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# Asymptotically synchronous chaotic orbits in systems of excitable elements

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Sufficient conditions are given for the emergence of asymptotically synchronous chaotic orbits in ensembles of identical dynamical systems with all-to-all interaction.

Very recently Pecora and Carroll[1] reported on the synchronization of the chaotic trajectories in dynamic systems of the type

$$\dot{v} = g(u,w)\,, \tag{1a}$$

$$\dot{w} = h(v,w)\,, \tag{1b}$$

$$\dot{w}' = h(v,w')\,, \tag{1c}$$

where the subsystems (1a) and (1b) taken together represent a dynamic system in a chaotic regime. Two identical subsystems (1b) and (1c) when considered as *nonautonomous*

$$\dot{w} = h(s(t),w) \tag{2}$$

with a chaotic driving function $s(t)$ where assumed to have all the Lyapunov exponents negative. It was conjectured and has been confirmed by numerical and "hardware" nonlinear circuit experiments that the subsystems (1b) and (1c) starting with different initial conditions may asymptotically synchronize in terms of the $w,w'$ variables. The observed synchronization was "structurally" stable with respect to small parametric perturbations of the subsystems. It was also pointed out that the condition of negativity of all the Lyapunov exponents of the auxiliary system (2) is only a necessary one for a synchronization.

We have been independently investigating the phenomenon of the complete (approximate) synchronization in networks of identical (or similar) "excitable elements," and have come to similar conclusions concerning the possibility of emerging *network* synchronized states ranging from equilibrium through periodic and quasiperiodic to chaotic for a parameter modulated, all-to-all, unidirectional interactions between units (this interaction may be, nevertheless, arbitrarily small along each link in the limit of an infinite system). Although the indisputable priority of "synchronous chaos" belongs to Pecora and Carroll, we still decided to present some of our results, which contain the *sufficient* condition for synchronization of networks with all-to-all interaction, and stress that a separate "master" driving system like (1a) may emerge as a *collective* network property. This mechanism seems to be of particular relevance for small, realistic neural networks exhibiting spontaneous activity.[2]

Possible neurophysiological implications of synchronized chaos were also pointed out by Pecora and Carroll.

To illustrate the main idea and to show that a synchronized chaos may occur naturally in populations of coupled dynamical systems let us consider a somewhat artificial example of coupled discrete-time maps

$$x_{n+1}^{(k)} = \mu Y_n(1-Y_n) + \beta(x_n^{(k)} - Y_n),\ \mu > 0,\ \beta \in (0,1),\ k = 1,\dots,N\,, \qquad Y_n \equiv \frac{1}{N}\sum_{k=1}^{N} x_n^{(k)}\,. \tag{3}$$

Here each separate map is coupled to the system *mean*, and the interactions are arranged in such a way that all the differences $x_n^{(k)} - Y_n$ converge to a single stable fixed point at zero. On the other hand, as is well known, the average $Y_n$ could remain chaotic and even ergodic (for $\mu = 4$). The whole system then has a "relative equilibrium" (a concept already introduced by Poincaré) and converges to a single chaotic orbit. This orbit still sensitively depends on the system's initial condition, and hence the whole system, though asymptotically synchronous, is still unpredictable.

To deal with the more "serious" case of continuous-time differentiable dynamical systems let us consider, first, an idealized population of identical differentiable dynamical systems (termed units) with a *separate* unit described by a low-dimensional equation of motion

$$\dot{x}^{(i)} = m(x^{(i)}),\ x^{(i)} \in R^s,\ i = 1,2,\dots,N\,, \tag{4}$$

where $x^{(i)}$ is an $s$-dimensional vector of the state variable of the $i$th unit, and $m$: $R^s \to R^s$ is a vector field governing the dynamics. We assume that each unit has single globally asymptotically stable equilibrium, but may be excited to interesting dynamics by impulses, drives, and networking. Let all units interact with input-output type of interaction, where a single unit receives separate signals dependent on the state variables of an input unit, integrates them linearly, and influences, in turn, other units by some output function of its state variables. This type of interaction is most appropriate for systems interacting via unidirectional "messages," and differs from systems with coupling functions depending on both state variables of interacting units, like typical microscopical physical systems with their two-particle interactions. Of course mutual interaction of two units in our system is still possible, but has to be realized by at least two independent interaction links. This seemingly simple restriction has important consequences for the system dynamics. Neuronal ensembles with their unidirectional propagation of signals in axons and axonal branches can be used below as standard examples of such systems. The simplest ensemble of this type is one with all-to-all identical interactions

$$\dot{x}^{(i)} = m(x^{(i)}) + \frac{\epsilon}{N} \sum_{j(\neq i)} f(x^{(j)}), \quad i = 1,2,\ldots,N, \tag{5}$$

where $f\colon R^s \to R^s$ is the "influence function," $f(0)=0$, initially assumed identical for all interacting units, where $\epsilon$ measures the interaction strength. The $N^{-1}$ factor is introduced to stress that the interaction can be considered weak in the large system limit. (Note obvious analogies with the "mean-field" type of kinetic models of statistical physics.) The system (5) has an evident symmetry with respect to the permutations of units. In other words, if $\varphi^{(1)}(t), \varphi^{(2)}(t), \ldots, \varphi^{(N)}(t)$ is a solution satisfying the initial conditions $\varphi^{(1)}(0) = x_0^{(1)}, \varphi^{(2)}(0) = x_0^{(2)}, \ldots, \varphi^{(N)}(0) = x_0^{(N)}$ then any permutation of the single units' orbits will be a solution. A solution to (5) with identical initial conditions for all units is then necessarily *synchronous*,

$$x^{(i)}(t) = \varphi(t), \quad \text{for all } i \tag{6}$$

as a simple consequence of the system's symmetry and the uniqueness theorem. We will be mostly interested in the asymptotically synchronous solutions, when

$$\lim_{t\to\infty} x^{(i)}(t) = \varphi(t), \quad \text{for all } i. \tag{7}$$

Let $\varphi^{(1)}(t), \varphi^{(2)}(t), \ldots, \varphi^{(N)}(t)$; $\varphi^{(1)}(0) = x_0^{(1)}, \varphi^{(2)}(0) = x_0^{(2)}, \ldots, \varphi^{(N)}(0) = x_0^{(N)}$ be a solution of the original system (5). Select a solution $\phi(t)$ of the *auxiliary* system

$$\dot{x} = m(x) + \frac{\epsilon(N-1)}{N} f(x), \quad x \in R^s \tag{8}$$

satisfying the initial condition $\phi(0) = (1/N)\sum_{i=1}^{N} x_0^{(i)}$. It is assumed that this system may have a chaotic attractor for sufficiently large $N$. We will investigate the condition under which the solution $\phi(t)$ is a relative equilibrium of the system (5): $\lim_{t\to\infty} x^{(i)}(t) = \phi(t)$ for all $i = i = 1, \ldots, N$. Consider yet another $N-1$ identical *nonautonomous* auxiliary equations

$$\dot{y}^{(i)} = m(\phi(t) + y^{(i)}) - m(\phi(t)) + \frac{\epsilon}{N}[f(\phi(t) + y^{(i)}) - f(\phi(t))]. \tag{9}$$

One may easily check that the systems (5), and (8) and (9) have equivalent linearization operators about $\phi(t)$ (related by a simple *time-independent* similarity transformation leading to the "mean plus deviations" coordinates). It follows that the stability results based upon invariant

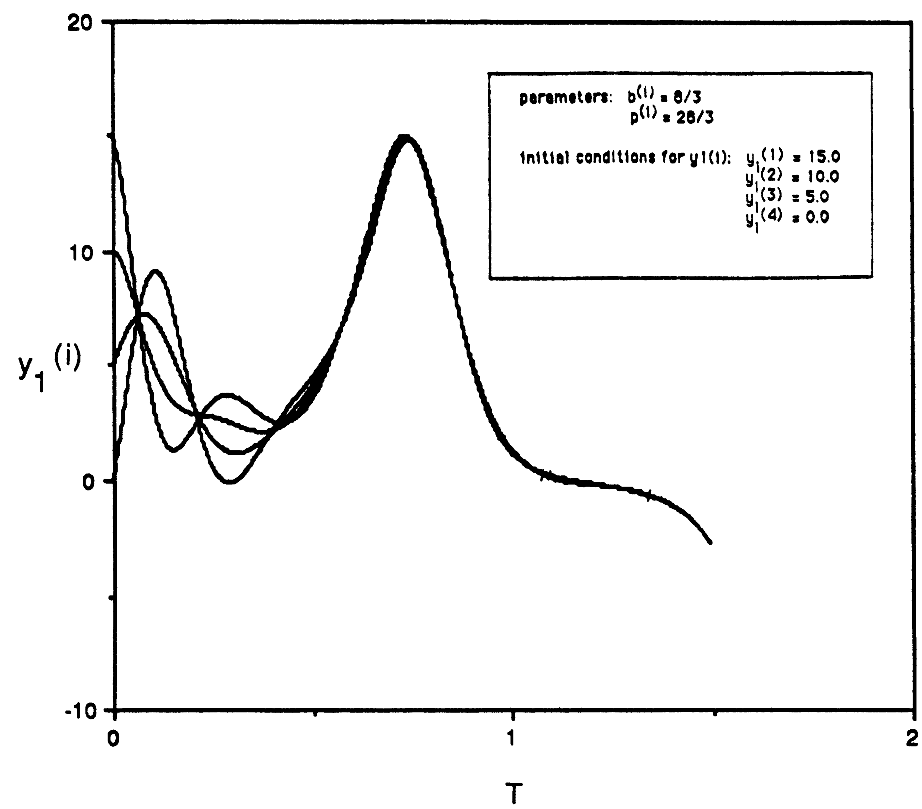


FIG. 1. Synchronization of four identical Lorenz units with different initial conditions. (Synchronization occurred in all three variables, but only the $y_1^{(i)}$ variables are shown.)

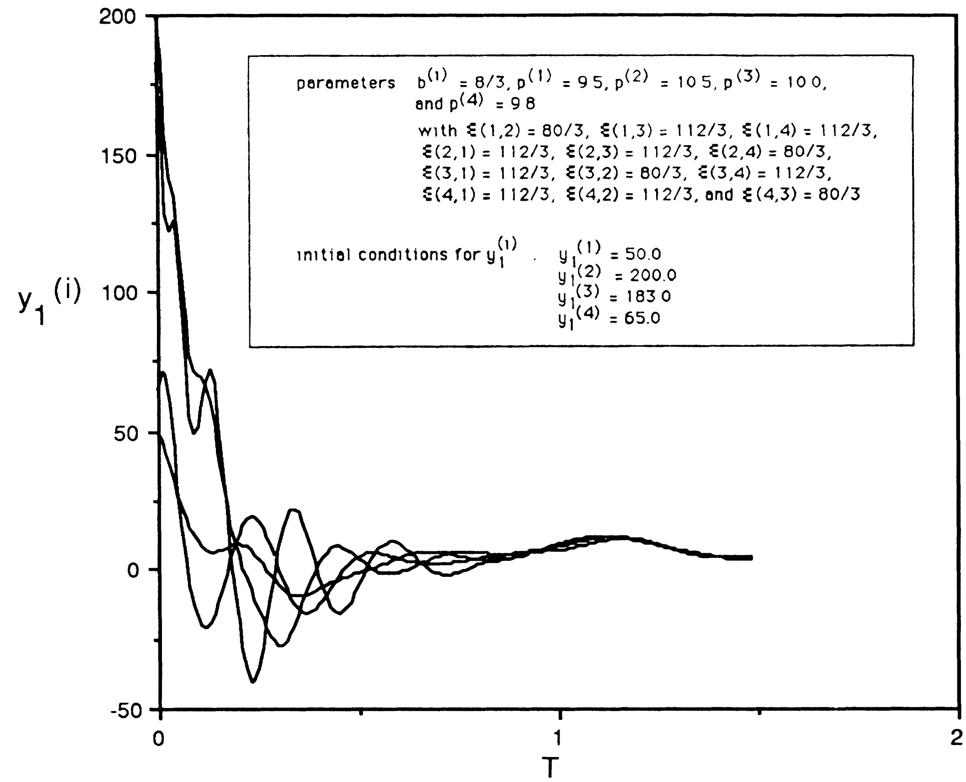


FIG. 2. Synchronization of $y_1^{(i)}$ for four Lorenz units with different parameter values and different coupling strengths.

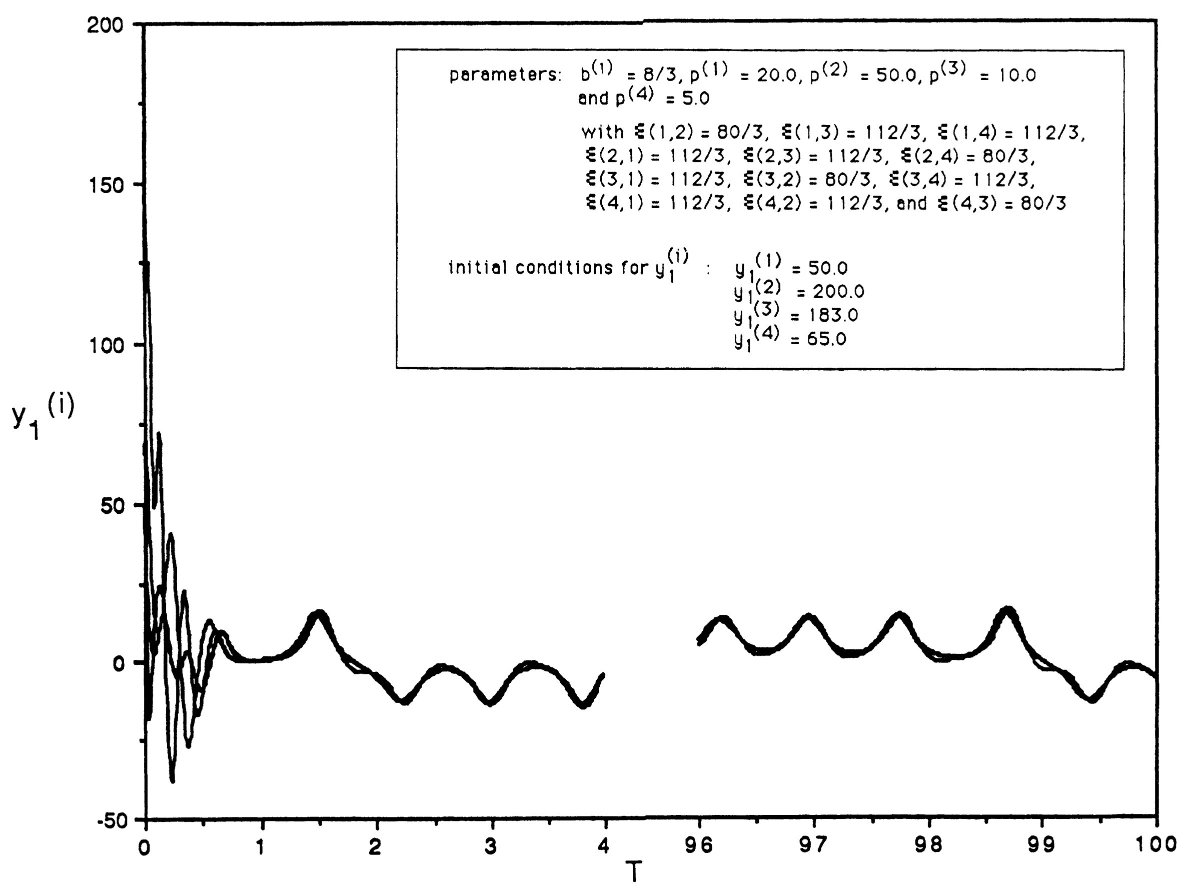


FIG. 3. Synchronization of $y_1^{(i)}$ for four Lorenz units with different parameter values and different coupling strengths (note the long-term synchronization of these units).

properties of the linearization operator (determinant, trace, and Lyapunov exponents) will be the same for both systems. Possible positive Lyapunov exponents occur then in the subsystem (8) subspace and systems like (9) are easier to analyze, as simply related to a single unit dynamics (4). Standard stability results indicate that $y^{(i)}=0$ correspond to a stable equilibrium if the linearization operator $L(t)$ of (8) and (9) is a constant stability matrix. This is the case when attractors of (8) reduces to a fixed point, and the single-unit "corrected" dynamics [with the $(\epsilon/N)f$ term] has a unique stable equilibrium. For periodic stable orbits of (8) this also remains true, if the characteristic exponents of $L(t)$ are all negative (Floquet theory). Difficulty occurs when $\phi(t)$ is a nonperiodic orbit possibly approaching (or staying on) a chaotic attractor. In general, it is not true that the negativity of the Lyapunov exponents for (9) implies the stability of $y^{(i)}=0$ solution. However, this will be the case if (9) is (forward) *regular*[3] on $\phi(t)$. For definition, see also Ref. 3; here we only quote a passage from this review: "Although regularity may be hard to verify for a particular system, we know that it happens in many cases involving a flow with an invariant probability measure. This is a key statement of Oseledec's famous multiplicative ergodicity theorem."

The above sketched proof of the synchronization to a periodic orbit in the system (5) is similar to the one given by Schnol[4] for a system of many generalized "oscillators" interacting with an oscillatory "medium." Schnol also deals with the case of "slightly different" oscillators. For equilibria, and for periodic orbits these results are readily transferable to our case if one considers "slightly different" (in $C^1$ norm) units or interactions. We were unable to obtain the same results in the "chaotic" case, although our computer simulations strongly indicate that this is the case. Lack of space does not allow us to present all of these simulations we carried out for networks of realistic neuromimes (Fitzhugh-Nagumo, Hodgkin-Huxley, and Chay type neurons— all with a novel "nonlinear filtertype" axosynaptic coupling—these results will be presented elsewhere). We present however, an example of a synchronization that is strikingly strong (even transients synchronize) and insensitive to structural changes in units and in the coupling parameters. This is a system of four Lorenz-type units described by

$$
\begin{aligned}
\dot{y}_1^{(i)} &= P^{(i)}(y_2^{(i)} - y_1^{(i)}) , \\
\dot{y}_1^{(i)} &= -y_1^{(i)}y_3^{(i)} - y_2^{(i)} + \frac{1}{N}\sum_{j\,(\neq i)} \epsilon^{(ij)} y_1^{(j)} , \\
\dot{y}_3^{(i)} &= y_1^{(i)}y_2^{(i)} - b^{(i)}y_3^{(i)}, \quad i=1,\ldots,4
\end{aligned}
\tag{10}
$$

with the results presented in Figs. 1–3 and commented upon in the figure captions.

The type of synchronization discussed above becomes even more striking if one considers evolving networks with interconnections growing in time, when for some critical density of links the whole network suddenly changes to a new dynamic state. We are currently investigating percolative transitions in networks with random interactions.

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[1] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).

[2] G. W. Gross and J. M. Kowalski, in *Neural Networks: Concepts, Applications, and Implementations*, edited by P. Antogrietti and V. Milutinovic (Prentice-Hall, Englewood Cliffs, NJ, in press), Vol. 3.

[3] *Lyapunov Exponents*, edited by L. Arnold and V. Wihstutz, Lecture Notes in Mathematics Vol. 1186 (Springer-Verlag, Berlin, 1985).

[4] E. E. Schnol, Prikl. Matem. Mekhan. (USSR) **51**, 9 (1987).