Scalar optical transmittance function as a path integral: Formal derivation

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The scalar optical transmittance function is given by the Fourier transform of a path integral. Here we try to express it directly by a path integral. Applications both to the calculation of complex amplitude distributions and "instantaneous" optical transformations in the paraxial region are developed.

Scalar paraxial optical transformations by operational methods have been an important subject in past years. Bacry and Cadilhac[1] derived the Sp$(4,R)$ as the symmetry group of the wave equation in the paraxial limit. Canonical transform kernels were introduced by Nazarathy and co-workers.[2] Dragt introduced Lie algebra techniques in the Hamiltonian treatment of geometrical optics.[3] Moreover, in the spirit of Dragt's ideas, Wolf developed nonlinear canonical transformations in optical phase space; in particular, "instantaneous" finite canonical transformations, which have no counterpart in mechanics.[4–6]

In this Brief Report we show how each optical component can be regarded as an element that performs an effective (thick element) or real (thin element) "instantaneous" transformation, which is connected to both the optical phase space and the idea of the transmittance function (TF) as a path integral representing an "instantaneous" optical propagator. The method is not restricted to Gaussian optics.

Path-integral (PI) formalism can be considered as complementary to that setup with a group-theoretical basis and canonical operators. Recently, a formal derivation of the optical propagator for the Helmholtz equation has been given,[7,8] where limiting cases such as a paraxial approximation have been obtained; moreover, in the paraxial region the Wiener-Feynman measure is restored and explicit calculations can be performed. Thus, the optical propagator for the gradient-index (GRIN) media was calculated and transmittance functions were inferred[9] as a PI. Nevertheless, a formal derivation of these TF's has not been given. In this work, we try to express the scalar TF directly by a PI. On the other hand, the vector TF could be also obtained by the Mukunda-Simon-Sudarshan[10] (MSS) formalism and by PI formalism:[11] both formalisms may be regarded as the vector generalization of the operational methods mentioned above.

The TF can be defined as a PI when the sum over paths is restricted to a subset of them determined by the initial conditions of the light rays. This definition is closely related to other path integrals such as the Hamiltonian ones where the sum over paths is restricted to those characterized by constants of motion (energy, momentum, etc.).[12]

The TF can be used to evaluate the complex amplitude distribution (CAD) at the output plane of the optical system if the input CAD is known. Neither the general kernels nor the propagation integral are required, because the TF supplies an "instantaneous" transformation of the CAD. The following PI expression has been formally derived for the scalar optical propagator:[7]

$$K(x_1,y_1,x_0,y_0;z_1) = \int Dx(z)Dy(z)\exp[ikS(x_1,y_1,x_0,y_0;z_1)] , \quad (1)$$

S being the geometrical optical path length and k the wave number. Thus, if the CAD $\Psi(x_0,y_0)$ is known at a transverse plane $z=0$, then the function Ψ at some plane $z=z_1>0$ satisfies the integral equation

$$\Psi(x_1,y_1,z_1)=\int_{R^2}K(x_1,y_1,x_0,y_0;z_1)\Psi(x_0,y_0)dx_0dy_0 . \quad (2)$$

The path functional S can be written as

$$S=\int_0^{z_1}L(x,y,\dot{x},\dot{y};z)dz , \quad (3)$$

where L is the optical Lagrangian given by

$$L(x,y,\dot{x},\dot{y};z)=n(x,y,z)(1+\dot{x}^2+\dot{y}^2)^{1/2} \quad (4)$$

for arbitrary media, and the dot represents differentiation with respect to z. In order to examine light propagation through an optical system we assume the following representation for the refractive index:

$$n(x,y,z)=n_0+\Delta n(x,y,z) . \quad (5)$$

This refractive index can be a real or effective one.[13] For instance, a GRIN medium has a real refractive index; nevertheless, for a conventional lens an effective refractive index can be used. Moreover, if we consider the paraxial region ($\Delta n/n_0 \ll 1$, $\dot{x},\dot{y} \ll 1$; note that $\dot{q}=p_q/n_0$, $q=x$ or y, and p_q is the optical momentum) the optical Lagrangian can be approximated by

$$L(x,y,\dot{x},\dot{y};z)\simeq(n_0/2)(\dot{x}^2+\dot{y}^2)+n(x,y,z) \quad (6)$$

and Eq. (1) becomes

$$K(x_1,y_1,x_0,y_0;z_1)=\int Dx(z)Dy(z)\exp\left[ik\int[(n_0/2)(\dot{x}^2+\dot{y}^2)+n(x,y,z)]dz\right] . \tag{7}$$

Note that Eq. (7) represents a PI with a well-defined Wiener-Feynman measure,[14,15] and corresponds to the kernel of a paraxial wave equation. We can now evaluate the optical propagator (7), only for a subset of paths with a fixed initial optical momentum $\dot{x}_0=c_x$, $\dot{y}_0=c_y$, by the following projection operator:

$$G(x_1,y_1,x_0,y_0;z_1)=\int K(x_1,y_1,x_0,y_0;z_1)\exp[-ikn_0(\dot{x}_0x_0+\dot{y}_0y_0)]dx_0dy_0 . \tag{8}$$

From Eq. (8) it follows that G is given by a Fourier transform which is widely used for projecting over a fixed energy in the Hamiltonian path integrals.[1,2] On the other hand, the optical propagator (7) can be rewritten as

$$K(x_1,y_1,x_0,y_0;z_1)=\int Dx(z)Dy(z)\exp(ikn_0z_1)\exp[ikn_0(\dot{x}_1x_1-\dot{x}_0x_0+\dot{y}_1y_1-\dot{y}_0y_0)/2] \times\exp\left[ikn_0\int[(\ddot{x}x+\ddot{y}y)/2+\Delta n)]dz\right] \tag{9}$$

where integration by parts in the phase integral and Eq. (5) has been used. Now, we perform the path integration as a sum over paths with fixed $\dot{x}_0,\dot{y}_0$, where we follow closely Gutzwiller's idea[16] related to fixed energy. Then integrating over all the values of $\dot{x}_0,\dot{y}_0$, Eq. (8) becomes

$$G(x_1,y_1,x_0,y_0;z_1)=\int d\dot{x}_0d\dot{y}_0\int dx_0dy_0\exp[-ikn_0(c_x\dot{x}_0+c_y\dot{y}_0)] \times\int_{\dot{x}_0=\text{const},\,\dot{y}_0=\text{const}}Dx(z)Dy(z)\exp(ikn_0z_1)\exp[ikn_0(\dot{x}_1x_1+\dot{x}_0x_0+\dot{y}_1y_1+\dot{y}_0y_0)/2] \times\exp\left[ikn_0\int[\ddot{x}x+\ddot{y}y/(2+\Delta n)]dz\right] . \tag{10}$$

The integrations over x_0,y_0 generate a factor proportional to $\delta(c_x+\dot{x}_0,c_y+\dot{y}_0)$, so that Eq. (10), after some algebra, can be rewritten as

$$G(x_1,y_1,x_0,y_0;z_1)=\int_{\dot{x}_0=\text{const},\,\dot{y}_0=\text{const}}Dx(z)Dy(z)\exp\left[\int[(n_0/2)(\dot{x}^2+\dot{y}^2)+n(x,y,z)]dz\right] . \tag{11}$$

Equation (11) is a PI on trajectories from (x_0,y_0) to (x_1,y_1), parametrized by the variable z. Only the paths with $\dot{x}_0=c_x$, $\dot{y}_0=c_y$ are considered in the integration. In short, expression (11) describes a functional integration with constraints[17] in the phase space. From the optical point of view, it may be regarded as a TF, as will be made clear later.

On the other hand, Eq. (8) can be used to perform explicit calculations of the TF in a direct way. Several paraxial cases can be studied, but for the sake of simplicity we consider the Gaussian region. It is well known that in the Gaussian approximation only the classical paths contribute to the PI;[15,9] therefore Euler's equations can be used to evaluate the PI. A Gaussian optical path length is given by

$$S=n_0z_1+(n_0/2)(\dot{x}^2+\dot{y}^2)-g^2(z)(x^2+y^2) \tag{12}$$

(linear terms are not considered because no new result would be obtained). Therefore from Euler's equations

$$\frac{\partial L}{\partial q}-\frac{d}{dz}\left[\frac{\partial L}{\partial \dot{q}}\right]=0,\quad q=x \text{ or } y \tag{13}$$

Eq. (7) becomes

$$K(x_1,y_1,x_0,y_0;z_1)=\frac{1}{F(z)}\exp(ikn_0z_1) \times\exp[ikn_0(\dot{x}_1x_1-\dot{x}_0x_0+\dot{y}_1y_1-\dot{y}_0y_0)/2] , \tag{14}$$

where $F(z)$ is given by[9]

$$F(z)=F_0\int\exp\left[-\frac{1}{2n_0}\int\nabla_\perp^2S\,dz\right] , \qquad \nabla_\perp^2S=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} . \tag{15}$$

Substituting Eq. (14) into Eq. (8) and performing the Fourier transform, it follows that

$$G(x_1,y_1,x_0,y_0;z_1)=\frac{1}{F(z)}\exp(ikn_0z_1)\exp[ikn_0(\dot{x}_1x_1-\dot{x}_0x_0+\dot{y}_1y_1-\dot{y}_0y_0)/2]\delta(c_x+\dot{x}_0,c_y+\dot{y}_0)\ , \tag{16}$$

which represents an "instantaneous" optical propagator. A useful application of this PI formulation is, for instance, the singular case when $B=0$ in the *ABCD* ray-transform matrix. The optical propagator for this singular case can be derived from Eq. (16) in a straightforward way. Likewise, operator techniques[1,2] have been used for deriving the transfer operator in the above-mentioned case. Finally, the usual TF can be derived from G as

$$t(x_1,y_1,x_0,y_0;z_1)=\int G(x_1,y_1,x_0,y_0;z_1)\Psi(x_0,y_0)dx_0dy_0[\Psi(x_0,y_0)]^{-1}$$
$$=\frac{1}{F(z)}\exp(ikn_0z_1)\exp[ikn_0(\dot{x}_1x_1+\dot{y}_1y_1)][\Psi(x_0,y_0)]^{-1}\ . \tag{17}$$

Now, we apply Eq. (16) to calculate the output CAD due to an input general Gaussian beam characterized by the initial condition $\dot{q}_0=q_0/U$, with $U\in\mathbb{C}$. The classical rays are given by

$$q=q_0[S_2+S_1/(n_0U)]=q_0F(z)\ , \tag{18}$$

where S_1 and S_2 are the axial and field rays.[9] Inserting Eq. (18) into Eq. (16) the output CAD is given by Eq. (2), where K is replaced by G. After some algebra one obtains

$$\Psi(x_1,y_1,z_1)=\frac{1}{F(z)}\exp(ikn_0z_1)\exp[ikn_0\dot{F}/F(x_1^2+y_1^2)]\ . \tag{19}$$

In short, a formal derivation of the TF as a PI with a constraint in the paraxial region has been derived. Likewise, a projection operator has been used to obtain this result and evaluate the general form of the Gaussian TF. This method can be used for more general optical systems (non-Gaussian systems), whose TF performs three-dimensional focusing. Results will be reported later. Finally, the method that has been developed can be applied to evaluate propagators associated to "instantaneous" transformations (such as arbitrary refractive surfaces) as a limiting case. Equation (16) makes clear this assertion. Moreover, it could be applied in other branches of physics, such as electron optics, diffraction, and so on.

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