PHYSICAL REVIEW A VOLUME 42, NUMBER 9 1 NOVEMBER 1990

Stability of solitary waves in a nonlinear birefringent optical fiber

R. J. Dowling*
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge, United Kingdom CB3 9EW
(Received 9 January 1990)

In this paper the pulse solutions to the pair of partial differential equations governing light in a nonlinear, birefringent fiber are considered. A direct proof of the nonintegrability of the equations is given and attention is restricted to the stationary solutions. These are parametrized in terms of the physical conserved quantities and the relationship between the conserved quantities indicates when certain pulse solutions become unstable. This theoretical prediction is tested against numerical simulations.

INTRODUCTION

In a birefringent "monomode" optical fiber there are two polarization modes that need to be considered.[1] The small birefringence arises from a geometric and a material contribution. The material contribution is caused by the strain within the glasses forming the core and cladding averaged over the cross section of the fiber with the average weighted according to the intensity of the light. The geometric contribution comes from the ellipticity of the core of the fiber breaking the cylindrical symmetry. We can model a large range of behavior in these fibers by considering the slowly varying amplitude of a single-frequency carrier wave. If we assume that both the coefficients of dispersion and the group velocities are identical for the two polarization modes, that the dispersion is purely second order, and that the only nonlinearity is a cubic Kerr effect, then the amplitudes of the envelopes of the two modes are governed by[2]

$$i\mathbf{e}_z+\mathbf{e}_{tt}+\chi\cdot\mathbf{e}+a(\mathbf{e}\cdot\mathbf{e}^*)\mathbf{e}+b(\mathbf{e}\cdot\mathbf{e})\mathbf{e}^*=0\ ,$$

where $\mathbf{e}$ is a vector of the complex amplitudes of the envelopes of the carrier beam, χ is the birefringence tensor, and a and b are the coefficients of the Kerr nonlinearity tensor. The tensor χ is real, symmetric, and traceless, so a real basis transformation will cast it into the form $\left(\begin{smallmatrix}\kappa & 0\\ 0 & -\kappa\end{smallmatrix}\right)$. The two modes parallel to these principal axes of χ are called the fast and slow modes. We normalize according to convention so that $a+b=1$, leaving the intensity in dimensionless units and giving the equations relating the fast and slow components of $\mathbf{e}=(u,v)$ as

$$iu_z+u_{tt}+\kappa u+a(|u|^2+|v|^2)u+(1-a)(u^2+v^2)u^*=0\ ,$$

$$iv_z+v_{tt}-\kappa v+a(|u|^2+|v|^2)v+(1-a)(u^2+v^2)v^*=0\ ,$$

where the subscripts t and z denote differentiation with respect to normalized time and distance down the fiber.

If we restrict attention to waves in a single principal mode (either $u=0$ or $v=0$), then we get the simple nonlinear Schrödinger equation for the nonzero mode

$$i\psi_z+\psi_{tt}\pm\kappa\psi+|\psi|^2\psi=0,\quad \psi=u,v\ .$$

Considerable analytical and numerical work has been done for cw light and various pulse initial conditions.[3-9] In their paper,[10] Tratnik and Sipe use a variant of Hirota's method[11] to generate a new exact solitary-wave solution to the coupled equations modeling a pulse of light with energy in both principal modes of the birefringence. This solution bifurcates from a single-mode soliton solution, which becomes unstable for large energies. They conjectured that the bifurcation point was the onset of the instability. The instability of single-mode solutions both for pulsed and cw light has been the target of much of the discussion. The development of the instability is a major change in the solution's behavior caused by a small change in the intensity of the solution past some threshold. This is the ideal behavior for a switch, and it is this switching that has generated the interest in the phenomenon.

This paper will outline a direct demonstration involving the conserved quantities of the coupled differential equations of the system's nonintegrability by inverse scattering. It will then isolate those conserved quantities whose existence is implied by the underlying physics and which can have any bearing on the dynamics of the solutions to the equations. It will use these to give a physical parametrization of the solutions and relate them with one another for the various families of solitary-wave solutions. These relations give a novel insight into the stability of the single-mode solitons. This is used to make a new prediction for the onset of instability of a class of solitary-wave solutions, which is confirmed by numerical experiment. Finally, the implications of this instability for the mixed-mode solitary wave are considered with special regard to the conjecture made by Tratnik and Sipe in Ref. 10.

CONSERVED QUANTITIES

The simple, single nonlinear Schrödinger equation, governing pulses in a single principal mode, has been extensively studied and is known to be integrable.[12] In particular, it has genuine soliton solutions (as opposed to solitary waves) which do not interact apart from translations as they pass through one another. This raises the question of the integrability of the coupled equations: can the same multisoliton behavior be seen involving both polarization modes? A Painlevé property can be defined for partial differential equations[13] analogous to its definition for ordinary differential equations,[14] which appears to hold for precisely those systems of equations which are integrable. It has been shown[15] that for $a \neq 1$ the coupled equations do not possess this property, strongly suggesting that they are not integrable.

A more direct proof of nonintegrability by inverse scattering can be given in terms of the conserved quantities. The equations have four obvious, physically meaningful, geometric symmetries, transformations which map solutions of the equations into other solutions of the equations. Furthermore, as the equation is Hamiltonian, Noether's theorem[16] assigns to each of these symmetries a corresponding conserved quantity taking the form of the integral of a multinomial in u, v, and their t derivatives.

The equations are invariant under a Galilean shift which physically corresponds to a change in the reference frequency of the carrier beam and which maps solutions according to

$$u(z,t) \mapsto e^{-i(1/4)V^2 z + i(1/2)Vt} u(z, t - Vz) ,$$

$$v(z,t) \mapsto e^{-i(1/4)V^2 z + i(1/2)Vt} v(z, t - Vz) ,$$

and to which Noether's theorem assigns the conserved quantity

$$I_1 = \int_{-\infty}^{+\infty} t(|u|^2 + |v|^2) - 2iz(u_t u^* + v_t v^*) dt . \qquad (1)$$

This invariant is the initial "center of mass" of the solution.

The equations are trivially invariant under the phase rotation

$$u(z,t) \mapsto e^{i\theta} u(z,t), \quad v(z,t) \mapsto e^{i\theta} v(z,t)$$

and have

$$I_2 = \int_{-\infty}^{+\infty} (|u|^2 + |v|^2) dt \qquad (2)$$

as the corresponding constant of the motion. In the context of optical fibers this invariant is the total energy of the solution.

The equations have no explicit dependence on t and hence

$$u(z,t) \mapsto u(z, t - \Delta t), \quad v(z,t) \mapsto v(z, t - \Delta t)$$

must be a symmetry. This t translation gives

$$I_3 = \int_{-\infty}^{+\infty} (u_t u^* + v_t v^*) dt \qquad (3)$$

as its conserved quantity, which can be thought of as the "momentum" of the solution.

Similarly, the equations have no explicit dependence on z so

$$u(z,t) \mapsto u(z - \Delta z, t), \quad v(z,t) \mapsto v(z - \Delta z, t)$$

must be a symmetry. This has the Hamiltonian integral as its conserved quantity:

$$\begin{aligned} I_4 = \int_{-\infty}^{+\infty} & -(|u_t|^2 + |v_t|^2) + \kappa(|u|^2 - |v|^2) \\ & + \tfrac{1}{2}(|u|^4 + |v|^4) + a|u|^2|v|^2 \\ & + \tfrac{1}{2}(1-a)[u^2(v^*)^2 + (u^*)^2 v^2] dt . \end{aligned} \qquad (4)$$

Note that the values of the initial center of mass (I_1) and the momentum (I_3) are irrelevant as a t translation can alter I_1 and a Galilean shift to a different reference frequency can alter I_3 without either one affecting the dynamics of the solution. However, the energy I_2 and the Hamiltonian I_4 cannot be altered in this way and this paper will closely link their values with the behavior of the stationary solutions of the equations.

The simple nonlinear Schrödinger equation remains invariant under the transformation

$$\psi \mapsto \alpha\psi, \quad t \mapsto \alpha^{-1} t, \quad z \mapsto \alpha^{-2} z .$$

However, the coupled system is not invariant, as it has a scale imposed on it by κ. To rescale the coupled system in a similar manner to the single equation it is necessary to scale

$$\kappa \mapsto \alpha^2 \kappa, \quad a \mapsto a$$

as well. In this way we can assign "orders" to all the integrands of the conserved quantities. By our choice of notation the integrand of I_n scales by α^n.

For integrable systems the inverse scattering transformation predicts an infinite sequence of independent conserved quantities. Each of these is the integral over the real line of a multinominal in u, v, and their derivatives, with one of each order.[17] Noether's theorem gives an explicit transformation of the partial differential equation corresponding to the integral of any multinomial in a function and its derivatives and states that the transformation derived in this way from a conserved integral must be a symmetry, i.e., it maps the differential equation onto itself and one solution onto another. We can generate the most general fifth-order multinomial with arbitrary coefficients and determine its corresponding transformation. The computations required are individually quite simple, but the expressions involved are very large and the REDUCE algebraic manipulation package was used to evaluate them. Requiring this transformation to be a symmetry of the coupled equations gives relations between the coefficients which, when resolved for the case $a \neq 1$, imply that $I_5 = \kappa I_3$, giving an algebraic proof that there is no independent fifth-order conserved quantity. This lack of a fifth-order conserved quantity, the existence of which would be implied by integrability through inverse scattering, explicitly demonstrates the nonintegrability of the system by this method. For the case $a = 1$ the system of partial differential equations reduces to Manakov's equations, which are known to be integrable.[18]

STATIONARY SOLUTIONS

Consider for a moment the $N = 1$ soliton of the nonlinear Schrödinger equation in its simplest form

$$\psi(t,z) = 2^{(1/2)} \alpha e^{i\alpha^2 z} \text{sech}(\alpha t) .$$

Note in particular the decoupling of t and z in the solution. Motivated by this, for a general set of coupled nonlinear Schrödinger equations we may look for solutions

$$\mathbf{e}(t,z)=e^{i\omega z}\tilde{\mathbf{e}}(t)\ ,$$

where ω is an arbitrary real number and for our particular coupled set of equations $\tilde{\mathbf{e}}(t)$ satisfies the ordinary differential equation

$$\tilde{\mathbf{e}}_{tt}+\chi\cdot\tilde{\mathbf{e}}+a(\tilde{\mathbf{e}}\cdot\tilde{\mathbf{e}}^*)\tilde{\mathbf{e}}+(1-a)(\tilde{\mathbf{e}}\cdot\tilde{\mathbf{e}})\tilde{\mathbf{e}}^*=\omega\tilde{\mathbf{e}}\ .$$

The only conserved quantities relevant to the behavior of solutions of a general set of coupled nonlinear Schrödinger equations are the energy and the Hamiltonian. The other two can be set to arbitrary values using the symmetries of the equation. The relevance of the Hamiltonian and the energy to the stationary solutions can be seen most clearly if we rewrite the ordinary differential equation as a variational problem:

$$\frac{\delta}{\delta\mathbf{e}}(H(\mathbf{e})-\omega E(\mathbf{e}))\Big|_{\mathbf{e}=\tilde{\mathbf{e}}}=0\ , \tag{5}$$

where we define the variational derivative of a functional $\mathcal{L}[\mathbf{e}]$ to be

$$\frac{\delta\mathcal{L}}{\delta\mathbf{e}}=\left[\frac{\delta\mathcal{L}}{\delta u},\frac{\delta\mathcal{L}}{\delta v},\frac{\delta\mathcal{L}}{\delta u^*},\frac{\delta\mathcal{L}}{\delta v^*}\right]$$

such that

$$\delta\mathcal{L}=\int_{-\infty}^{+\infty}\left[\delta u(x)\frac{\delta\mathcal{L}}{\delta u}+\delta v(x)\frac{\delta\mathcal{L}}{\delta v}+\delta u(x)^*\frac{\delta\mathcal{L}}{\delta u^*}+\delta v(x)^*\frac{\delta\mathcal{L}}{\delta v^*}\right]dt\ ,$$

which for simplicity we write as an inner product $\langle\delta\mathbf{e},\delta\mathcal{L}/\delta\mathbf{e}\rangle$. This variational formulation of the equation immediately gives us a very powerful result: if a family of stationary solutions are continuously parametrized by a set of variables then the values of the energy integral and of the Hamiltonian integral depend only on ω.

Suppose $\delta(H(\mathbf{e})-\omega_0E(\mathbf{e}))=0$ has solutions $\mathbf{e}(t)=\tilde{\mathbf{e}}(t;\omega_0,\lambda)$ where λ may well be vector valued. Define $\tilde{H}(\omega,\lambda)=H(\tilde{\mathbf{e}}(t;\omega,\lambda))$ and $\tilde{E}(\omega,\lambda)=E(\tilde{\mathbf{e}}(t;\omega,\lambda))$. We have

$$\left\langle\frac{\partial\tilde{\mathbf{e}}}{\partial\lambda}(\omega_0,\lambda)\Big|_{\lambda=\lambda_0},\frac{\delta}{\delta\mathbf{e}}(H(\mathbf{e})-\omega_0E(\mathbf{e}))\Big|_{\mathbf{e}=\tilde{\mathbf{e}}(\omega_0,\lambda_0)}\right\rangle=0$$

as the second element of the inner product is zero by (5). By the linearity of the inner product we have

$$\left\langle\frac{\partial\tilde{\mathbf{e}}}{\partial\lambda}(\omega_0,\lambda)\Big|_{\lambda=\lambda_0},\frac{\delta}{\delta\mathbf{e}}(H(\mathbf{e}))\Big|_{\mathbf{e}=\tilde{\mathbf{e}}(\omega_0,\lambda_0)}\right\rangle=\omega_0\left\langle\frac{\partial\tilde{\mathbf{e}}}{\partial\lambda}\Big|_{\lambda=\lambda_0},\frac{\delta}{\delta\mathbf{e}}(E(\mathbf{e}))\Big|_{\mathbf{e}=\tilde{\mathbf{e}}(\omega_0,\lambda_0)}\right\rangle.$$

The first element of each inner product is the variation of a function as its parameter λ is varied. The second is the variation of the value of a functional as the function it is evaluated on varies. Therefore, by its definition, the inner product of the two expressions is the evaluation by the chain rule of the variation of the value of the functional as the parameter is varied, i.e.,

$$\frac{\partial\tilde{H}}{\partial\lambda}(\omega_0,\lambda)\Big|_{\lambda=\lambda_0}=\omega_0\frac{\partial\tilde{E}}{\partial\lambda}(\omega_0,\lambda)\Big|_{\lambda=\lambda_0}\ . \tag{6}$$

In the same manner we may deduce

$$\left\langle\frac{\partial\tilde{\mathbf{e}}}{\partial\omega}(\omega,\lambda_0)\Big|_{\omega=\omega_0},\frac{\delta}{\delta\mathbf{e}}(H(\mathbf{e})-\omega_0E(\mathbf{e}))\Big|_{\mathbf{e}=\tilde{\mathbf{e}}(\omega_0,\lambda_0)}\right\rangle=0\ ,$$

$$\left\langle\frac{\partial\tilde{\mathbf{e}}}{\partial\omega}(\omega,\lambda_0)\Big|_{\omega=\omega_0},\frac{\delta}{\delta\mathbf{e}}(H(\mathbf{e}))\Big|_{\mathbf{e}=\tilde{\mathbf{e}}(\omega_0,\lambda_0)}\right\rangle=\omega_0\left\langle\frac{\partial\tilde{\mathbf{e}}}{\partial\omega}\Big|_{\omega=\omega_0},\frac{\delta}{\delta\mathbf{e}}(E(\mathbf{e}))\Big|_{\mathbf{e}=\tilde{\mathbf{e}}(\omega_0,\lambda_0)}\right\rangle\ ,$$

i.e.,

$$\frac{\partial\tilde{H}}{\partial\omega}(\omega,\lambda_0)\Big|_{\omega=\omega_0}=\omega_0\frac{\partial\tilde{E}}{\partial\omega}(\omega,\lambda)\Big|_{\omega=\omega_0}\ . \tag{7}$$

If we now calculate the cross derivative of $\tilde{H}$ from (6) and (7) we see that

$$\frac{\partial^2\tilde{H}}{\partial\omega\partial\lambda}(\omega,\lambda)\Big|_{\substack{\lambda=\lambda_0\\ \omega=\omega_0}}=\frac{\partial\tilde{E}}{\partial\lambda}(\omega,\lambda)\Big|_{\substack{\lambda=\lambda_0\\ \omega=\omega_0}}+\omega_0\frac{\partial^2\tilde{E}}{\partial\omega\partial\lambda}(\omega,\lambda)\Big|_{\substack{\lambda=\lambda_0\\ \omega=\omega_0}}\ , \tag{6$'$}$$

$$\frac{\partial^2\tilde{H}}{\partial\lambda\partial\omega}(\omega,\lambda)\Big|_{\substack{\omega=\omega_0\\ \lambda=\lambda_0}}=\omega_0\frac{\partial^2\tilde{E}}{\partial\lambda\partial\omega}(\omega,\lambda)\Big|_{\substack{\omega=\omega_0\\ \lambda=\lambda_0}}\ . \tag{7$'$}$$

Equating these two expressions for the cross derivative gives

$$\frac{\partial \tilde{E}}{\partial \lambda}(\omega,\lambda)\bigg|_{\substack{\omega=\omega_0\\ \lambda=\lambda_0}}=0 ,$$

i.e., the energy of the solution is not a function of any parameter other than ω. From this result and either of (6) or (7), we derive

$$\frac{\partial \tilde{H}}{\partial \lambda}(\omega,\lambda)\bigg|_{\substack{\omega=\omega_0\\ \lambda=\lambda_0}}=0 ,$$

so that the Hamiltonian integral is similarly a function of ω only. If the function $E(\omega)$ is invertible then we can substitute for ω in any expression for a solution to give a physical parametrization and we may calculate the function $H(E)$ relating the two relevant physical quantities of the system.

If we restrict our attention to the particular coupled set we are considering here we can see that the ordinary differential equation clearly has the single-mode solutions corresponding to the solitons of the single nonlinear Schrödinger equation

$$u=2^{1/2}\exp(i\omega z)A_1\operatorname{sech}(A_1 t),\quad v=0 \quad \text{(fast mode)}$$

$$u=0,\quad v=2^{1/2}\exp(i\omega z)A_2\operatorname{sech}(A_2 t) \quad \text{(slow mode)}$$

where $A_1=(\omega-\kappa)^{1/2}$ and $A_2=(\omega+\kappa)^{1/2}$. These solutions, however, are parametrized in terms of ω, which we may replace. We calculate E for these stationary solutions to get

$$E_{\text{fast}}=4(\omega-\kappa)^{1/2},\quad E_{\text{slow}}=4(\omega+\kappa)^{1/2} ,$$

and can substitute for ω to get a more physically meaningful parametrization in terms of the energy of the pulse:

$$u=8^{-1/2}E\exp\{i[(E/4)^2+\kappa]z\}\operatorname{sech}(Et/4) ,$$
$$v=0 \quad \text{(fast mode)}$$
$$u=0,\quad v=8^{-1/2}E\exp\{i[(E/4)^2-\kappa]z\}\operatorname{sech}(Et/4)$$
$$\text{(slow mode)} .$$

In addition, Tratnik and Sipe[10] found a family of mixed-mode solutions

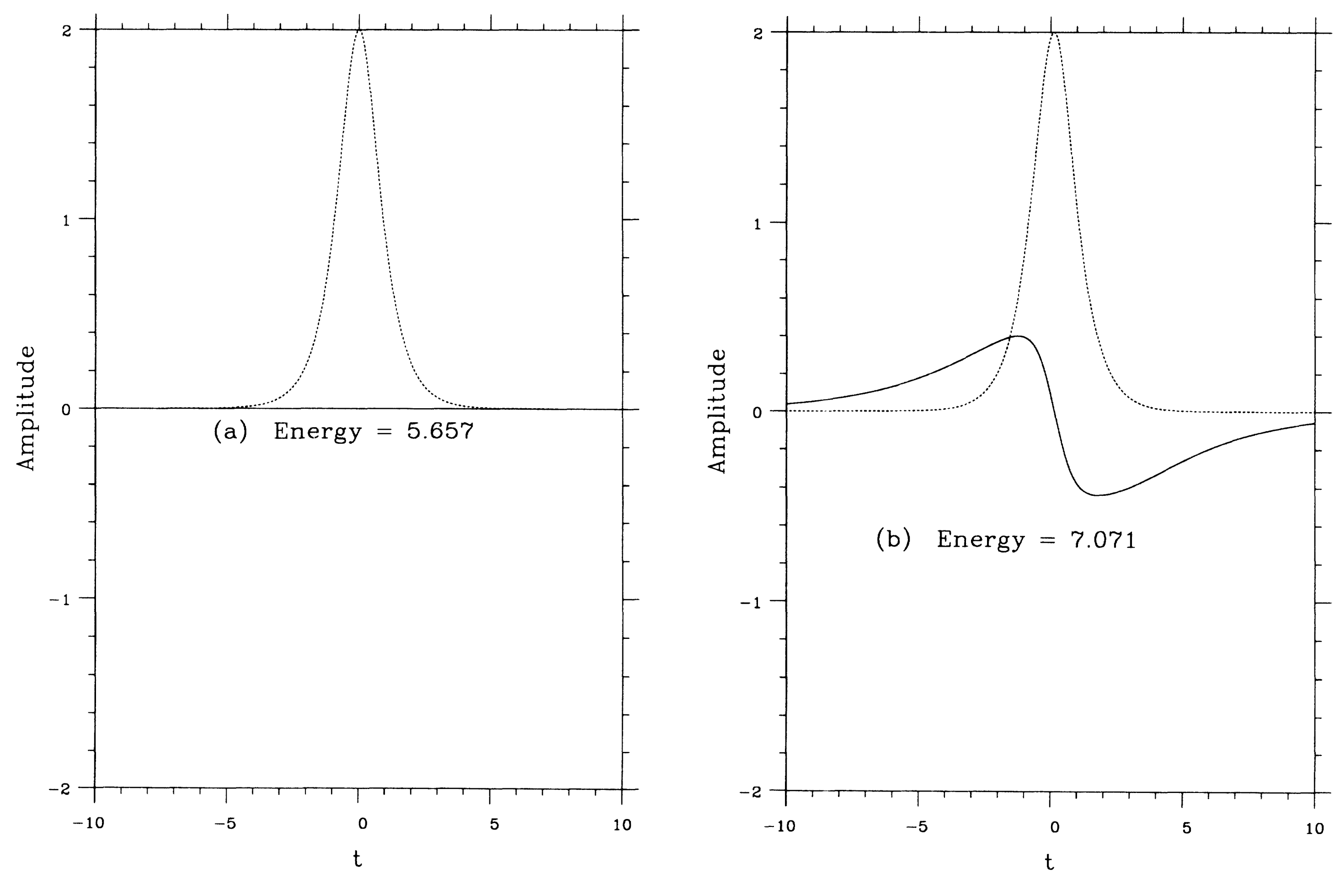

FIG. 1. Plots of the fast and slow components of a mixed-mode solution for energies equal to and above the bifurcation point and $t_1=t_2=0$. Note that in graph (a) (at the bifurcation point) the mixed-mode solution is exactly coincident with the pure fast-mode solution. For all these plots $\kappa=1$. The solid and dotted lines give the fast-and slow-mode components, respectively. Both energy and time are measured in normalized, dimensionless units. (a) $E=5.657\approx E_{\text{crit}}$, (b) $E=7.07$, (c) $E=8.49$, (d) $E=9.90$, (e) $E=11.3$.

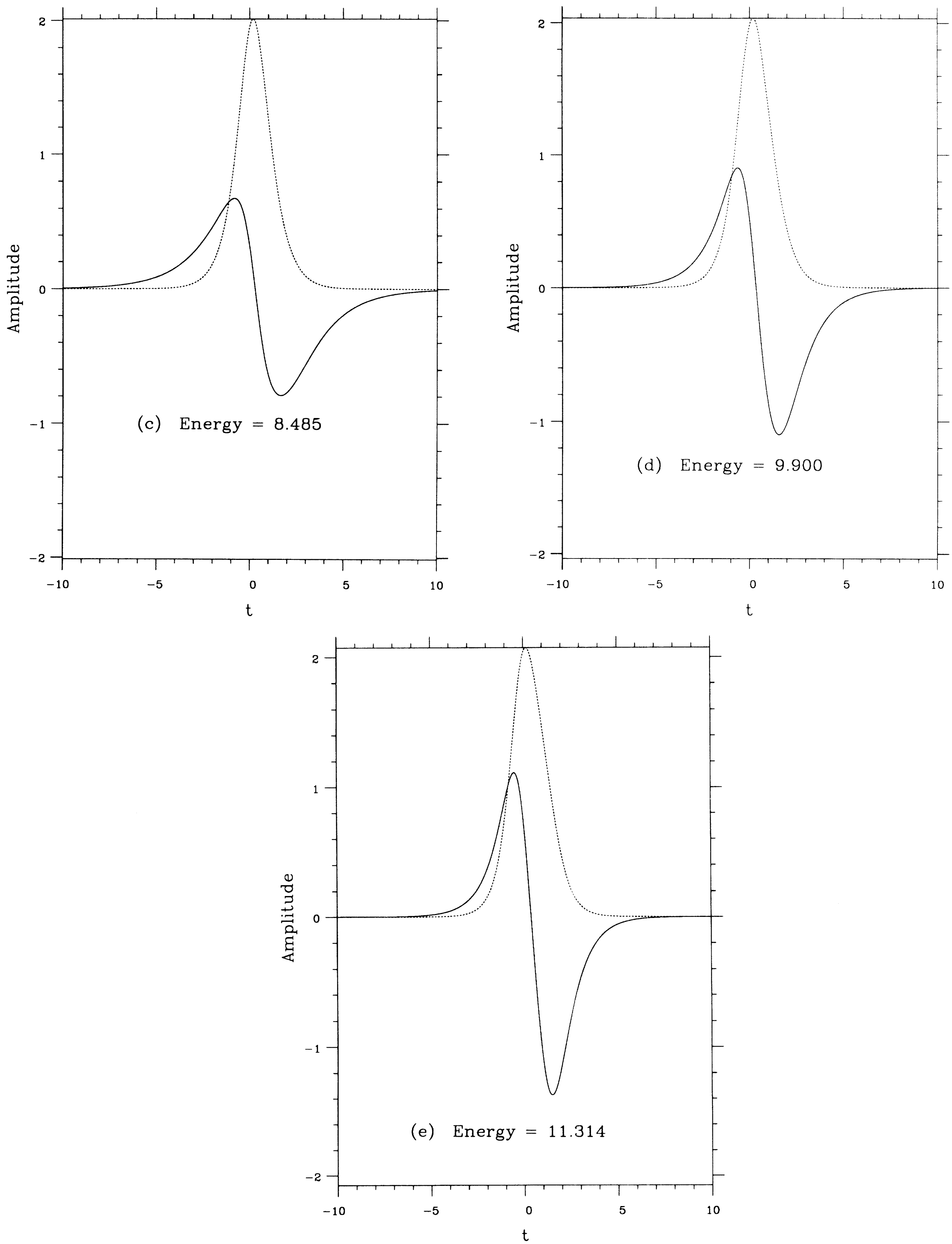

FIG. 1. (*Continued*).

$$u=\frac{8^{1/2}A_1\exp[A_1(t-t_1)+i\omega z]\{1+[(A_1-A_2)/(A_1+A_2)]\exp[2A_2(t-t_2)]\}}{1+\exp[2A_2(t-t_1)]+\exp[2A_1(t-t_2)]+[(A_1-A_2)/(A_1+A_2)]^2\exp[2A_1(t-t_1)+2A_2(t-t_2)]},$$

$$v=\frac{8^{1/2}A_2\exp[A_2(t-t_2)+i\omega z]\{1-[(A_1-A_2)/A_1+A_2)]\exp[2A_1(t-t_1)]\}}{1+\exp[2A_2(t-t_1)]+\exp[2A_1(t-t_2)]+[(A_1-A_2)/(A_1+A_2)]^2\exp[2A_1(t-t_1)+2A_2(t-t_2)]}.$$

These solutions are bound states of two single-mode solitons, one in each principal mode, with energies giving them the same value of ω. The two solitons have fixed positions given by t_1 and t_2. In Ref. 10 the special form of this solution, given by the adapted Hirota's method, was used to demonstrate that the total energy of the solution did not depend on t_1-t_2. This is a simple example of the general result above. However it is derived, we may use the independence of E from t_1 and t_2 to calculate the energy of the mixed-mode solution in the limiting case of $|t_1-t_2|\rightarrow\infty$ when the mixed mode solution is simply two well-separated single-mode pulses and the energy is the sum of the energies of the two soliton solutions

$$E(\omega)=4[(\omega-\kappa)^{1/2}+(\omega+\kappa)^{1/2}].$$

We may invert this to give a physical parametrization of the mixed-mode stationary solution

$$\omega(E)=\frac{E^4+1024\kappa^2}{64E^2}.$$

Note that the constraint that $\omega>\kappa$ for A_1 to be real gives these solutions a minimum energy $E_{\min}=4(2\kappa)^{1/2}$. When the energy takes precisely this threshold value the mixed-mode solution and the slow-mode solution are coincident (Fig. 1). Furthermore, for all mixed-mode solutions the slow-mode component will have at least this amount of energy.

The independence of the values of the energy and Hamiltonian from parameters other than ω is only valid for continuous parametrizations. The three types of solution we have, fast mode, slow mode, and mixed mode, are distinct and the functions $E(\omega)$, $H(\omega)$, and $H(E)$ need not be the same for them all, and indeed are not. We may calculate the function $H(E)$ for all three types of solution to get

$$H_{\text{fast}}(E)=\frac{E(E^2+48\kappa)}{48},$$

$$H_{\text{slow}}(E)=\frac{E(E^2-48\kappa)}{48},$$

$$H_{\text{mixed}}(E)=\frac{E^4+3072\kappa^2}{192E}.$$

Figure 2 gives the graphs of these three functions. Note that each point of the line corresponding to the mixed-mode solution represents an infinite family of solutions parametrized by t_1-t_2 and that the point where the mixed-mode solution bifurcates from the slow mode can be seen at $E=4(2\kappa)^{1/2}$. Only the curve corresponding to $H_{\text{slow}}(E)$ has a minimum, which is at $E=4\kappa^{1/2}$, a factor of $2^{1/2}$ lower than the bifurcation energy. As the ordinary differential equation has a variational representation (5) of extremizing H for a fixed value of E, such a turning point might be expected to correspond to some change in the behavior of the solutions. We will see that what it corresponds to is the loss of stability of the solution.

NUMERICAL STUDIES OF STABILITY

Numerical studies have been performed[3] wherein various N-soliton pulses were launched near the two modes and the effect of varying κ, the measure of the birefringence of the fiber, was observed. Pulses in the fast mode were always stable, but the slow mode solutions were sometimes subject to a bimodal instability. It was

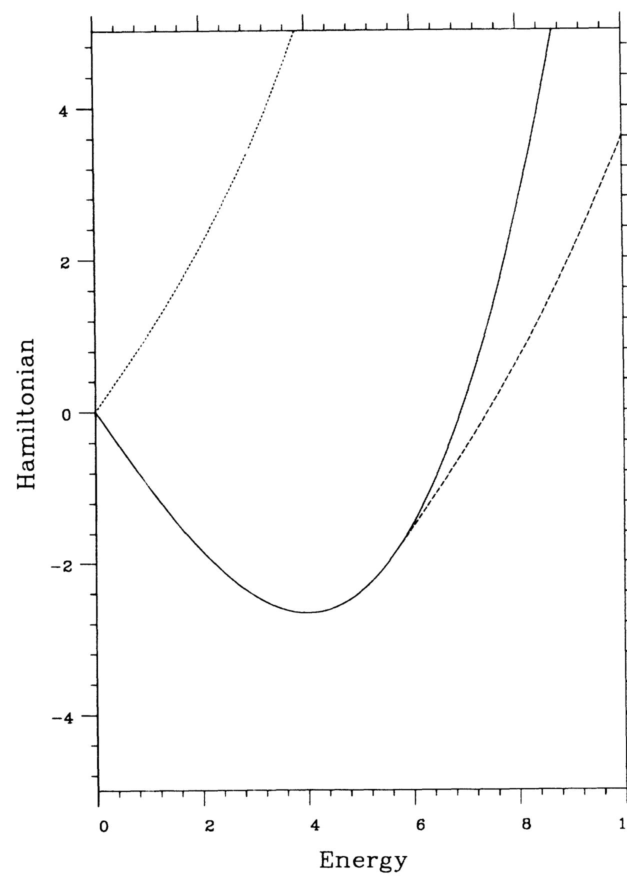

FIG. 2. Graphs of the Hamiltonian integral vs the energy integral for the three types of stationary solution. This particular figure is plotted for $\kappa=1$, but is typical of all the graphs. Solid line, slow mode; dotted line, fast mode; dashed line, mixed mode.

found that increasing κ stabilized pulses and that the critical values of κ needed rose like N^2. It was also observed that the collapse of multisoliton $N > 1$ pulses in the unstable mode led to the emission of a small amount of radiation and the creation of a single, very narrow soliton in the stable mode containing the majority of the energy of the original pulse.

In this paper we are restricting attention to the stationary waves, and rather than consider multisoliton pulses of fixed width, we considered only single soliton pulses with various values of E, and hence various different widths. To permit us to observe any instability due to the presence of the other mode, the pulses were launched slightly off the principal modes

$$u(t,0)=8^{-1/2}E\,\mathrm{sech}(Et/4)\cos\theta\ ,$$

$$v(t,0)=8^{-1/2}E\,\mathrm{sech}(Et/4)\sin\theta\ .$$

Pulses launched near the fast mode ($\theta \approx 0$) were observed to remain stable for all values of E, keeping most of their energy in the original mode as illustrated in Fig. 3. However, when the pulses were launched near the slow mode ($\theta \approx \pi/2$), this stability only persisted up to a critical value of E. Beyond this value the multimode instability caused the energy to oscillate between both modes. Figure 4 shows the propagation of pulses just above the critical energy and Fig. 4(b) illustrates the nature of the bimodal instability. These numerical experiments were done for a range of values of a and κ to experimentally determine the energy at which the solutions become unstable for a variety of parameters. This critical energy is found to be $4\kappa^{1/2}$ within the limits of numerical experiment, which is also the value of E which minimizes $H_{\mathrm{slow}}(E)$, confirming our expectation that this minimum ought to have a bearing on the behavior of the solution. Note that, contrary to Tratnik and Sipe's conjecture, the instability in the slow mode occurs at a lower energy than the bifurcation energy for the mixed-mode solution.

We have also performed similar numerical experiments with the mixed-mode stationary solutions. None of these

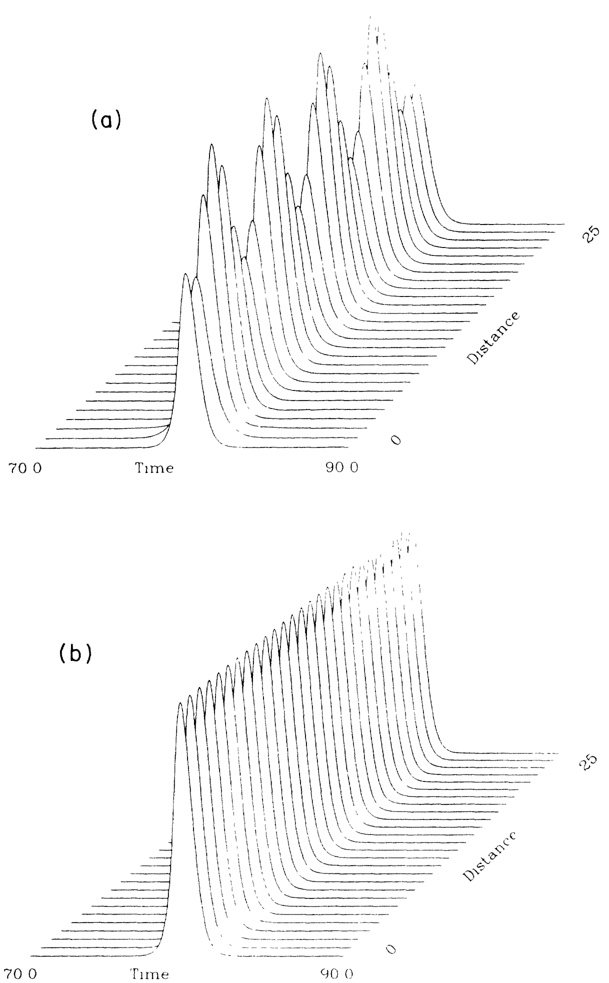

FIG. 3. Normalized intensities of the two components of a pulse launched near the fast mode. (a) Slow mode. (N.B. the scale on this plot is very much greater than on the slow-mode plot. It contains less than 1% of the total energy.) (b) Fast mode.

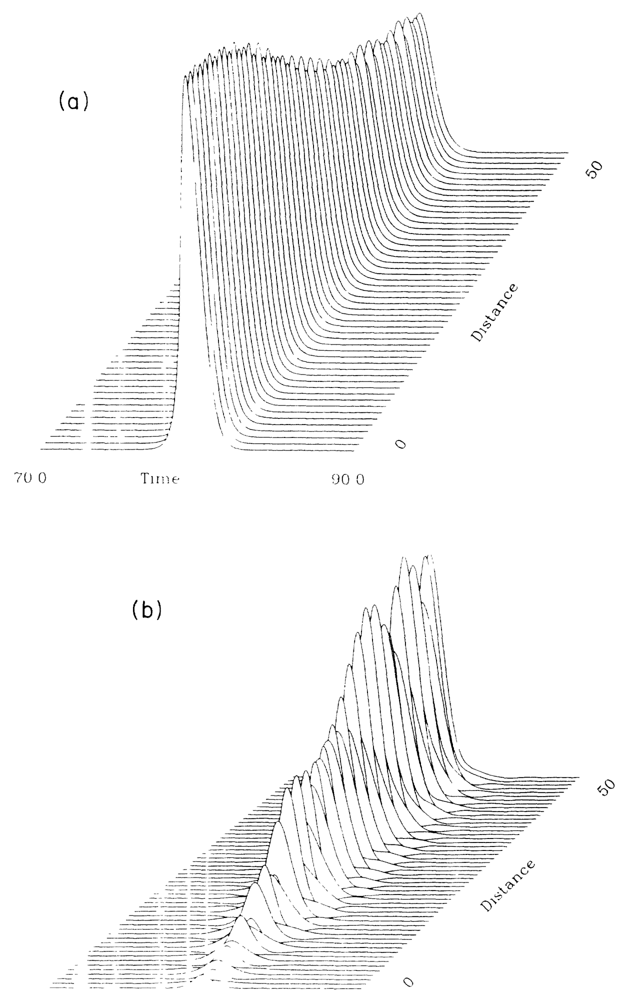

FIG. 4. Normalized intensities of the two components of a pulse launched near the slow mode with energy above the instability threshold. (Unlike Fig. 3 the scales on these two plots are equal.) (a) Slow mode. (b) Fast mode.

have been found to be stable. This is not particularly surprising since for large values of the separation of the fast- and slow-mode components, $t_1 - t_2$, the slow mode component always has an energy greater than that at which it becomes unstable and the fast mode solution is too far separated to be able to stabilize it. Similarly for energies just above the bifurcation energy, $E = 4(2\kappa)^{1/2}$, and for arbitrary values of $t_1 - t_2$, the solution is virtually identical to the unstable pure slow-mode soliton, and again will be unstable itself. Numerically we find no stabilization of the slow-mode component by the fast-mode component for any value of $t_1 - t_2$ in the mixed-mode stationary solutions, and as the energy in the slow mode exceeds the instability threshold, all the mixed-mode solutions seem to be unstable.

CONCLUSION

In this paper we have considered the coupled nonlinear Schrödinger equations for a simple model of a birefringent optical fiber. The conserved quantities of the equation have been used to demonstrate that it is not integrable by the method of inverse scattering.

The stabilities of both the single-mode stationary solutions corresponding to the single-soliton solutions of the simple nonlinear Schrödinger equation, and Tratnik and Sipe's family of mixed-mode solutions were considered and related to the values of the energy and Hamiltonian integrals for the particular types of solution. This provided a very simple, analytic means to determine the onset of instability for a pulse involving only two scalar quantities. The technique should be contrasted with linear perturbation schemes, which often cannot be solved analytically, though of course there are exceptions,[19] and which tend to lead to very complicated numerical problems. Furthermore, it is a global analysis, not relying on local perturbations around the stationary solution.

Tratnik and Sipe's result that the energy of the mixed-mode solution, which is a bound state of two single-mode solitons, is independent of the separation of those two solitons has been shown to be a special case of a far more general result. An analytical expression has been given for the energy at which the slow-mode solution becomes unstable. This energy is a factor of $2^{1/2}$ below the energy at which the mixed-mode solution is created. This suggested that the mixed-mode solutions are unstable, which has been numerically confirmed for a wide variety of parameters.

ACKNOWLEDGMENTS

This work was supported by the United Kingdom Science and Engineering Research Council and the British Telecom Research Laboratories under Cooperative Awards in Science and Engineering Grant No. 87501300. The author would like to thank Dr. David Wood for the BTRL and Professor Nigel Weiss of the Department of Applied Mathematics and Theoretical Physics, University of Cambridge for many useful conversations and much helpful advice.

---

*Electronic address: rjd4@uk.ac.cam.phx.

[1]I. P. Kaminow, IEEE J. Quantum Electron. **17**, 15 (1981).
[2]A. D. Boardman and G. S. Cooper, J. Opt. Soc. Am. **5**, 403 (1988).
[3]K. J. Blow, N. J. Doran, and D. C. Wood, Opt. Lett. **12**, 202 (1987).
[4]F. Florańczyk and R. Tremblay, Phys. Lett. A **141**, 34 (1989).
[5]S. Trillo, S. Wabnitz, E. M. Wright, and G. I. Stegeman, Opt. Commun. **70**, 166 (1989).
[6]G. Winful, Opt. Lett. **11**, 33 (1986).
[7]C. R. Menyuk, Opt. Lett. **12**, 614 (1987).
[8]C. R. Menyuk, J. Opt. Soc. Am. B **5**, 392 (1988).
[9]S. Trillo, S. Wabnitz, E. M. Wright, and G. I. Stegeman, Opt. Lett. **13**, 692 (1988).
[10]M. V. Tratnik and J. E. Sipe, Phys. Rev. A **38**, 2011 (1988).
[11]R. Hirota, *Solitons,* edited by R. K. Bullough and P. J. Caudrey (Springer-Verlag, Berlin, 1980), Chap. 5.
[12]V. E. Zacharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys.—JETP **34**, 62 (1972)].
[13]J. Weiss, M. Tabor, and G. Carnevale, J. Math. Phys. **24**, 522 (1983).
[14]E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956), Chap. XII.
[15]R. Sahadevan, K. M. Tanizhmani, and M. Laksmanan, J. Phys. A **19**, 1783 (1986).
[16]E. Noether, Trans. Trans. Theory Stat. Phys. **1**, 183 (1971).
[17]L. D. Fadeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer, New York, 1987), Chap. 1, Sec. 4.
[18]S. V. Manakov, Zh. Eksp. Teor. Fiz. **65**, 505 (1973) [Sov. Phys.—JETP **38**, 248 (1974).
[19]E. M. Wright, G. I. Stegeman, and S. Wabnitz, Phys. Rev. A **40**, 4455 (1989).