Perturbation theory for solitons in optical fibers

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Using a singular perturbation expansion, we study the evolution of a Raman loss compensated soliton in an optical fiber. Our analytical results agree quite well with the numerical results of Mollenauer, Gordon, and Islam [IEEE J. Quantum Electron. **QE-22**, 157 (1986)]. However, there are some differences in that our theory predicts an additional structure that was only partially seen in the numerical calculations. Our analytical results do give a quite good qualitative and quantitative check of the numerical results.

I. INTRODUCTION

One of the most useful features of soliton theory is the practicality of using it for the approximate description of physical phenomena. An excellent example of such an application is the proposed use of solitons as bit carriers in optical fibers.[1–4] Compensating for normal losses by the use of Raman gain,[5–6] one can obtain propagation over thousands of kilometers with bit rates in the 10-GHz range.[7]

The first application of perturbation techniques to soliton propagation was done in 1976 by this author.[8] Interestingly enough, the equation studied then was the damped nonlinear Schrödinger equation; almost exactly the same one that we shall study here. Actually, those results could have been easily used here, just in the form as they were given then. However, we have been able to slightly generalize the original theory and it is this generalization that we shall emphasize here.

Since that time, there has been a tremendous amount of work on perturbations of soliton systems. Among the notable results was the development of a Green's-function approach by Keener and McLaughlin,[9] which was applied to the sine-Gordon equation by McLaughlin and Scott.[10] Karpman and Maslov[11,12] devised techniques for handling perturbations of the Korteweg–de Vries equation. Kaup and Newell[13] studied the same problem and also summarized many of the results for perturbations of integrable systems. Recently, the entire area has been excellently and very comprehensively reviewed by Kivshar and Malomed.[14]

What we shall present here is the application of an expanded perturbation theory for single-soliton propagation to a problem of current interest. And with a minimal effort, the same theory can easily be extended to handle any array of well-separated solitons, as one can expect in an optical fiber. The extended perturbation theory is based on an extension of the inverse scattering transform (IST) perturbation theory first developed in 1976.[8] That theory was developed in the hope that one could alternate between coordinate space and scattering space in some way so as to extend those results into higher order. However, the calculations beyond the first order rapidly became extremely complex, and nothing more has been done in this direction. What we have achieved here is a perturbation expansion which is based on a regular perturbation expansion about the nonlinear one-soliton solution. It differs only slightly from the 1976 theory, but in a small way which could be quite important for optical pulse propagation. In that problem, one finds relatively low amplitude and broad pulses to be more stable.[7] In that limit, a perturbation expansion becomes accurate if we decouple the soliton's amplitude from the soliton's width. This could not be done with the 1976 theory, but can be done here. Although it may seem that this theory entirely bypasses the IST, nevertheless we will be using a very fundamental and key feature that is a result of the existence of the IST for the model integrable system. And that is there exists a closed set of closed-form functions in which the perturbation may be expanded.[15] For a general nonlinear solitary wave, such a system does not exist.

In Sec. II, we present the general singular perturbation expansion for the nonlinear Schrödinger (NLS) equation out to first order. In Sec. III, we present the closure of the eigenstates of an operator L, in which the solution can best be expanded. In Sec. IV, we apply these results to the perturbed NLS, obtaining the solution for the evolution of the soliton parameters and the continuous spectrum. In Sec. V, we apply these results to the case of a Raman pumped soliton propagating in an optical fiber. We compare our results for the evolution of the soliton's area with the numerical results of Mollenauer, Gordon, and Islam.[7] We demonstrate that our analytic results from this theory agree quite well with the numerical results.

II. PERTURBED NLS

The general form of the perturbed NLS is

$$i\partial_t q = -\partial_x^2 q + 2rq^2 + \epsilon R[q,r] \ , \tag{1a}$$

$$i\partial_t r = \partial_x^2 r - 2r^2 q - \epsilon R^*[q,r] \ , \tag{1b}$$

where $r = -q^*$ and ϵ is our expansion parameter. We take the zeroth-order solution to be

$$q_0 = \frac{Ae^{i\alpha}}{\cosh\theta} \ . \tag{2}$$

In the absence of any perturbations, (2) is an exact solution if

$$A=2\eta\ , \tag{3}$$

$$\theta=2\eta(x-\bar{x}) \tag{4}$$

$$\alpha=-2\xi(x-\bar{x})+\bar{\alpha}\ , \tag{5}$$

where η and ξ are the constants of the motion and

$$\partial_t\bar{x}=-4\xi\ , \tag{6}$$

$$\partial_t\bar{\alpha}=4(\eta^2+\xi^2)\ . \tag{7}$$

Now we expand q and r in a singular perturbation expansion. It shall be most convenient to choose θ as our spatial coordinate. We take

$$q=\frac{Ae^{i\alpha}}{\cosh\theta}+\epsilon q_1+\cdots\ , \tag{8a}$$

$$r=-\frac{Ae^{-i\alpha}}{\cosh\theta}+\epsilon r_1+\cdots\ , \tag{8b}$$

and consider q_1 and r_1 to be functions of θ and $\tau(=t)$. θ is chosen to be of the form in (4). The perturbation will cause η and $\bar{x}$ to shift as a function of time. Thus we allow η to depend on the slow time $[\eta=\eta(\tau_1=\epsilon\tau)]$. For $\bar{x}$, it will be more convenient to use the trick of replacing τ with $\tau_1(=\epsilon\tau)$ in

$$\bar{x}=v\tau+\bar{x}_0=(v\tau_1)/\epsilon+\bar{x}_0\ . \tag{9}$$

In the latter form, $\bar{x}$ is not dependent on the fast time, but only on the slow time. Thus we take

$$\bar{x}=\bar{x}_{-1}\epsilon^{-1}+\bar{x}_0+\bar{x}_1\epsilon+\cdots \tag{10}$$

as the expansion for $\bar{x}$ with the coefficients only dependent on the slow time. Similarly for α, we choose (5) to be true with $\bar{\alpha}$,

$$\bar{\alpha}=\bar{\alpha}_{-1}\epsilon^{-1}+\bar{\alpha}_0+\bar{\alpha}_1\epsilon+\cdots\ , \tag{11}$$

where the coefficients are again independent of the fast time.

Taking all the above and expanding (1), we have in zeroth order

$$\partial_{\tau_1}\bar{x}_{-1}=-4\xi\ , \tag{12a}$$

$$\partial_{\tau_1}\bar{\alpha}_{-1}=4\xi^2+4\eta^2\ , \tag{12b}$$

and in first order

$$i\partial_\tau v+4\eta^2Lv=F\ , \tag{13}$$

$$v=\begin{bmatrix} e^{-i\alpha}q_1 \\ -e^{i\alpha}r_1\end{bmatrix}\ , \tag{14}$$

$$F=\begin{bmatrix} \mathcal{R} \\ -\mathcal{R}^*\end{bmatrix}\ , \tag{15}$$

$$\mathcal{R}=e^{-i\alpha}R+\frac{1}{\cosh\theta}[A\partial_{\tau_1}\bar{\alpha}_0+2\xi A\partial_{\tau_1}\bar{x}_0-i\partial_{\tau_1}A-A\theta(\partial_{\tau_1}\xi)/\eta]+i\frac{\sinh\theta}{\cosh^2\theta}A[\theta(\partial_{\tau_1}\eta)/\eta-2\eta\partial_{\tau_1}\bar{x}_0]+\frac{2A(4\eta^2-A^2)}{\epsilon\cosh^3\theta}\ . \tag{16}$$

In (13), the operator L is

$$L=\sigma_3(\partial_\theta^2-1)+\frac{2}{\cosh^2\theta}(2\sigma_3+i\sigma_2)\ , \tag{17}$$

where $\sigma_1,\sigma_2,\sigma_3$ are the Pauli spin matrices.

Given R, we want a solution for v. Obviously from (13) we had best expand in eigenstates of L. These we give in Sec. III, along with their closure and adjoints.

III. EIGENSTATES OF L

These eigenstates are simply related to the "squared eigenfunctions."[8,15] However, no formal connection with the IST is required here, so we shall simply start anew and only present the essential results. Consider the problem

$$L\psi=\lambda\psi\ , \tag{18}$$

where L is given by (17) and ψ is a column matrix. One may verify that a solution of (18) is

$$\psi=e^{ik\theta}\left[1-\frac{2ike^{-\theta}}{(k+i)^2\cosh\theta}\right]\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\frac{e^{ik\theta}}{(k+i)^2\cosh^2\theta}\ , \tag{19}$$

with the eigenvalue

$$\lambda_\psi=k^2=1\ . \tag{20}$$

Given this solution, one may show that a second solution of (18) is

$$\bar{\psi}=\sigma_1\psi\ , \tag{21}$$

whose eigenvalue is

$$\lambda_{\bar{\psi}}=-(k^2+1)\ . \tag{22}$$

These two solutions, for all real k, are the continuous spectrum of L.

Since L is self-adjoint, these solutions are also the adjoint solutions. However, for inner products, it is more convenient to use the set

$$\phi=e^{-ik\theta}\left[1-\frac{2ike^{\theta}}{(k+i)^2\cosh\theta}\right]\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\frac{e^{-ik\theta}}{(k+i)^2\cosh^2\theta}\ , \tag{23}$$

$$\bar{\phi}=\sigma_1\phi\ , \tag{24}$$

whose eigenvalues are

$$\lambda_\phi=-(k^2+1),\quad \lambda_{\bar{\phi}}=(k^2+1)\ . \tag{25}$$

The bound states have an eigenvalue of zero and $k=+i$. They are a linear combination of the functions

$$\phi_e=\frac{1}{\cosh\theta}\begin{bmatrix}1\\-1\end{bmatrix},\quad \phi_o=\frac{\sinh\theta}{\cosh^2\theta}\begin{bmatrix}1\\1\end{bmatrix}, \tag{26}$$

where $e(o)$ refers to the even (odd) part of the solution. Since they are bound states, it follows that

$$L\phi_e=0=L\phi_o\ . \tag{27}$$

Since the eigenfunctions have a double pole at the bound state $k=i$ [see (19) and (23)], we must include two more states for closure.[15] These are "derivative states" and are proportional to the two states $\theta\phi_e$ and

$$\chi=\frac{\theta\tanh\theta-1}{\cosh\theta}\begin{bmatrix}1\\1\end{bmatrix}. \tag{28}$$

The L operator acting on these states gives

$$L\chi=-\phi_e,\quad L(\theta\phi_e)=-\phi_o\ . \tag{29}$$

From the Wronskian relation and the above definitions, one may verify that the only nonzero inner products are the following:

$$\langle\phi(k')|\sigma_3|\bar\psi(k)\rangle=2\pi a^2\delta(k-k')\ , \tag{30a}$$

$$\langle\bar\phi(k')|\sigma_3|\psi(k)\rangle=-2\pi a^2\delta(k-k')\ , \tag{30b}$$

$$\langle\theta\phi_e|\sigma_3|\phi_o\rangle=2=\langle\phi_o|\sigma_3|\theta\phi_e\rangle\ , \tag{30c}$$

$$\langle\phi_e|\sigma_3|\chi\rangle=-2=\langle\chi|\sigma_3|\phi_e\rangle\ , \tag{30d}$$

where

$$a=\frac{k-i}{k+i}\ , \tag{31}$$

and the inner product is defined by

$$\langle u|m|v\rangle=\int_{-\infty}^{\infty}d\theta[u(\theta)]^T m(\theta)v(\theta)\ , \tag{32}$$

with no complex conjugations involved and where $[\ ]^T$ indicates the matrix transpose. By direct integration, one may verify the closure relation

$$\sigma_3\delta(\theta-\theta')=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,a^{-2}(|\bar\psi\rangle\langle\phi|-|\psi\rangle\langle\bar\phi|)+\tfrac{1}{2}(|\phi_o\rangle\langle\theta\phi_e|+|\theta\phi_e\rangle\langle\phi_o|-|\phi_e\rangle\times\langle\chi|-|\chi\rangle\langle\phi_e|) \tag{33}$$

in an obvious notation.

IV. FIRST-ORDER SOLUTION

We start by expanding v in the closure of the eigenfunctions of L;

$$v=\int_{-\infty}^{\infty}dk[g(k)\psi(k)+\bar g(k)\bar\psi(k)]+g_e\phi_e+g_o\phi_o+h_e\chi+h_o\theta\phi_e\ . \tag{34}$$

We also expand F [Eq. (15)] similarly

$$F=F_{\rm ext}-2i\eta A(\partial_{\tau_1}\bar x_0)|\phi_o\rangle+A(\partial_{\tau_1}\bar\alpha_0+2\xi\partial_{\tau_1}\bar x_0)|\phi_e\rangle+2A(4\eta^2-A^2)\epsilon^{-1}|\nu\rangle+i\frac{A}{\eta}(\partial_{\tau_1}\eta)|\chi\rangle-\frac{A}{\eta}(\partial_{\tau_1}\xi)|\theta\phi_e\rangle-i\eta[\partial_{\tau_1}(A/\eta)]|\sigma_3\phi_e\rangle\ , \tag{35}$$

where

$$F_{\rm ext}\begin{vmatrix}e^{-i\alpha}R\\-e^{i\alpha}R^*\end{vmatrix}, \tag{36}$$

and $|\nu\rangle$ is the state

$$|\nu\rangle=\frac{1}{\cosh^3\theta}\begin{bmatrix}1\\-1\end{bmatrix}. \tag{37}$$

Note that $|\nu\rangle$ and $|\sigma_3\phi_e\rangle$ are *not* linearly independent states.

Inserting (34) and (35) into (13) gives

$$\partial_{\tau_1}h_e=2\partial_{\tau_1}A-A\frac{\partial_{\tau_1}\eta}{\eta}+\tfrac{1}{2}i\langle\phi_e|\sigma_3|F_{\rm ext}\rangle\ , \tag{38a}$$

$$\partial_{\tau_1}h_0=i\frac{A}{\eta}\partial_{\tau_1}\xi-\tfrac{1}{2}i\langle\phi_o|\sigma_3|F_{\rm ext}\rangle\ , \tag{38b}$$

$$\partial_{\tau_1}g_e+4\eta^2 ih_e=-iA\left[\partial_{\tau_1}\bar\alpha_0+2\xi\partial_{\tau_1}\bar x_0+\frac{2}{\epsilon}(4\eta^2-A^2)\right]+\tfrac{1}{2}i\langle\chi|\sigma_3|F_{\rm ext}\rangle\ , \tag{39a}$$

$$\partial_{\tau_1}g_0-i4\eta^2h_0=-2\eta A\partial_{\tau}\bar x_0-\tfrac{1}{2}i\langle\theta\phi_e|\sigma_3|F_{\rm ext}\rangle\ , \tag{39b}$$

$$\partial_{\tau}g-4\eta^2(k^2+1)ig=\frac{1}{2\pi a^2}\eta\partial_{\tau_1}(A/\eta)\langle\bar\phi|\sigma_3|\sigma_3\phi_e\rangle+\frac{iA(4\eta^2-A^2)}{\pi a^2\epsilon}\langle\bar\phi|\sigma_3|\nu\rangle+\frac{i}{2\pi a^2}\langle\bar\phi|\sigma_3|F_{\rm ext}\rangle\ , \tag{40a}$$

$$\partial_{\tau}\bar g+4\eta^2(k^2+1)i\bar g=-\frac{1}{2\pi a^2}\eta\partial_{\tau_1}(A/\eta)\langle\phi|\sigma_3|\sigma_3\phi_e\rangle-\frac{iA(4\eta^2-A^2)}{\pi a^2\epsilon}\langle\phi|\sigma_3|\nu\rangle-\frac{i}{2\pi a^2}\langle\phi|\sigma_3|F_{\rm ext}\rangle \tag{40b}$$

upon using

$$\langle\phi_e|\sigma_3|\sigma_3\phi_e\rangle=4\ , \tag{41}$$

$$\langle\chi|\sigma_3|\nu\rangle=-2\ , \tag{42}$$

with all other inner products of ϕ_e, ϕ_o, $\theta\phi_e$, and χ with $\sigma_3\theta_e$ and ν vanishing. Now, one should observe in (38) that if h_e and h_0 are initially zero, we may maintain them zero for all τ simply by requiring η and ξ to evolve according to

$$\frac{\partial_{\tau_1}\eta}{\eta}=2\frac{\partial_{\tau_1}A}{A}+\frac{i}{2A}\langle\phi_e|\sigma_3|F_{\rm ext}\rangle\ , \tag{43a}$$

$$\partial_{\tau_1}\xi=\frac{\eta}{2A}\langle\phi_o|\sigma_3|F_{\rm ext}\rangle \ . \tag{43b}$$

If this is not done, then in general h_e and h_0 will become secular in τ_1. Similarly, we may take g_e and g_0 to be zero by taking

$$\partial_{\tau_1}\bar{x}_0=-\frac{i}{4A\eta}\langle\theta\phi_e|\sigma_3|F_{\rm ext}\rangle \tag{44a}$$

$$\partial_{\tau_1}\bar{\alpha}_0=-2\xi\partial_{\tau_1}\bar{x}_0-\frac{2}{\epsilon}(4\eta^2-A^2)+\frac{1}{2A}\langle\chi|\sigma_3|F_{\rm ext}\rangle \ . \tag{44b}$$

Then integration of (40) will give the solution for g and $\bar{g}$.

Note that A is still arbitrary. We have not constrained it in any way. And outside of A being near 2η [see (40)], we are still free to specify it.

Finally, we shall give the specific results for a damped soliton where

$$R=-i\gamma q \ , \tag{45}$$

which by (36) gives

$$F_{\rm ext}=-i\gamma A|\sigma_3\phi_e\rangle-i\gamma\epsilon|v\rangle \ . \tag{46}$$

In the above, γ may be a function of time. The evaluation of the inner products in (43) and (44) then gives

$$\partial_{\tau_1}(A^2e^{2\Gamma}/\eta)=0 \ , \tag{47a}$$

$$\partial_{\tau_1}\xi=0 \ , \tag{47b}$$

$$\partial_{\tau_1}\bar{x}_0=0 \ , \tag{47c}$$

$$\partial_{\tau_1}\bar{\alpha}_0=-2(3\eta^2-A^2)/\epsilon \ , \tag{47d}$$

where

$$\Gamma=\int_0^{\tau_1}\gamma(s)ds \ . \tag{48}$$

For the evaluation of (40), we will need

$$\langle\phi|\sigma_3|\nu\rangle=\langle\bar{\phi}|\sigma_3|\nu\rangle=\frac{\pi(k-i)^2}{4\cosh[(\pi/2)k]} \ , \tag{49a}$$

$$\langle\bar{\phi}|\sigma_3|\sigma_3\phi_e\rangle=-\langle\phi|\sigma_3|\sigma_3\phi_e\rangle=\frac{a\pi}{\cosh[(\pi/2)k]} \ , \tag{49b}$$

which then gives

$$\partial_\tau g+\epsilon\gamma g-4i\eta^2(k^2+1)g=\eta\frac{\partial_{\tau_1}(A/\eta)+\gamma(A/\eta)}{2a\cosh[(\pi/2)k]}+\frac{iA(k+i)^2(4\eta^2-A^2)}{4\epsilon\cosh[(\pi/2)k]} \ . \tag{50}$$

The equation for $\bar{g}$ follows from the symmetry

$$\bar{g}(k)=g^*(-k^*) \tag{51}$$

Once g is obtained, then q can be constructed from (8), (14), (19), (21), and (34), giving

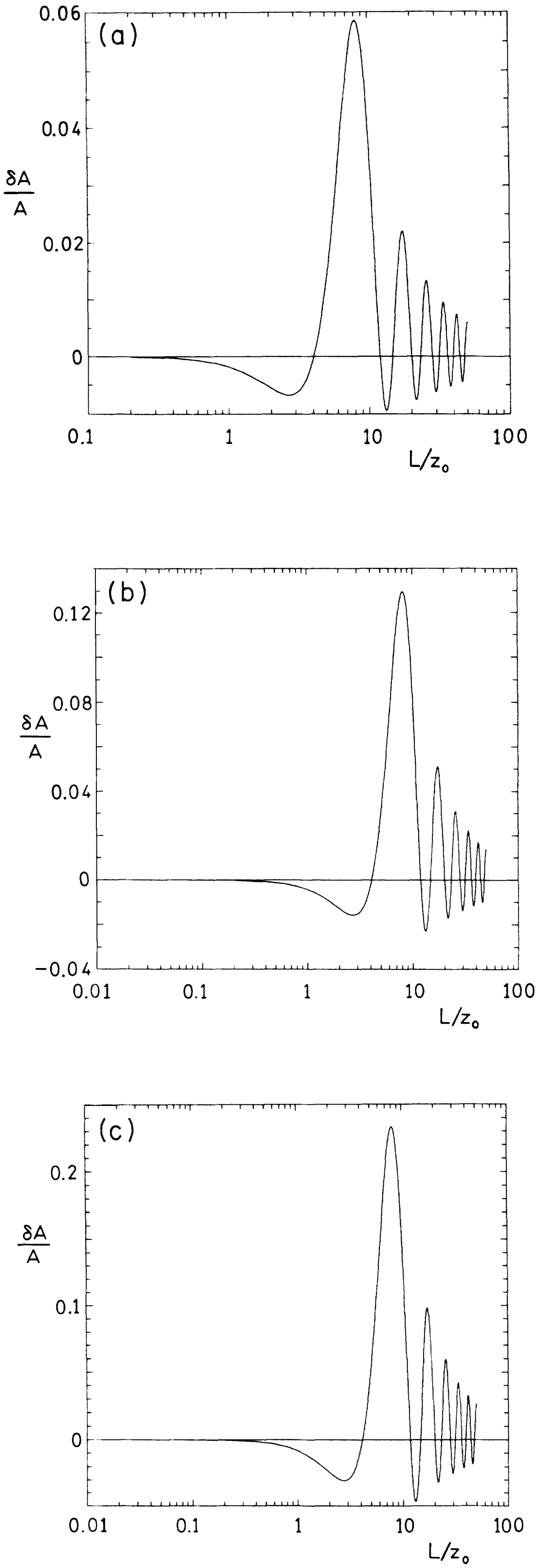

FIG. 1. First-order perturbation results for the fractional change in the area $\delta A/A$ of a Raman loss compensated soliton at the end of exactly one period for a period of (a) 30 km, (b) 40 km, and (c) 50 km. These curves are similar to the numerical calculations in Fig. 6 of Ref. 7, except that the x axis is the mirror image of that in Ref. 7.

$$q=\frac{Ae^{i\alpha}}{\cosh\theta}+\epsilon e^{i\alpha}\int_{-\infty}^{\infty}dk\,\bar{g}(k)e^{ik\theta}\left[1-\frac{2ike^{-\theta}}{(k+i)^2\cosh\theta}\right]$$
$$+\epsilon e^{i\alpha}\int_{-\infty}^{\infty}dk\frac{e^{ik\theta}}{(k+i)^2\cosh^2\theta}[\bar{g}(k)+g(k)]+\theta(\epsilon^2)\ , \qquad (52)$$

where α is given by (5), (10)–(12), and (47).

V. EVOLUTION OF SOLITON AREA

Let us now specialize the previous results to an optical soliton propagating in an optical fiber. We take

$$\xi=0\ . \qquad (53)$$

Obviously the area of q in (52) is complex although the phase α is now independent of θ. Simple straightforward integration gives

$$\int_{-\infty}^{\infty}q\,d\theta=A\pi e^{i\alpha}+2\pi\epsilon e^{i\alpha}a^2(k=0)\bar{g}(k=0)$$
$$+\epsilon e^{i\alpha}\int_{-\infty}^{\infty}dk\frac{\pi k}{(k+i)^2\sinh[(\pi/2)k]}$$
$$\times[g(k)-\bar{g}(k)]\ . \qquad (54)$$

Due to (51), the integral over k in (54) is pure imaginary and is 90° out of phase with the zeroth-order term $A\pi$. This part we ignore. Thus we only need $\bar{g}(k=0)$.

It is now time to choose how to couple the amplitude and the width. Based on the numerical observations,[7] the width of the soliton remains more constant than the amplitude. Thus we choose

$$\partial_{\tau_1}\eta=0\ , \qquad (55)$$

whence by (47a)

$$A=2\eta e^{-\Gamma}\ . \qquad (56)$$

Letting $g(k=0)=g_0A$, then (52) becomes

$$\partial_\tau g_0-4i\eta^2g_0=-\frac{i}{\epsilon}\eta^2(1-e^{-2\Gamma})\ , \qquad (57)$$

whose solution is

$$g_0=-\frac{i\eta^2}{\epsilon}\int_0^\tau(1-e^{-2\Gamma(\epsilon s)})e^{4i\eta^2(\tau-s)}ds\ , \qquad (58)$$

with $\Gamma(\tau)$ being the slow-time integral of the damping $\gamma(\tau)$ as given by (48). From (54), we have

$$\int_{-\infty}^{\infty}q\,d\theta=e^{i\alpha}\pi A[1+2\epsilon g_0+O(\epsilon^2)]\ , \qquad (59)$$

so it is the real part of g_0 that we want.

For the Raman compensated soliton, the damping has a complex form,[7] which is

$$\Gamma=\frac{\alpha_s}{2}\left[\tau-\tfrac{1}{2}L-\frac{L}{2}\frac{\sinh[\alpha_p(\tau-L/2)]}{\sinh(\alpha_pL/2)}\right] \qquad (60)$$

for $0<\tau<L$. It is periodic in τ of period L, where L is the distance between repeaters and τ is the distance along the fiber. The constants α_p and α_s for quartz fibers have the values of

$$\alpha_s=0.042/\text{km},\quad \alpha_p=0.067/\text{km}\ . \qquad (61)$$

Now it is simply a matter of numerically evaluating g_0 in (58). In order to compare with the results in Ref. 7, we have the relationship

$$z_0=\frac{\pi}{16\eta^2} \qquad (62)$$

between our η and their characteristic length z_0. In Fig. 1, we present our results for the standard quartz fiber Raman compensated case characterized by (61), for the three repeater distances of interest, $L=30$, 40, and 50 km. Here $\delta A/A$ is the real part of the fractional change in the area given in Eq. (59). If one compares these results to the numerical calculations presented in Fig. 6 of Ref. 7, one observes some remarkable similarities as well as some differences. First, the heights of the first peak are all almost identical to those in Ref. 7. Second, the depth of the first minimum is likewise almost identical as well as the asymptotic shape for small $L/z_0(\rightarrow 0)$. However, in our results, the first zero crossing occurs very close to $L/z_0=4$ for all three distances, whereas in Ref. 7, it varies from $L/z_0=3.5$ for $L=30$ km down to $L/z_0=2.5$ for $L=50$ km. Furthermore, the structure to the left of the main peak from the numerical calculations in Ref. 7 lacks the details that we find, simply because they did not give sufficient detail for resolving the structure of their oscillations. However, comparison with our Fig. 1 indicates that the oscillations are indeed real and should be present.

One must remember that our calculation is only a first-order calculation, and that the second-order terms could possibly be important, particularly in the $L=50$-km case. However, from the close agreements in the peaks and in the general trends, we suspect that, in general, the second-order contributions are probably small, if not insignificant. But even if these second-order terms were significant, still our first-order perturbation results do give a quick and excellent qualitative check of the numerical results.

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