Noise-induced instability

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We have found that the regular motion of an integrable system becomes erratic when a weak white noise is added. The interaction between dynamic and stochastic forces gives rise to an erratic power spectrum, positive Lyapunov exponents, and nonzero correlation exponent independent of embedding dimensions. Since the weak noise drives the systems to unstable fixed points that the dynamic force cannot reach, we call the onset of this erratic behavior the noise-induced instability. The results also indicate that the nonzero correlation exponents independent of embedding dimensions are not necessarily related to the deterministic chaos and that caution should be used in applying this technique to distinguish chaos from noise.

All macroscopic physical systems are subject to fluctuations or noise. Effects of stochastic noise on dynamic systems are an intriguing subject and have received growing attention. Much of this work has been covered in the book series edited by Moss and McClintock.[1] Under certain conditions adding noise to a dynamic system can cause qualitative changes. An interesting example is the stochastic resonance discovered by Benzi, Sutera, and Vulpiani, where the interaction between stochastic and dynamic forces drives the system into an oscillatory state.[2] For systems which exhibit deterministic chaos, adding a weak noise can either delay or expedite the onset of chaos.[3] In this article we report an interesting situation where in the absence of noise the system has a stable oscillatory motion [Fig. 1(a)], while by adding a weak noise the system begins to show erratic behavior [Fig. 1(b)]. Since in the absence of noise the motion of the system is regular and only with the help of the stochastic force can it reach the unstable fixed points, we call the onset of the erratic behavior the noise-induced instability.

This system has an erratic power spectrum, a positive Lyapunov exponent, and bears some resemblance to noise-perturbed chaos such as having a nonzero correlation exponent independent of embedding dimensions, yet it is different from the deterministic chaos because it does not have a nonlinear map structure usually found in the dissipative chaos. Thus these results not only show that stochastic force can interact with dynamic force and drive the system to a chaoslike behavior, but also exhibit that some techniques used for distinguishing chaos from noise are not necessarily related to chaos and cautions should be used in applying these techniques in examining time series obtained experimentally.

We studied a Brownian rotor driven by an external force. In the limit of strong friction the equation of motion can be expressed by the following Langevin equation:

$$\gamma \frac{\partial \theta}{\partial \tau} = \epsilon_1 \sin\omega\tau \sin\theta + F_s g_w , \tag{1}$$

where γ is the friction coefficient, ϵ_1 is the strength of the external driving force, F_s is the strength of the stochastic force, and g_w is a Gaussian white noise. Let t $(=\omega\tau)$ be the dimensionless time, ϵ $(=\epsilon_1/\omega\gamma)$ be the dimensionless amplitude of the external force, $\sqrt{2D}$ $(=F_s/\sqrt{\omega}\gamma)$ be the reduced strength of the noise, and $x=\theta$, mod$[-\pi,\pi]$; Eq. (1) can then be reduced to a simpler form:

$$\frac{\partial x}{\partial t} = \epsilon \sin t \sin x + \sqrt{2D}\, g_w , \tag{2}$$

where $\langle g_w(t)\rangle = 0$ and $\langle g_w(t_1) g_w(t_2)\rangle = \delta(t_1 - t_2)$. In the absence of the stochastic force the motion is integrable:

$$x = 2\arctan\left[\tan\frac{x_0}{2} e^{\epsilon(1-\cos t)}\right] , \tag{3}$$

where x_0 is the initial position of the particle.

When noise is present, we numerically integrated the stochastic differential equation to find the evolution of the system. We have modified the integration algorithm developed by Mannella and Palleschi[4] to the case where the driving force is time dependent.[5] The numerical integration is accurate to the order of $h^{5/2}$ where h is the time step of integration (we used $h=\pi/100$ in present calculations). We then analyzed the time series by measuring the power spectrum and the Lyapunov exponent. The Lyapunov exponent is determined from the time series of more than 600 periods of external force by using the algorithm developed by Wolf *et al.*[6] Numerically measuring the Lyapunov exponent in the presence of strong noise is an intriguing issue and the algorithm given by Wolf *et al.* only applied to the case of weak noise. Following their procedure, we kept the noise at a very low level and used an appropriate small-distance cutoff in choosing the initial separation when measuring the Lyapunov exponents, and we were able to ignore the effect of pure noise. Therefore only larger scale fluctuations caused by the interaction between noise and the dynamical force were detected.

Typical power spectra for different external forces are given in Fig. 2. We have kept the noise level fixed at $D = 0.001$ and increased the amplitude of dynamic force from $\epsilon = 0.3$ to 5.0. For weak external force such as $\epsilon = 0.3$, we saw a strong peak at the driving frequency sitting in a weak-noise background. When the amplitude of the driving force increased, instead of seeing an increase of power spectrum in the driving frequency and its harmonics, we saw an increase over a broad range of frequencies. At $\epsilon = 3.0$ one can see the peak at the main frequency has become quite broad, and at $\epsilon = 5.0$ the peak is buried among the other frequencies and the whole spectrum is quite erratic.

Since the noise level was kept the same for all cases, the high level of the noisy spectrum at the broad range of frequencies is the result of interaction of the dynamic force and weak noise. The log-log plot of the power spectrum for $\epsilon = 5.0$, $D = 0.001$ [Fig. 2(d)] shows almost level distribution for low frequency and a falloff in a power-law behavior with $P(f) \sim f^{-4}$ in the high-frequency region, indicating it is different from a white noise.

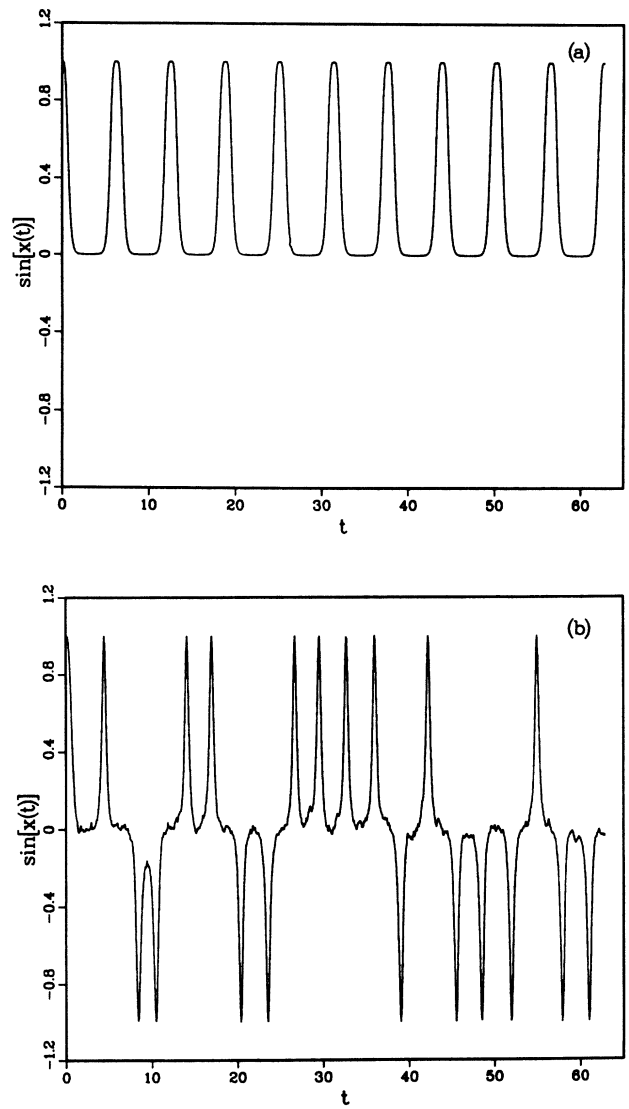

FIG. 1. Trajectories of a particle $x(t)$ (plotted as $\sin[x(t)]$ vs t), in the absence and presence of noise. (a) The particle follows a periodic motion in the absence of noise, $\epsilon = 5.0$, $D = 0$. (b) For the same dynamic force, adding weak noise made the motion quite erratic, $\epsilon = 5.0$, $D = 0.001$.

Sigeti and Horsthemke stated that the power spectrum of $x(t)$ would fall off faster than f^{-2n} when $x(t)$ is $(n-1)$ times differentiable.[7] For a first-order Langevin equation like the present case their theory predicted the power spectrum should fall off as f^{-2}. This result would be observed if there were no strong interaction between the driving force and the stochastic force. The observed falloff as f^{-4} instead of f^{-2} showed the strong interaction between the driving force and the stochastic force and the energy of the driving force has been shifted to neighboring frequencies. It might still be possible to observe some f^{-2} falloff at even higher frequencies, but normally these regions will be flat in real experiments because of the background noise.

The Lyapunov exponents were determined from time series of over 600 periods of the external forces. The measured Lyapunov exponents for various noise levels (D) were given in Fig. 3. For weak noise and small external forces the measured exponent is almost zero since we have filtered out the noise by selecting an appropriate cutoff. After a certain value of ϵ the exponent increases quickly. Compared with the measured power spectrum given in Fig. 2 we found that the Lyapunov exponent became significantly positive when the power spectrum showed substantially noisy background in addition to the peak of the driving frequency. As the Lyapunov exponent increased the spectrum became even more erratic and the peak of the driving frequency eventually disappeared.

The observed positive Lyapunov exponent is the result of the intriguing interaction between the stochastic force and dynamic force, more specifically, the interaction between the noise and the unstable points. In the absence of noise there are two fixed points in the system ($x = 0$ and π). The interesting feature is that they are constantly switching their stabilities. In one-half of a period when $\sin t > 0$, $x = 0$ is the unstable fixed point and $x = \pi$ is the stable fixed point and the opposite situation holds when $\sin t < 0$. While dynamic force moves the particle back and forth, it can never drive the particle to pass the two fixed points. This can be easily seen from Eq. (3) as x approaches π only when the amplitude of dynamic force reaches infinity. (This is different from the situation with a pendulum where the pendulum can pass from an oscillatory mode through the hyperbolic fixed point to a rotating mode by increasing its energy.) Thus the particle, as if it climbs an increasingly steep slope, cannot reach the two fixed points by any finite driving force unless it is initially located on these points. Therefore the fixed points, though unstable for half of a driving period, cannot cause any bifurcations of the trajectory of a particle and its motion stays periodic. The stochastic noise, on the other hand, is not subject to the potential barrier formed by external forces and can drive a particle all over the phase space [this can be seen more clearly in the distribution of

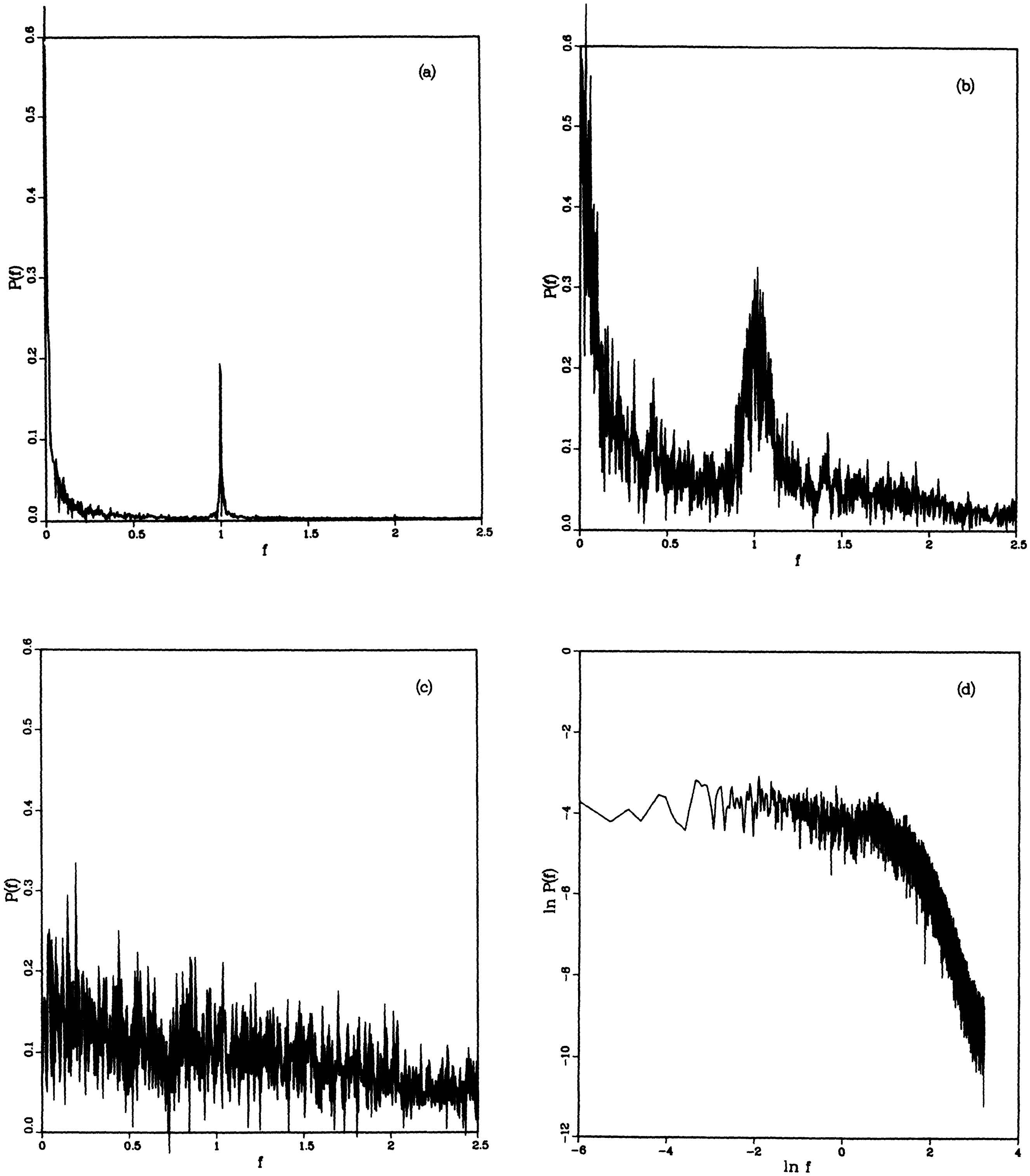

FIG. 2. Power spectra gained by analyzing the time series $x(t)$ show that the interplay between the dynamic and the stochastic forces causes erratic spectra. Each figure gives an average of power spectra for three trajectories of 200 periods long. (a) $D=0.001$, $\epsilon=0.3$. This figure represents a situation of a small driving force and a weak noise. The coupling between these two forces is weak and the spectrum shows a strong peak in the driving frequency and a weak noise background. (b) $D=0.001$, $\epsilon=3.0$. When the driving force increases the coupling between the noise and the dynamic force becomes stronger, and the power spectrum shows a broad peak in the driving frequency and the noise level in other frequencies is quite high. (c) $D=0.001$, $\epsilon=5.0$. For even stronger driving force the power spectrum becomes erratic and the peak in the driving frequency disappears completely. (d) Log-log plot of power spectrum for $D=0.001$ and $\epsilon=5.0$.

the probability density function in Fig. 5(b)]. With the help of noise, the particle can sometime reach the fixed point and when it is unstable, the motion of the particle becomes uncertain afterwards. When the noise drives the particle frequently to the unstable fixed points one will measure a strong positive Lyapunov exponent.

The unstable fixed points and the noise are two essential parts which cause the onset of erratic behavior. Equation (2) represents a Brownian particle under two rotating waves traveling in the opposite directions $[\cos(t-x)$ and $\cos(t+x)]$. If we change the dynamic force by using only one traveling wave,

$$\frac{\partial x}{\partial t}=\frac{\epsilon}{2}\cos(t-x)+\sqrt{2D}\,g_w\ , \tag{4}$$

then the addition of noise will only create some perturbations for most parameters except for $\epsilon=2$. Indeed we found the Lyapunov exponents for the case of one traveling wave [Eq. (4)] are in the order of zero for most cases except for $\epsilon=2$ because there is a bifurcation point at $\epsilon=2$, $x=t$.[8]

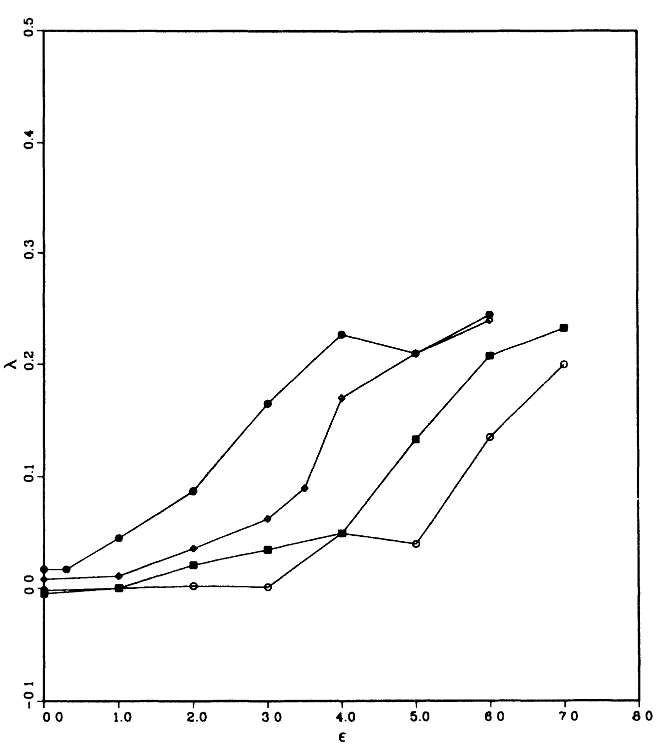

FIG. 3. Measured Lyapunov exponents from the time series for various driving forces and noise levels. The Lyapunov exponents are almost zero for very weak noise and small driving force (e.g., $D=0.000\,001$, $\epsilon<3$) and become significantly positive for stronger dynamic forces. When the noise levels increase, a weaker dynamic force is needed to cause this transition. Notice that the curves move almost evenly leftward when the noise level increases every tenfold. The open circles represent the case $D=0.000\,001$, the solid squares represent $D=0.000\,01$, the diamonds represent $D=0.0001$, and the solid circles represent the case of $D=0.001$. The initial conditions are set at $x_0=\pi/2$, except for $D=0.001$ and $D=0.000\,001$, where both $x_0=\pi/2$ and $x_0=0.1\pi$ are used and their averages are plotted. The difference between these two initial conditions is not significant.

Grassberger and Procaccia have suggested a scheme to characterize the strange attractors in chaotic dynamic systems by measuring the correlation exponent which describes the variation of the correlation integral as the length increases.[9] For a random noise the correlation exponent will increase and equal the embedding dimension when the latter increases. For deterministic chaos, the exponent will have the same positive value when the embedding dimension increases, and the correlation exponent equals zero when the system has a regular motion. For the deterministic chaos perturbed by weak noise Ben-Mizrachi, Procaccia, and Grassberger[10] suggested there will be a crossover from a region where the exponent is depending on the embedding dimension to a region where the exponent stays the same as the embedding dimension increases. The measured correlation integral for different embedding dimensions is plotted in Fig. 4. Indeed we observed that at small length scale the correlation exponent (the slope of the log-log plot) increased as the embedding dimension increased, indicating the dominance of noise at the small length scale. For the large length scale we found that slope appeared the same (~ 0.3) for various embedding dimensions (2,3,4,5), exhibiting some scaling behavior. However the present system is not a noise-perturbed deterministic chaos and this shows that the scaling behavior independent of embedding dimensions is not necessarily related to the deterministic chaos, and a caution should be taken in using the correlation integral to determine whether there is underlying chaos in a time series. The idea of Ref. 10 only

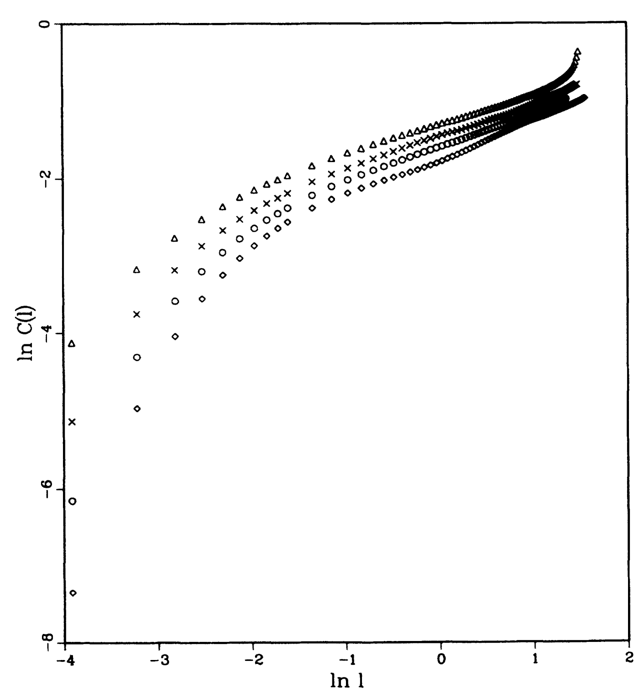

FIG. 4. Log-log plot of the correlation integral $C(l)$ as a function of l for various embedding dimensions. The triangles represent the embedding dimension $d=2$, the crosses $d=3$, the circles $d=4$, and the diamonds $d=5$.

holds for the systems where the noise does not cause dramatic changes and just makes the trajectories fuzzier, when there are strong interaction between noise and dynamic forces the correlation integral may show scaling behavior independent of embedding dimension. A similar situation has been reported in Ref. 11 where the correlation integrals were found independent of embedding dimensions but the underlying dynamics is not a deterministic chaos.

Perhaps a more powerful tool to detect the deterministic chaos is the one-dimensional map. The one-dimensional map created from the time series is plotted in Fig. 5(a). Here we see that the motion of the particle spends a significant amount of time jumping between 0 and π (shown as a high concentration of points around the center and corners in the map). Around these locations there is a spread of points.

The created one-dimensional map did not have a nonlinear map structure usually found in the deterministic chaos and thus the motion bore some signature of stochasticity. And it is because of this gradual spreading of points around the fixed points that the correlation integral shows some scaling behavior in the larger length scale.

The probability density distribution of x in Fig. 5(a) is given in Fig. 5(b). In the absence of noise the motion is periodic and will appear as a section of a straight line in the diagonal direction $[X(t)=X(t+\tau)]$ and will be limited to the section from $x_{\min}$ to $x_{\max}$ in a 1D map. The corresponding distribution can be calculated from Eq. (3):

$$P(x)=\frac{\sqrt{1+(dt/dx)^2}}{\int_{x_{\min}}^{x_{\max}}\sqrt{1+(dt/dx)^2}dx} \quad (5)$$

for $x_{\min}<x<x_{\max}$ and $P(x)=0$ otherwise. $(dt/dx)^2$ can be easily calculated from Eq. (3). Note this distribution is limited to the section from $x_{\min}$ to $x_{\max}$ with $x_{\min}$ and $x_{\max}$ depending on the initial condition x_0, and is always limited within one-half of the phase space (0 to π or $-\pi$ to 0). Bulsara, Schieve, and Jacobs[3(a)] have observed that noise tends to "smooth" the probability density function, and this has made a greater region of phase space available to the system. In the present case we observed not only a greater spreading of the probability density function, but also a change of the distribution structure. While the noise-free distribution is limited to a section of phase space $x_{\min}$ to $x_{\max}$ with higher probability near $x_{\min}$ and $x_{\max}$, the distribution in the presence of noise covers the whole phase space with high probability near the fixed points 0 and π.

Since the onset of this erratic motion is related to the probability for the particle to reach the unstable fixed points, the critical strength ϵ_c and the strength of stochastic force (D) are related through the following condition:

$$2\arctan\left|\tan\frac{x_0}{2}e^{2\epsilon_c}\right|+C\sqrt{D\pi}\approx\pi \ . \quad (6)$$

The first part of the left-hand side represents the maximum distance traveling under the external force only, the second part represents a typical drifting distance due to the noise over the time period π, and C is some constant.

A simple calculation then gives the critical strength ϵ_c which satisfies the following relation:

$$\epsilon_c\approx C-\tfrac{1}{4}\ln D \ . \quad (7)$$

Thus we expect that for an even weaker noise we will still observe the similar chaotic behavior but it will occur at a larger ϵ_c, with ϵ_c increasing in proportion to $-\ln D$. Indeed we found in Fig. 3 the curves for different D shift

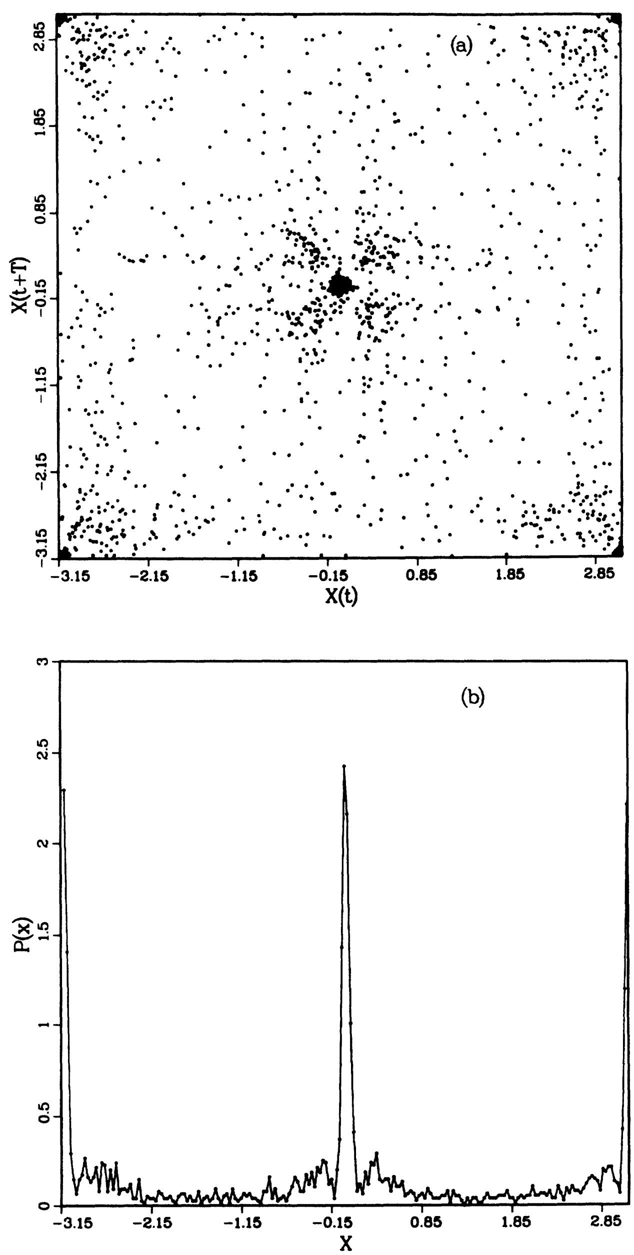

FIG. 5. (a) The one-dimensional map created from the time series. The time interval T is equal to one period of the driving force, and 2400 points are included in this plot. (b) The probability density distribution of x in the 1D map given in (a).

evenly to the right when D decreases every tenfold. It is easy to see that Eq. (6) can also give an estimate for the critical strength of external force for the mean first-passage time (MFPT) to drop to the order of π and this indicates a transition from a diffusion-dominant region to a region where the dynamic force determines the MFPT. Around the critical external driving force the MFPT will exhibit a sudden drop. We found indeed this is the case. In Fig. 6 we plotted the MFPT versus ϵ for different noise levels. The MFPT were measured numerically through more than 1600 realizations for $x_0=\pi/2$. We found the MFPT to drop to the order of π at different ϵ for various D, and the critical values ϵ_c are very close to the values where the Lyapunov exponents became substantially positive and saturate. In Fig. 7 we plotted the critical values of external forces for various noise levels. It can be seen that the transitions measured from Lyapunov exponents and the MFPT are quite close and they decrease linearly with $\ln D$ as Eq. (7) indicated, though the slope is larger than $\frac{1}{4}$.

In Ref. 12 we studied the level statistics in the Floquet spectrum of the Fokker-Planck equations corresponding to the Langevin equations [Eqs. (1) and (4)] in this paper. We found the level spacings of the Floquet spectrum exhibit different statistics for these two systems. The system with two traveling waves [Eq. (1)] exhibited level repulsions in the Floquet spectrum and this happened when the MFPT showed a significant drop similar to what we found in this paper. This shows that different level statistics in Fokker-Planck equations do lead to different behaviors of individual particles and the level repulsions in Fokker-Planck equations correspond to an interesting interplay between the stochastic and dynamic forces such as the noise-induced instability we found in this paper.

In summary we found an interesting case where the motion of an integrable system became erratic when a weak noise was added. This erratic behavior is the result of the interaction between the noise and the dynamic force around unstable fixed points. The necessary condition for deterministic chaos is the existence of saddle-node bifurcation points where a particle will move towards the bifurcation point along the stable direction and then is driven apart in the unstable direction. In the present case though it is virtually a one-dimensional system and the fixed points only have one eigenvalue, the eigenvalue is oscillatory. When the eigenvalue is negative the particle is driven towards the fixed points and with the help of the stochastic force the motion becomes uncertain when the eigenvalue switches to positive and the fixed point becomes unstable. Thus it has an equivalent hyperbolic situation as in the deterministic chaos, and it bears some resemblance to noise-perturbed chaos such as positive Lyapunov exponents and nonzero correlation exponents. But as its motion is limited in one dimension it does not have a strange attractor as in the deterministic chaos and the motion shows erratic oscillations between two fixed points. The interesting part of this system is that it provides an example that a weak noise can drive a system to unstable fixed points which cannot be reached

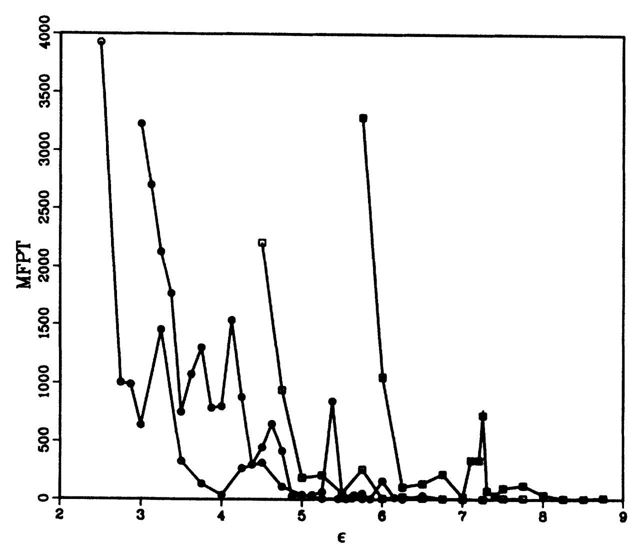

FIG. 6. The measured mean first-passage time (MFPT) to reach $x=\pi$ vs the driving forces for various noise levels. The MFPT is measured through more than 1600 realizations for $x_0=\pi/2$. Notice the transition from the diffusion-dominant region to a drifting region and the critical strengths of the external force increase almost evenly for every tenfold decrease in the diffusion coefficient. The solid squares represent the MFPT for $D=0.000\,001$, the open squares $D=0.000\,01$, the solid circles denote MFPT for $D=0.0001$, and the open circles $D=0.001$.

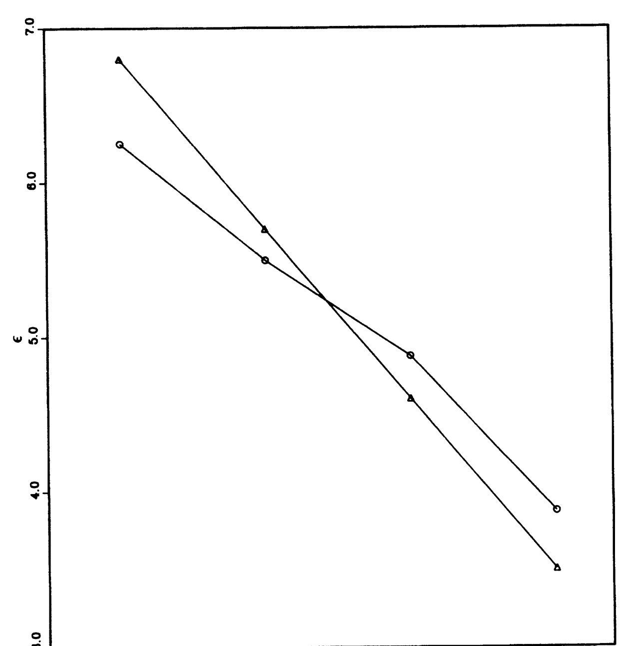

FIG. 7. Critical strengths of the driving forces, ϵ vs the noise level (diffusion constant D). The open circles are the results from the measurements of MFPT. The triangles are determined from the measurement of the Lyapunov exponents. The slope for the MFPT is close to $\frac{1}{3}$, while the slope from the Lyapunov exponents is slightly larger.

by the dynamic force so the addition of weak noise would cause a dramatic change in the motion. The study also raises concerns about the methods now existing in distinguishing noise and chaos such as using the correlation integral to separate the noise effect and chaos, as we see that the nonzero correlation exponent independent of the embedding dimensions is not necessarily related to deterministic chaos. This kind of noise-induced instability should be found in other systems where similar interaction between noise and fixed points occurs, and is worthy of attention.

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$$x(h)-x(0)=\sqrt{2D}Z_1+fh+f_x\sqrt{2D}Z_2+\tfrac{1}{2}(f_xf+f_t)h^2+f_{xx}DZ_3\ ,$$

with

$$Z_1=\int_0^h \xi(t)dt=\sqrt{h}\,Y_1\ ,$$

$$Z_2=\int_0^h\left[\int_0^t \xi(s)ds\right]dt=h^{3/2}(Y_1/2+Y_2/2\sqrt{3})\ ,$$

and

$$Z_3=\int_0^h\left[\int_0^t \xi(s)ds\int_0^t \xi(y)dy\right]dt\approx(h^2/3)(Y_1^2+Y_{3+}\tfrac{1}{2})\ .$$

Y_1,Y_2,Y_3 are three uncorrelated Gaussian variables with average zero and standard deviation 1.

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