Quantum analysis of light propagation in a parametric amplifier

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We develop a quantization procedure for treating the propagation of light. This formalism is particularly effective in a dispersive nonlinear medium characterized by its macroscopic linear and nonlinear polarizability. We demonstrate it by analyzing the propagation of light in a multimode degenerate parametric amplifier.

I. INTRODUCTION

The recent experiments on nonclassical states of light have called for a full quantum analysis of the electromagnetic field.[1] Surprisingly, whereas this analysis has been fully carried out for fields inside cavities, there are still difficulties related to propagating fields, especially inside a dispersive medium. To our knowledge, this problem has not yet been solved in a fully consistent quantum manner for a nonlinear dispersive medium, when many modes are excited.

The standard method[2] consists of writing the Hamiltonian in a given volume V, and demanding periodicity in space. For propagating fields, the space evolution is then replaced by a time evolution, by identifying the time and space variable by the equation $z=ct$. The length of the nonlinear medium is then replaced by an effective interaction time.[3–6] This method has two main limitations. The first one is that, by identifying the space evolution with time evolution, we lose one variable. As mentioned by Shen,[3] this formalism can therefore describe only steady-state propagation. The second problem is that this procedure cannot be applied rigorously to a dispersive medium, where each frequency propagates at a different velocity.

Another widely successful approach consists of using an operator form of the classical Maxwell equations for the fields, and imposing at some stage commutation relations (CR) for a set of creation-annihilation operators.[7–9] However, the imposed CR for this set of operators have not been fully justified with respect to the canonical quantization procedure, and the physical interpretation of these operators has not been emphasized.

Our approach aims at presenting a somewhat more rigorous (even though still phenomenological) basis for these equations, and at giving a physical interpretation of these operators. In Sec. II, we present the main ideas of our approach. We develop the mathematical formalism in Sec. III for the case of a free field in the vacuum, and analyze the case of a linear dielectric medium in Sec. IV. We then turn to the nonlinear case, and treat the combined creation and propagation of light in a degenerate parametric amplifier in Sec. V. We conclude in Sec. VI.

II. MAIN IDEAS

Let us first identify the tools we have at our disposal. We work in the Heisenberg picture, and consider, e.g., the electric field $\hat{E}(z,t)$ (all our operators shall be denoted by a caret). In all this work we consider only the one-dimensional case, with fields propagating in the $+z$ direction only, and with an electric and magnetic fields orthogonal to each other and to the direction of propagation. For notational simplicity, we take the cross section of the beam to be equal to unity, and drop it in the calculations. All our operators are therefore defined relative to a unit cross section.

In quantum field theory, we know that the generator for time evolution is the Hamiltonian $\hat{H}$ so that, e.g., $\hat{E}(z,t)$ satisfies the equation of motion

$$i\hbar\frac{\partial\hat{E}(z,t)}{\partial t}=[\hat{E}(z,t),\hat{H}]\ , \qquad (1)$$

and the generator for space propagation is the momentum operator,[1,3,10] $\hat{G}$:

$$-i\hbar\frac{\partial\hat{E}(z,t)}{\partial z}=[\hat{E}(z,t),\hat{G}]\ . \qquad (2)$$

These equations are of course correct for any operator, and therefore give a complete description of the time and space evolution of the field.

In the usual field theory, we define [in the one-dimensional (1D) case]

$$\hat{H}=\int_0^L\hat{\mathcal{H}}(z)dz\ , \qquad (3)$$

where $\hat{\mathcal{H}}(z)$ is the Hamiltonian (or energy) density, and L is the quantization length. By doing so, we immediately run into a problem. On the one hand we know that the time dependence of the field $\hat{E}(z,t)$ is the same inside and outside a dielectric medium (the frequency remains constant), and on the other hand we also know that the energy density of the field changes, so that $\hat{H}$ should depend on the medium. To solve this problem, Abram[1,10] suggested changing the quantization volume accordingly, so that $\hat{H}$ would remain constant. This approach runs into trouble when the refraction index depends on the frequency, so that each mode should be quantized in a different volume.

Our approach is based on the fact that, whereas the energy *density* depends on the medium, the energy *flux*, i.e., the flux of the Poynting vector, does not. This leads us to the realization that the important quantity seems to be the flux, and not, as is usually assumed, the density. This point has been already emphasized by Abram[10] and by

Caves and Crouch.[9] Surprisingly enough, the classical theory is in agreement with this point. Indeed, the classical analysis of nonlinear effects like second harmonic generation or parametric down conversion[11] shows that, in order to obtain simple equations of propagation, it is useful to introduce a quantity proportional to the photon-flux amplitude (its square is proportional to the photon flux). Remembering that the phenomenological approach that consists of taking the classical equations and quantizing them[7–9] is indeed very successful in analyzing the experiments, we shall take this fact as a starting point, and define, e.g., our Hamiltonian in terms of the flux of energy.

Since we still need a Hamiltonian that describes an energy, we have to define it by the integration of the energy flux at a given point z_0 over a given time T. Therefore the second main point of our approach is to replace the space integration of a density over a given quantization length L by a time integration of a flux over a given time T. This means that, instead of demanding space periodicity of the fields, we shall demand time periodicity.

The third point of our approach is based on the fact that, as mentioned above, whereas the wave vector of a monochromatic field does depend on the medium, the frequency does not. Therefore, instead of writing the field in terms of spatial modes and time-dependent mode operators, we shall write it in terms of temporal modes and space-dependent mode operators. The advantage is that the temporal modes remain the same inside or outside a dielectric medium, and that the space evolution of the mode operators can now be obtained by means of the momentum operator. Indeed, this seems to be more suitable to the propagation problem: we define all the quantities at a given plane $z=z_0$ for all times, and try to obtain the propagation towards $z \geq z_0$. Let us now formalize these principles, and show that they indeed enable us to give a quantum-mechanical description of the propagation.

III. QUANTIZATION IN THE VACUUM

As a first step, let us quantize the field in the vacuum, to show that our method is consistent and that we recover all the known results. We shall then use it in a linear dielectric medium, and later in a nonlinear one.

Instead of quantizing the field in a large volume V (or in the 1D case in a large length L), and demanding spatial periodicity, we assume a time periodicity T of the field (T should be much larger than any relevant time, and will be taken to infinity at the end of the calculations). Then, instead of writing the field in terms of spatial modes (thus performing a Fourier analysis of the space variable z into wave vectors k_m), we write it in terms of temporal modes (perform a Fourier analysis of the time variable t into *discrete* frequencies ω_m, where $\omega_m = 2m\pi/T$). We specialize to the case of plane waves propagating in direction $+z$ only, with an electric and a magnetic field orthogonal to each other and to the direction of propagation, and for simplicity treat them as scalars. The electric field can be written as

$$\hat{E}(z,t)=\hat{E}^{(+)}(z,t)+\hat{E}^{(-)}(z,t) ,$$

where

$$\hat{E}^{(+)}(z,t)=\sum_m \left[\frac{\hbar\Omega_m}{2\epsilon_0 cT}\right]^{1/2}[\hat{a}(z,\omega_m)e^{-i\omega_m t}]$$

and

$$\hat{E}^{(-)}(z,t)=\hat{E}^{(+)}(z,t)^\dagger , \tag{4}$$

where $\hat{a}(z,\omega_m)$ and their conjugates form a set of localized creation and annihilation operators [instead of the usual $\hat{a}(k_m,t)$]. In this equation, the cross section of the beam should be introduced in the denominator to obtain to usual units for the electric field. For notational simplicity, and since we have chosen the cross section to be unity, we discard it in this and all other equations. By using these operators, we go from the usual initial-condition approach to the quantization to a new boundary-condition approach, which is the analog of the classical theory.[12]

We also write the magnetic field

$$\hat{B}(z,t)=\sum_m \left[\frac{\hbar\omega_m}{2\epsilon_0 cT}\right]^{1/2}\left[\frac{1}{c}\right][\hat{a}(z,\omega_m)e^{-i\omega_m t}+\text{H.c.}] . \tag{5}$$

Following the classical theory, we now write the normal ordered Poynting vector

$$\mathbf{S}(z,t)=\frac{1}{\mu_0}[\hat{E}^{(-)}(z,t)\hat{B}^{(+)}(z,t)+\text{H.c.}]\mathbf{z} , \tag{6}$$

where $\mathbf{z}$ is a unit vector in the z direction. Since we took the cross section of the beam to be unity, the intensity of the Poynting vector gives the energy flux

$$\hat{S}(z,t)=\sum_{m,m'}\frac{\hbar}{2T}(\omega_m\omega_{m'})^{1/2} \times[\hat{a}^\dagger(z,\omega_m)a(z,\omega_{m'})e^{i(\omega_m-\omega_{m'})t}+\text{H.c.}] , \tag{7}$$

and after integration over the period T, we obtain the Hamiltonian

$$\hat{H}(z)\equiv\int_{t_0}^{t_0+T}\hat{S}(z,t)dt = \sum_m(\hbar\omega_m)\hat{a}^\dagger(z,\omega_m)\hat{a}(z,\omega_m) . \tag{8}$$

Let us emphasize that, since all the frequencies are multiples of $2\pi/T$, we do not need to take the limit $T\to\infty$ at this stage. Indeed, as in the usual case with the space integration, the period T has to be kept when working with the operators, and the limit can be taken for physical quantities (quantum averages).

The interpretation of Eq. (8) is now straightforward. The operator

$$\hat{N}(z_0,\omega_m)\equiv\hat{a}^\dagger(z_0,\omega_m)\hat{a}(z_0,\omega_m) \tag{9}$$

is the number operator of the photons of frequency ω_m going through the plane $z=z_0$ during a period T. Multiplying each operator $\hat{N}(z_0,\omega_m)$ by the corresponding energy of each photon, and summing over the frequencies gives the total energy going through the z_0 plane during the period T. The $\hat{a}^\dagger(z,\omega_m)$ and $\hat{a}(z,\omega_m)$ are therefore the creation and annihilation operators creating or annihilating one phonon of frequency ω_m at point z during the period T. We now state the equal-space commutation relations (ESCR):

$$[\hat{a}(z,\omega_i),\hat{a}^\dagger(z,\omega_j)]=\delta_{i,j} \tag{10}$$

(these ESCR have already been used by Caves and Crouch[9]). In order to relate our approach to the canonical quantization procedure, we have to check that these ESCR are consistent with the usual equal-time commutation relations (ETCR). We shall do this in Sec. IV for the more general case of a field in a dielectric medium. The operator $\hat{a}^\dagger(z,\omega_i)$ can now be applied to the vacuum in order to create various states. For example, the state $\hat{a}^\dagger(z_0,\omega_i)|0\rangle$ is the plane wave in which one photon of frequency ω_i goes through the plane $z=z_0$ during the period T. It is therefore an eigenstate of $\hat{N}(z_0,\omega_i)$ with eigenvalue 1. In this respect, even though the operator is taken at $z=z_0$, this photon is not localized at z_0, which would be in contradiction with the fact that it also have a well-defined frequency, but is delocalized over the length $L=cT$, exactly as in the usual approach. In a fashion completely similar to the usual case, we can also create multimode states, coherent states, etc. The difference is that the $|n\rangle$ state, for example, is not interpreted as n photons in a given volume V, or rather in our case in a given length L, but as n photons going through a given plane during time T. Of course, in the vacuum, the two interpretations are equivalent once we write $L=cT$. However, we shall see that the new interpretation is much more useful in a dispersive medium. It is now straightforward to check that the CR [Eq. (10)] lead to the correct Heisenberg equations of motion, for example,

$$i\hbar\frac{\partial\hat{E}(z,t)}{\partial t}=[\hat{E}(z,t),\hat{H}(z)] .$$

For future use, we can also define a photon flux amplitude,

$$\hat{e}(z,t)=\sum_m \frac{1}{\sqrt{T}}[\hat{a}(z,\omega_m)e^{-i\omega_m t}+\text{H.c.}] , \tag{11}$$

and a photon flux operator,

$$\hat{I}(z,t)=\hat{e}^{(-)}(z,t)\hat{e}^{(+)}(z,t) . \tag{12}$$

We shall use these operators in the analysis of the detection process. Of course, we immediately find the relationship with the number operator:

$$\hat{N}(z)=\int_0^T\hat{I}(t)dt=\sum_m \hat{N}(z,\omega_m) , \tag{13}$$

where $\hat{N}(z,\omega_m)$ is defined in Eq. (9).

To find the spatial equations of motion, we now have to define the momentum operator. Following the same approach for the Hamiltonian, we shall define it as the momentum flux, integrated over the period T. The definition that is relevant for quantum optics is the Minkovski one:[10] classically the momentum *density* is $\mathbf{p}(z,t)=\mathbf{D}(z,t)\times\mathbf{B}(z,t)$, where $\mathbf{D}(z,t)$ is the electric displacement field and is given by $\mathbf{D}(z,t)=\epsilon_0\mathbf{E}(z,t)+\mathbf{P}(z,t)$, $\mathbf{P}(z,t)$ being the polarization of the medium. In the vacuum, of course, the polarization being zero, $\mathbf{D}$ is just proportional to the electric field, but we shall write the momentum in terms of $\mathbf{D}$ in order to facilitate the transition to a dielectric medium. Quantum mechanically, we write the normal ordered momentum density operator

$$\mathbf{g}_0(z,t)=[\hat{D}^{(-)}(z,t)\hat{B}^{(+)}(z,t)+\text{H.c.}]\mathbf{z} \tag{14}$$

($\mathbf{z}$ is again a unit vector in the z direction), and the momentum flux is therefore

$$\hat{g}(z,t)=[\hat{D}^{(-)}(z,t)\hat{B}^{(+)}(z,t)+\text{H.c.}]c . \tag{15}$$

We can also write it in a more general form, which will remain correct in a dielectric medium:

$$\hat{g}(z,t)=[\hat{D}^{(-)}(z,t)\hat{E}^{(+)}(z,t)+\text{H.c.}] . \tag{16}$$

Similar to the Hamiltonian, its integration over T gives the momentum operator $\hat{G}(z)$:

$$\hat{G}(z)\equiv\int_{t_0}^{t_0+T}\hat{g}(z,t)dt . \tag{17}$$

In the vacuum, $\hat{D}=\epsilon_0\hat{E}$, and using Eqs. (4) and (5), we can easily perform the integration and obtain

$$\hat{G}(z)=\sum_m(\hbar k_m)\hat{a}^\dagger(z,\omega_m)\hat{a}(z,\omega_m) , \tag{18}$$

where $k_m=\omega_m/c$ is the wave vector. As with the Hamiltonian [Eq. (8)] the interpretation of $\hat{G}(z)$ is straightforward: the number of photons times their momentum, which gives the full momentum of the field.

For our formalism to be consistent, the ESCR [Eq. (10)] need to be conserved. This can be easily checked, but we shall postpone it till the next section, where we shall treat the case of a linear dielectric medium, the vacuum being just a particular case thereof.

IV. QUANTIZATION IN A LINEAR DIELECTRIC MEDIUM

We now turn to the case of a linear dielectric medium. In this medium, the operators $\hat{a}(z,\omega_m)$ and $\hat{a}^\dagger(z,\omega_m)$, which are related to the *flux,* do not change (if there is no reflection at the boundary, the photon flux is the same inside and outside a dielectric medium). However, the field operators do change according to the refraction index $n(\omega_m)$ of the medium[10] (classically, we have $E_{\text{in}}=E_{\text{out}}/\sqrt{n}$, $b_{\text{in}}=B_{\text{out}}\sqrt{n}$, n being the refraction index) and we obtain

$$\hat{E}(z,t)=\sum_m\left[\frac{\hbar\omega_m}{2\epsilon_0 cTn(\omega_m)}\right]^{1/2}[\hat{a}(z,\omega_m)e^{-i\omega_m t}+\text{H.c.}] ,$$

$$\hat{B}(z,t)=\sum_m\left[\frac{\hbar\omega_m n(\omega_m)}{2\epsilon_0 c^3T}\right]^{1/2}[\hat{a}(z,\omega_m)e^{-i\omega_m t}+\text{H.c.}] ,$$

$$\hat{D}(z,t)=\sum_m\left[\frac{\hbar\omega_m\epsilon_0}{2cTn(\omega_m)}\right]^{1/2}\times n(\omega_m)^2[\hat{a}(z,\omega_m)e^{-i\omega_m t}+\text{H.c.}] . \tag{19}$$

Since the Poynting vector in the dielectric is the same as in the vacuum, the Hamiltonian $\hat{H}(z)$ [Eq. (8)] remains the same. This is in full agreement with the known fact that the frequency of the light in a dielectric medium is the same as in the vacuum. We now turn to the momentum operator $\hat{G}_l(z)$ (the index l meaning a linear medium) [Eqs. (16) and (17)], and after straightforward calculation obtain

$$\hat{G}_l(z)=\sum_m (\hbar k_m)\hat{a}^{\dagger}(z,\omega_m)\hat{a}(z,\omega_m) , \tag{20}$$

where now $k_m=n(\omega_m)\omega_m/c$ is the wave vector in the medium. The expression for $\hat{G}_l(z)$ in the dielectric medium is exactly the same as in the vacuum, with only a new definition of the wave vector.

It is now easy to check that the ESCR are conserved. We use the equation for space propagation [Eq. (2)], and get

$$\frac{\partial \hat{a}(z,\omega_i)}{\partial z}=\frac{i}{\hbar}[\hat{a}(z,\omega_i),\hat{G}_l(z)]=ik_i\hat{a}(z,\omega_i) , \tag{21}$$

so that

$$\frac{\partial}{\partial z}[\hat{a}(z,\omega_i),\hat{a}^{\dagger}(z,\omega_j)]=0 . \tag{22}$$

Integrating Eq. (21), we obtain the z dependence of the operator $\hat{a}(z,\omega_i)$:

$$\hat{a}(z,\omega_i)=\hat{a}(0,\omega_i)e^{ik_iz} , \tag{23}$$

and use it to calculate commutation relations at different points:

$$[\hat{a}(z,\omega_i),\hat{a}^{\dagger}(z',\omega_j)]=\delta_{i,j}e^{ik_i(z-z')} . \tag{24}$$

Another important test for our approach is to check the consistency with the canonical quantization procedure. In a dielectric medium, the two canonical conjugate fields are the vector potential $\hat{A}(z,t)$ and $-\hat{D}(z,t)$.[13] Therefore, we now write the vector potential operator (classically $E=-\partial A/\partial t$)

$$\hat{A}(z,t)=(-i)\sum_m \left[\frac{\hbar}{2\epsilon_0 cT\omega_m n(\omega_m)}\right]^{1/2} \times[\hat{a}(z,\omega_m)e^{-i\omega_m t}-\text{H.c.}] , \tag{25}$$

and use it to calculate the ETCR of the two conjugates fields. Using Eqs. (19) and (25), and the CR for the creation and annihilation operators at different points [Eq. (24)], we easily obtain

$$\begin{aligned}[\hat{A}(z,t),-\hat{D}(z',t)]&=[\hat{A}^{(-)}(z,t),-\hat{D}^{(+)}(z',t)]\\&\quad+[\hat{A}^{(+)}(z,t),-\hat{D}^{(-)}(z',t)]\\&=i\hbar\sum_m \frac{n(\omega_m)}{cT}e^{ik_m(z-z')} .\end{aligned} \tag{26}$$

By taking now the limit $T\to\infty$

$$\frac{1}{T}\sum_m \to\frac{1}{2\pi}\int d\omega , \tag{27}$$

we obtain

$$[\hat{A}(z,t),-\hat{D}(z',t)]=i\hbar\delta(z-z') , \tag{28}$$

which is the canonical ETCR for the one-dimensional case. This approach, consisting of assuming a set of CR for the creation and annihilation operators, and deriving from it the canonical ETCR is widely used in quantum field theory. The difference is that, since we use localized creation and annihilation operators, we assumed a set of ESCR for these operators, and derived the canonical ETCR. This shows that our quantization procedure is consistent with the canonical procedure, at least in the linear case. It can be used to calculate propagation properties of nonclassical light in a linear medium. One has to specify the state at an initial point (e.g., $z=0$), and it is then possible to calculate any quantum average at any other point. This is in agreement with Maxwell's equations: knowing both the electric and magnetic fields at the boundary is enough to determine the evolution of the electromagnetic field. Let us emphasize here that, since we are working in a Heisenberg-like picture, all the space and time dependence is in the operators. Another important point is that this formalism can accommodate explicitly time-dependent problems (pulses), but combining, at the initial point, states with different frequencies.

To demonstrate the use of this approach for nonlinear interactions, we now analyze the generation and propagation of light in a degenerate parametric amplifier.

V. QUANTIZATION IN A NONLINEAR MEDIUM

To treat the nonlinear case, we have to rely on further approximations. By developing the field into temporal modes, we assume *a priori* that the time dependence of each mode is known, and is not changed by the interaction. Moreover, we still describe our various fields by Eqs. (19) and (25), which means that the relationship between the fields at a given point is still given by the linear polarization only. The nonlinearity therefore only couples various temporal modes, and influences their propagation properties. This corresponds to the slowly varying approximation that is used in the classical treatment of wave propagation, and is therefore correct only for small nonlinearities.

To treat the parametric amplifier, we first separate the field into two parts: the pump field $\mathcal{E}(z,t)=\frac{1}{2}(|\mathcal{E}|e^{-i(\omega_p t-k_p z)}+\text{c.c.})$ is taken as a monochromatic nondepleted classical field, of frequency ω_p, and the down-converted field $\hat{E}(z,t)$, which is nonmonochromatic but centered about the frequency $\omega_0=\omega_p/2$. This implies that the nonlinear crystal satisfies the phase matching condition at ω_0: $n(\omega_p)=n(\omega_0)$ (as shown later, the bandwidth will be fixed by phase mismatch at frequencies away from ω_0). The Hamiltonian operator is still given by Eq. (8), so that the time evolution of the fields at a fixed point remains the same. The momentum operator, however, changes, due to the nonlinear polarization added to the displacement field $\hat{D}(z,t)$. In our case, we consider only the second-order polarization (classically $P^{(2)}=\chi^{(2)}EE$, where $\chi^{(2)}$ is the second-order nonlinear polarizability). Therefore, the positive-frequency part of the polarization that will contribute to the displacement field of the down-converted light is

$$\hat{P}_{nl}^{(+)}(z,t)=\chi^{(2)}\mathcal{E}^{(+)}(z,t)\hat{E}^{(-)}(z,t) .$$

From Eq. (16), we now obtain the nonlinear part of the momentum flux operator:

$$\hat{g}_{nl}(z,t)=\chi^{(2)}[\mathcal{E}^{(+)}(z,t)\hat{E}^{(-)}(z,t)\hat{E}^{(-)}(z,t)+\text{H.c.}] , \tag{29}$$

and integration over t gives the momentum operator [Eqs. (17) and (19)]:

$$\hat{G}_{nl}(z)=\sum_m \frac{\lambda(\epsilon_m)\hbar}{4}[\hat{a}^\dagger(z,\omega_0+\epsilon_m)\times\hat{a}^\dagger(z,\omega_0-\epsilon_m)e^{ik_pz}+\text{H.c.}] ,$$

where

$$\epsilon_m\equiv\omega_m-\omega_0$$

and

$$\lambda(\epsilon_m)\equiv\frac{\chi^{(2)}|\mathcal{E}|}{\epsilon_0 c}\left[\frac{\omega_0+\epsilon_m}{n(\omega_0+\epsilon_m)}\frac{\omega_0-\epsilon_m}{n(\omega_0-\epsilon_m)}\right]^{1/2} \tag{30}$$

is the coupling constant between the different modes (all the frequencies are, of course, integer multiples of $2\pi/T$). We can now add the linear and nonlinear parts, obtain the full $\hat{G}(z)=\hat{G}_l(z)+\hat{G}_{nl}(z)$, and use Eqs. (2), (10), (20), and (30) to calculate the space dependence of, e.g., operator $\hat{a}(z,\omega_0+\epsilon_m)$:

$$\frac{\partial\hat{a}(z,\omega_0+\epsilon_m)}{\partial z}=ik(\epsilon_m)\hat{a}(z,\omega_0+\epsilon_m)+\frac{i\lambda(\epsilon_m)}{2}e^{ik_pz}\hat{a}^\dagger(z,\omega_0-\epsilon_m) , \tag{31}$$

where $k(\epsilon_m)$ is the wave vector at frequency $\omega_0+\epsilon_m$, and we used the fact that $\lambda(\epsilon_m)=\lambda(-\epsilon_m)$. This equation is similar to the one obtained by Caves and Crouch,[9] and used by Crouch[14] in his analysis of a parametric amplifier. Our derivation of this equation is based on the use of the momentum operator, and can be better justified in terms of standard quantum theory. Equation (31) is also analogous to the classical equation of propagation in a parametric amplifier obtained by Yariv for two modes.[11] Here we have an infinity of modes, but they are only coupled two by two ($\omega_0-\epsilon_m$ with $\omega_0+\epsilon_m$). An interesting point is that, even in the classical treatment of Yariv,[11] it was found more convenient to work with photon flux amplitudes. The main reason is that, with this choice, he obtained $\lambda(\epsilon_m)=\lambda(-\epsilon_m)$, which made the integration of the coupled differential equations easier. In the quantum case, this equality has much more meaning, since it is the condition for the conservation of CR [Eq. (10)] along the z axis, which is necessary for the consistency of the quantum model, and which is very easily obtained from Eq. (31). To solve Eq. (31), we need to introduce the phase mismatch $\Delta k(\epsilon_m)\equiv k_p-k(\epsilon_m)-k(-\epsilon_m)$ at frequencies away from the central one ω_0. We find

$$\Delta_k(\epsilon_m)=-\left[2\frac{\partial n}{\partial\omega}\bigg|_{\omega_0}+\omega_0\frac{\partial^2 n}{\partial\omega^2}\bigg|_{\omega_0}\right]\frac{\epsilon_m^2}{c} , \tag{32}$$

and, after straightforward integration,[11] obtain

$$\hat{a}(z,\omega_0+\epsilon_m)=[\mu(z,\epsilon_m)\hat{a}(0,\omega_0+\epsilon_m)+i\nu(z,\epsilon_m)\hat{a}^\dagger(0,\omega_0-\epsilon_m)]\times e^{i[\Delta k(\epsilon_m)z/2+k(\epsilon_m)z]} , \tag{33}$$

where $\mu(z,\epsilon_m)$ and $\nu(z,\epsilon_m)$ are defined differently according to the value of ϵ_m. For ϵ_m satisfying $\lambda(\epsilon_m)\geq|\Delta K(\epsilon_m)|$, we have

$$\mu(z,\epsilon_m)=\cosh s(\epsilon_m)z-i\frac{\Delta k(\epsilon_m)}{2s(\epsilon_m)}\sinh s(\epsilon_m)z ,$$

$$\nu(z,\epsilon_m)=\frac{\lambda(\epsilon_m)}{2s(\epsilon_m)}\sinh s(\epsilon_m)z , \tag{34}$$

and

$$s(\epsilon_m)=\tfrac{1}{2}[|\lambda(\epsilon_m)^2-\Delta k(\epsilon_m)^2|]^{1/2} .$$

The definitions are completely similar in the case $\lambda(\epsilon_m)\leq|\Delta k(\epsilon_m)|$, with only the replacement of $\cosh s(\epsilon_m)z$ and $\sinh s(\epsilon_m)z$ by $\cos s(\epsilon_m)z$ and $\sin s(\epsilon_m)z$. It is easy to check that $|\mu|^2-|\nu|^2=1$, so that these equations describe the well-known Bogoliubov transformation for squeezing. The down-converted field is therefore in a multimode squeezed state. We immediately see from Eqs. (33) and (34) that when $|\Delta k(\epsilon_m)|\gg\lambda(\epsilon_m)$, the coupling between the frequencies $\omega_0+\epsilon_m$ and $\omega_0-\epsilon_m$ is negligible, so that the corresponding modes evolve as in a linear medium, with no amplification. This justifies our assumption that the phase mismatch at frequencies away from ω_0 fixes the bandwidth of the down-converted field.

To check the consistency of our approach in this case, we can again calculate the ETCR for the two conjugates fields, $\hat{A}(z,t)$ and $-\hat{D}(z,t)$. The calculations in this case are somewhat cumbersome, and we outline them in the Appendix. The result is that, in the slowly varying approximation, the ETCR are still verified, so that our approach is still consistent with the canonical quantization procedure.

To obtain experimental results, we now have to link the fields inside the crystal to the fields at the detector. Following the usual approach in quantum optics,[5,6,15] we assume that our detectors respond to photon flux, and not to energy flux, and therefore use the photon-flux amplitude operator $\hat{e}(z,t)$ defined in Eq. (11) and the photon-flux operator $\hat{I}(z,t)$ defined in Eq. (12), whose average at the detector gives the intensity. Since these operators are written only in terms of the creation and annihilation operators, their expression is the same inside and outside the medium (conservation of the flux), and therefore all the averages at the detector can be also taken at the end of the crystal. We now specialize to a nonlinear crystal of length l, with only the strong pump input (the down-converted light is therefore in the vacuum state). We neglect reflections at the end of the crystal, and also any losses during the propagation to the detector. Using Eqs. (12) and (33), we obtain the intensity at the detector:

$$\langle 0|\hat{I}(l,t)|0\rangle=\frac{1}{2\pi}\int d\epsilon\,\nu(l,\epsilon)^2\ , \tag{35}$$

where we now have taken the limit $(1/T)\sum_m \rightarrow (1/2\pi)\int d\epsilon$.

The quantum properties of this light are best analyzed by means of a standard two-port homodyne detection scheme,[15] where the light is mixed at a beam splitter with a strong coherent local oscillator (LO) $\varepsilon(z,t)$ of frequency ω_0, the photocurrents obtained in the two branches are subtracted, and a frequency analysis of the resulting current is performed. A slight difficulty here is that the phase of each frequency of the squeezed light is different, so that we cannot assume that the usual homodyne detection scheme measures a given quadrature of the field (the relative phase of the local oscillator depends on the frequency). We can, however, use our approach,[15] linking the noise in the detectors to various averages of the field. We assume a strong local oscillator with infinite coherence time, detectors with unity efficiency, and discard the propagation from the parametric amplifier to the detectors by taking equal optical lengths for both paths. In this case, the noise in the photocurrent is given by Eq. (13) of Ref. 15:

$$N^2(\eta)=N_0^2[1+y(\eta)]\ , \tag{36}$$

where $N(\eta)$ is the noise at frequency η, N_0 is the standard quantum limit (SQL), and $y(\eta)=\int g_S(\tau)e^{-i\eta\tau}d\tau$ is the Fourier transform of the correlation function $g_S(\tau)$ given by

$$g_S^{(2)}(\tau)=\frac{\langle \varepsilon^{(-)}(t)\varepsilon^{(-)}(t+\tau)\rangle\,\langle \hat{e}^{(+)}(t+\tau)\hat{e}^{(+)}(t)\rangle}{\langle \varepsilon^{(-)}(t)\varepsilon^{(+)}(t)\rangle}+\frac{\langle \varepsilon^{(-)}(t)\varepsilon^{(+)}(t+\tau)\rangle\,\langle \hat{e}^{(-)}(t+\tau)\hat{e}^{(+)}(t)\rangle}{\langle \varepsilon^{(-)}(t)\varepsilon^{(+)}(t)\rangle}+\text{c.c.} \tag{37}$$

In this equation, the squeezed field is taken at the end of the crystal, and we did not write explicitly the space parameter. The correlation function $g_S(\tau)$ is normalized with respect to the LO strength, so that the LO enters only via the phase $\varphi_{\rm LO}$:

$$\begin{aligned}\frac{\langle \varepsilon^{(-)}(t)\varepsilon^{(-)}(t+\tau)\rangle}{\langle \varepsilon^{(-)}(t)\varepsilon^{(+)}(t)\rangle}&=e^{i(2\omega_0 t+\omega_0\tau+2\varphi_{\rm LO})}\ ,\\ \frac{\langle \varepsilon^{(-)}(t)\varepsilon^{(+)}(t+\tau)\rangle}{\langle \varepsilon^{(-)}(t)\varepsilon^{(+)}(t)\rangle}&=e^{-2i\omega_0\tau}\ .\end{aligned} \tag{38}$$

We now have to calculate the two two-time averages of the squeezed light. Using Eqs. (11), (33), and (34), and replacing the sum by an integral as in Eq. (35), we obtain

$$\begin{aligned}\langle \hat{e}^{(-)}(t+\tau)\hat{e}^{(+)}(t)\rangle&=\frac{1}{2\pi}\int \nu(l,\epsilon)^2e^{i(\omega_0+\epsilon)\tau}d\epsilon\ ,\\ \langle \hat{e}^{(+)}(t+\tau)\hat{e}^{(+)}(t)\rangle&\\ &=\frac{i}{2\pi}\int \mu(l,\epsilon)\nu(l,\epsilon)e^{-i[2\omega_0 t+(\omega_0+\epsilon)\tau-k_pl]}d\epsilon\ .\end{aligned} \tag{39}$$

It is now easy to calculate $g_S(\tau)$ and its Fourier transform $y(\eta)$:

$$\begin{aligned}y(\eta)=2\nu(l,\eta)^2-2|\mu(l,\eta)|\nu(l,n)&\\ \times\sin[2\varphi_{\rm LO}+\theta(l,\eta)+k_pl]\ ,&\end{aligned} \tag{40}$$

where $\varphi_{\rm LO}$ is the phase of the LO and $\theta(l,\eta)$ is the phase of $\mu(l,\eta)$. We recover the well-known result that the noise can be reduced under the SQL by an adequate choice of the LO phase. We analyze this more precisely in Fig. 1, where we draw the noise power as a function of the frequency of analysis η. If we fix the phase of the LO so as to get a noise minimum at $\eta=0$ (dotted line), the phase differences between the different frequencies of the squeezed light reduce the noise-reduction bandwidth. It is, however, possible to obtain a noise reduction over a broader bandwidth (solid line) by changing the phase of the LO according to the frequency of analysis, so as to remain at the minimum. The same type of curve (solid line) has been obtained by Crouch,[14] by using the usual interpretation of homodyne detection in terms of the field quadratures.

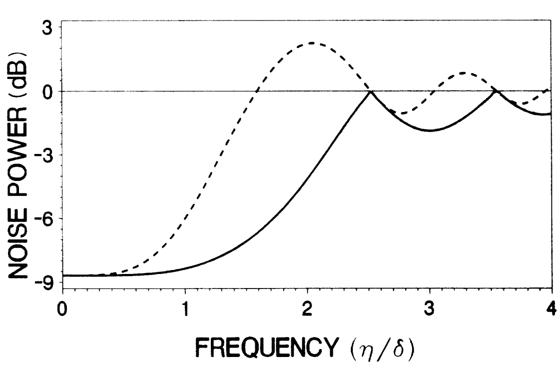

FIG. 1. Noise power as a function of the frequency η. The dotted curve is obtained by fixing the phase of the LO so as to get a minimum at $\eta=0$. The solid curve is the absolute minimum, obtained by varying the phase of the LO according to the frequency of analysis. The parameters are the following: $\lambda l=1$, where λ is the coupling constant, taken as independent of the frequency [Eq. (30)], and l is the length of the crystal; δ is the frequency scale of the spectrum, and is defined by $\Delta k(\epsilon)=-\lambda(\epsilon/\delta)^2$ and Eq. (32).

VI. CONCLUSION

In this work we have presented a new formalism that describes in a full quantum-mechanical way the propagation of light in a linear and nonlinear dispersive medium. In the slowly varying approximation, this formalism is consistent with the canonical quantization procedure. We have demonstrated its efficiency by analyzing the degenerate parametric amplifier, and finding the noise-reduction properties of the emitted light. The main drawback of this formalism is that it is for the moment restricted to one-dimensional problems. Work is currently under way to try to extend it to the three-dimensional case. We hope that this approach can be used in many other cases dealing with propagation in quantum optics.

APPENDIX

Here we outline the calculations of the canonical ETCR in the parametric amplifier. We assume that at each point the two conjugate fields $\hat{A}(z,t)$ and $-\hat{D}(z,t)$ are still given by Eqs. (19) and (25), and use the expression for $\hat{a}(z,\omega_0+\epsilon_m)$ given in Eq. (33) in order to calculate the CR. We need to calculate two kinds of terms: $[\hat{A}^{(-)}(z,t),-\hat{D}^{(+)}(z',t)]$, and also $[\hat{A}^{(+)}(z,t),-\hat{D}^{(+)}(z',t)]$. This second term is identically zero in the linear case, but here it is not necessarily so, since during the propagation the $\hat{a}$ operators get mixed with $\hat{a}^\dagger$ operators. However, since both operators are positive-frequency ones, this term has a very rapid time dependence (at $2\omega_0$), and therefore does not contribute to the commutator. After straightforward calculations using Eqs. (19), (25), and (33), we obtain for the first term

$$[\hat{A}^{(-)}(z,t),-\hat{D}^{(+)}(z',t)] = \frac{i\hbar}{2T}\sum_m \left[\frac{n(\omega_m)}{c}\right] f_{\epsilon_m}(z-z')e^{ik(\epsilon_m)(z-z')} , \quad \text{(A1)}$$

where $f_{\epsilon_m}(z-z')$ is again defined according to the value of ϵ_m [as in Eq. (34)]. For $\lambda(\epsilon_m)\geq|\Delta k(\epsilon_m)|$, we have

$$f_{\epsilon_m}(z-z') = \left[\cosh s(\epsilon_m)(z-z') + i\frac{\Delta k(\epsilon_m)}{2s(\epsilon_m)}\sinh s(\epsilon_m)(z-z')\right] \times e^{-i[\Delta k(\epsilon_m)/2](z-z')} , \quad \text{(A2)}$$

where $\Delta k(\epsilon_m)$ and $s(\epsilon_m)$ are defined in Eqs. (32) and (34), respectively. For $\lambda(\epsilon_m)<|\Delta k(\epsilon_m)|$, the expression is exactly similar, with only the replacement of cosh and sinh by cos and sin, respectively. We now take the usual limit $T\to\infty$ [Eq. (27)], and obtain

$$[\hat{A}^{(-)}(z,t),-\hat{D}^{(+)}(z',t)] = \frac{i\hbar}{2}\left[\frac{1}{2\pi}\right]\int f_\epsilon(z-z')e^{ik(\epsilon)(z-z')}dk , \quad \text{(A3)}$$

where $k(\epsilon)=n(\epsilon)\omega/c$. It can easily be shown that $f_\epsilon(z-z')$ is a very slowly varying function of ϵ, so that it can be approximated by its value at $\epsilon=0$ and taken out of the integral. The final result is therefore

$$[\hat{A}^{(-)}(z,t),-\hat{D}^{(+)}(z',t)] = \frac{i\hbar}{2}\delta(z-z') , \quad \text{(A4)}$$

so that the overall ETCR is still given by Eq. (28). The fact that the ETCR can be recovered only approximately is a bit unsettling, and is probably due to the various approximations we had to use.

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