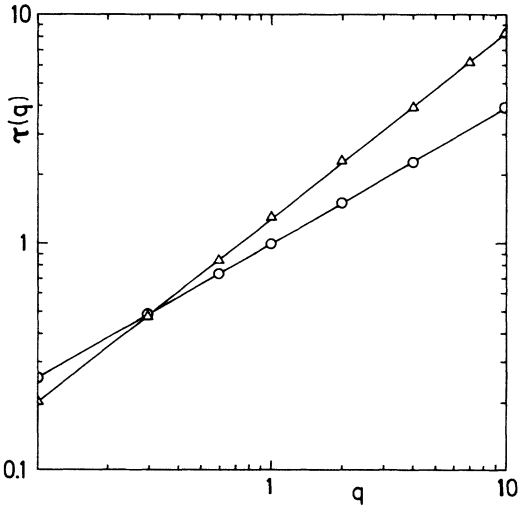

FIG. 3. Plot of $\tau(q)$ vs q for percolation clusters (△) and self-avoiding walks (○). The exponent γ from (2) is calculated from the slopes. To obtain the results for SAW's, we considered chains of 1000 sites on a square lattice and averaged over 2000 configurations.

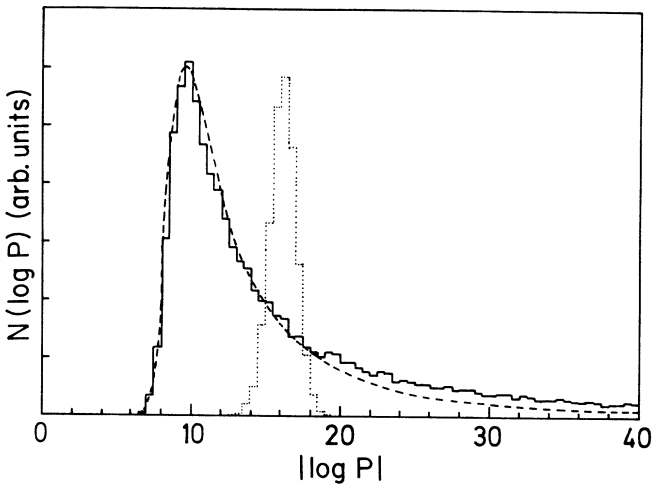

FIG. 4. Plot of the histogram $N(\log P)$ vs $|\log P|$ for fixed r and t (solid line) and for fixed l and t (dotted line). The chosen representative values are $r=30$, $l=80$, and $t=1000$. Similar results were obtained for other values of r, l, and t. The dashed line represents the theoretical results, Eqs. (1), (3a), and (3b) with $b=457$, and $|\log P_0|=4$. According to Eqs. (3a) and 3(b) we used $\alpha=2.6$ and $\beta=3.6$.

agreement with the prediction (3c). In Fig. 3 we also plotted, as a test, $\tau(q)$ for SAW's in $d=2$ obtained by simulation at time $t=1000$, where d_w already reached its asymptotic value. The slope is $\gamma=0.6$, as predicted by (3c).

From the moments one can deduce the relative width $\Delta\equiv(\langle P^2\rangle-\langle P\rangle^2)/\langle P\rangle^2$ of the histogram $N(\log P)$ in l and r space. In l space, Δ is constant (similar to $\Delta=0$ in SAW), while in r space Δ increases exponentially with r. Numerical results for percolation in $d=2$ are shown in Fig. 4. The theoretical results for the histogram, Eqs. (1), (3a), and (3b) (dashed line), are in excellent agreement with the numerical results.

We can argue that the multifractal behavior obtained here for random walks on linear fractals and percolation clusters is more general and occurs on all random fractal structures for which (5) is valid, if their chemical dimension d_l is smaller than the fractal dimension d_f. In this case, it is always possible to find short Euclidean distances r separated by large chemical distances l, and the distribution function $\langle P(r|l)\rangle$ scales in a similar form as (5). The convolutional integral (4) then leads to the same type of multifractal behavior discussed above for linear fractals and percolation clusters.

For $d_l=d_f$, on the other hand, which is the case, e.g., for diffusion-limited aggregates, both chemical and Euclidean distance scale the same and $\langle P(r|l)\rangle$ becomes a δ function at large distances. Consequently, also $N(\log P)$ and $\langle P^q\rangle$ scale the same in *both* l and r space, leading to $\gamma=1$ for $d_l=d_f$ and the multifractal scaling disappears. This conclusion can be also drawn from (3c), anticipating that (3c) holds for arbitrary random fractal structures. Using the general relation $d_w^l=d_w d_l/d_f$, we can write (3c) as $\gamma-1=(d_f-d_l)d_w/(d_w-1)d_f$ and the deviation of γ from unity becomes proportional to the difference between

the chemical and the fractal dimension.

In summary, we have found that diffusion on a large variety of random fractal structures is characterized by a multifractal spectrum. The multifractal behavior in r space originates from the convolution of two comparatively narrow distributions in l space, and the logarithmic relation between l and P. Our analytic calculations of the multifractal spectrum are exact for fractal structures generated by random walks; they are plausible for general linear fractal structures and percolation clusters, and have been confirmed by computer simulations. The problem of diffusion on random fractals represents the first example of a complex physical system where the intriguing occurrence of multifractality has received a simple and rigorous answer.

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[5]Previous work was restricted to the first moment $\langle P(r,t)\rangle$ that plays a central role in describing diffusion on random structures [S. Alexander and R. Orbach, J. Phys. (Paris), Lett. **43**, 1652 (1983); A. Blumen, J. Klafter, and G. Zumofen, in *Optical Spectroscopy of Glasses,* edited by I. Zschokke (Reidel, Dordrecht, 1986), p. 199; B. O'Shaughnessy and I. Procaccia, Phys. Rev. Lett. **55**, 455 (1985); R. A. Guyer, Phys. Rev. A **32**, 2324 (1984)]. It is also relevant to several other problems of interest such as quantum localization [A. B. Harris and A. Aharony, Europhys. Lett. **4**, 1355 (1987)], or self-avoiding walks on fractals [A. Aharony and A. B. Harris, J. Stat. Phys. **54**, 1091 (1989)]; for a review see Ref. 10.

[6]Linear fractals are topologically linear paths consisting of consecutively ordered segments. Although, when viewed geometrically, the path can intersect itself, it can still be defined uniquely as a linear chain of ordered segments. Special cases of linear fractals are random walks and self-avoiding walks in d dimensions (see also Ref. 9). Percolation clusters occur, e.g., in random resistor networks at the critical resistor concentration. In contrast to linear fractal structures, percolation clusters have loops and dangling ends on all length scales [see, e.g., D. Stauffer, *Introduction to Percolation Theory* (Taylor and Francis, London, 1985)].

[7]The multifractal features described by (1) and (2) are distinct from the multifractal behavior found in kinetic aggregation or voltage drops in percolation, where the scaling parameter is the size of the system and $\tau(q)$ approaches a straight line for $q\rightarrow\infty$. The scaling parameter here is $\langle P(r,t)\rangle$, which is similar to the study of intermittency in chaotic dynamical systems [G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987)]. Using the $f(\alpha)$ formalism it can be shown that in our case of diffusion on random fractals $\alpha_{\min}=0$ and $\alpha_{\max}=\infty$. The $f(\alpha)$ spectrum changes from $-\infty$ at $\alpha=0$ to 1 for $\alpha\rightarrow\infty$. The crossover from negative to positive values of $f(\alpha)$ has an interesting interpretation that will be discussed elsewhere.

[8]The chemical distance l between two sites on the fractal is the shortest path on the structure connecting them; the chemical dimension d_l characterizes how the mass scales with $l, M\sim l^{d_l}$. $P(l,t)$ is the probability to find the walker in chemical (topological) distance l from the origin, for a given configuration [see, e.g., S. Havlin and R. Nossal, J. Phys. A **17**, L427 (1984)].

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