Nonlinear theory of the degenerate quantum-beat laser: Lasing without inversion

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(Received 22 June 1990)

We present the nonlinear theory for lasing without population inversion in the degenerate Λ-type quantum-beam laser by using the density-operator method. The master equation for the degenerate quantum-beat laser is derived, from which the equation of motion for the mean photon number is obtained. The conditions for lasing without population inversion are discussed, which are the same for linear and nonlinear theories. The corresponding Fokker-Planck equation is obtained, and the possibility of diffusion coefficient reduction is discussed.

I. INTRODUCTION

Lasing is usually accompanied by population inversion. The requirement of the population inversion is due to the stimulated absorption.

Lasing without population inversion was suggested some time ago by utilizing the splitting of emission and absorption spectra caused by atomic recoil.[1,2] The recoil splitting will be large enough to have practical usage only for very high-frequency light, e.g., x rays. A most interesting possibility of obtaining noninversion lasing or amplification has recently been proposed by Harris[3] and studied by him and others.[4–7] Harris analyzes the difference between the emission and absorption spectra of a three-level atom due to Fano interferences.[8,9] The quantum-beat laser[10–12] concept was originally advanced as a means of quenching spontaneous-emission noise. The lasing medium in such a device consists of three-level atoms with the upper two (closely spaced) levels driven by a coherent microwave field. It has been shown in recent papers that lasing without population inversion can be reached in the degenerate Λ-type quantum-beat laser system[13,14] with two lower levels instead of two upper levels. The Λ-type degenerate quantum-beat laser (Fig. 1) can display gain even when only a small fraction of the atoms are in the upper level $|a\rangle$. However, in that paper[13] we used the perturbation method (keeping to the second order of the coupling constant) and only worked out a linear theory for lasing without population inversion in the Λ-type degenerate quantum-beat laser. In this paper, we present the nonlinear theory for the lasing without population inversion using a density-operator method.

II. THE MODEL AND SOLUTION

Consider a three-level atom, as shown in Fig. 1, inside a cavity with frequency ν. The atom has three levels, the upper level $|a\rangle$ and two lower levels $|b\rangle$ and $|c\rangle$. The transitions between $|a\rangle$ and $|c\rangle$ and between $|a\rangle$ and $|b\rangle$ are induced by the light field (cavity mode), while the two lower levels, $|b\rangle$ and $|c\rangle$, are strongly coupled by an external microwave of frequency ν_μ and the corresponding Rabi factor is $\Omega e^{-i\phi}$ where Ω is the Rabi frequency and ϕ is the phase of the microwave. The Hamiltonian for the system is[11,15]

$$H=\hbar\nu a^\dagger a+\sum_{a,b,c}\hbar\omega_\alpha\sigma_\alpha+V\ , \tag{1}$$

$$V=\hbar g_1 a^\dagger\sigma_1+\hbar g_2 a^\dagger\sigma_2+\tfrac{1}{2}\hbar\Omega e^{i(\nu_\mu t+\phi)}\sigma_\mu+\text{H.c.}\ , \tag{2}$$

where $a^\dagger$ (a) is the creation (annihilation) operator for the light field, $\sigma_\alpha=|\alpha\rangle\langle\alpha|$ ($\alpha=a,b,c$), $\sigma_1=|c\rangle\langle a|$, $\sigma_2=|b\rangle\langle a|$, and $\sigma_\mu=|c\rangle\langle b|$, g_1 and g_2 are atom-field coupling constants.

First we transform into the interaction picture. The interaction Hamiltonian in the interaction picture is

$$V^I=V_1^I+V_2^I\ , \tag{3}$$

$$V_1^I=\hbar g_1 a^\dagger\sigma_1 e^{-i\Delta t}+\hbar g_2 a^\dagger\sigma_2 e^{i\Delta t}+\text{H.c.}\ , \tag{3a}$$

$$V_2^I=\tfrac{1}{2}\hbar\Omega e^{i\phi}\sigma_\mu+\text{H.c.}\ , \tag{3b}$$

where $\omega_{ac}+\omega_{ab}=2\nu$ and $\omega_{bc}=\nu_\mu$ ($\omega_{\alpha\beta}=\omega_\alpha-\omega_\beta$) have been assumed and $\Delta=\frac{1}{2}\omega_{bc}$. The equations of motion for the state vector and density operator in the interaction

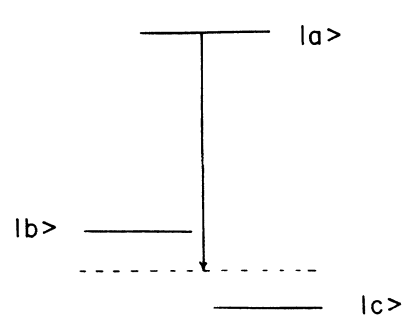

FIG. 1. Energy-level diagram for degenerate Λ-type quantum-beat laser.

picture are

$$\frac{d}{dt}|\psi^I\rangle=-\frac{i}{\hbar}V^I|\psi^I\rangle\ , \tag{4}$$

$$\frac{d}{dt}\dot{\rho}^I=-\frac{i}{\hbar}(V^I\rho^I-\rho^I V^I)\ . \tag{5}$$

Second, we make the following unitary transformation $\tilde{U}$, transforming into a second interaction picture[11] from the interaction picture:

$$\begin{aligned}\tilde{U}&=\exp[-(i/2)\Omega(e^{i\phi}\sigma_\mu+e^{-i\phi}\sigma_\mu^\dagger)t]\\&=|a\rangle\langle a|+(|c\rangle\langle c|+|b\rangle\langle b|)\cos\frac{\Omega}{2}t\\&\quad-i(e^{i\phi}\sigma_\mu+e^{-i\phi}\sigma_\mu^\dagger)\sin\frac{\Omega}{2}t\ .\end{aligned} \tag{6}$$

The equations of motion for the state vector and density operator become

$$\frac{d}{dt}|\tilde{\psi}\rangle=-\frac{i}{\hbar}\tilde{V}|\tilde{\psi}\rangle\ , \tag{7}$$

$$\frac{d}{dt}\dot{\tilde{\rho}}=-\frac{i}{\hbar}(\tilde{V}\tilde{\rho}-\tilde{\rho}\tilde{V})\ , \tag{8}$$

with $|\tilde{\psi}\rangle=\tilde{U}^\dagger|\psi^I\rangle$ and $\tilde{\rho}=\tilde{U}^\dagger\rho^I\tilde{U}$. Here the interaction Hamiltonian $\tilde{V}$ is

$$\begin{aligned}\tilde{V}&=\tilde{U}^\dagger V_1^I\tilde{U}\\&=\hbar g_1 a^\dagger e^{-i\Delta t}\left[\sigma_1\cos\frac{\Omega}{2}t+i\sigma_2 e^{-i\phi}\sin\frac{\Omega}{2}t\right]\\&\quad+g_2 a^\dagger e^{i\Delta t}\left[\sigma_2\cos\frac{\Omega}{2}t+i\sigma_1 e^{i\phi}\sin\frac{\Omega}{2}t\right]+\text{H.c.}\end{aligned} \tag{9a}$$

We adjust the Rabi frequency Ω so that $\Omega=\omega_{bc}=2\Delta$, and consequently $\tilde{V}$ becomes

$$\tilde{V}=a^\dagger(G_1\sigma_1+G_2\sigma_2)+\text{H.c.}\ , \tag{9b}$$

where G_1 and G_2 are effective coupling constants

$$G_1=\tfrac{1}{2}(g_1-g_2e^{i\phi})\ , \tag{10a}$$

$$G_2=\tfrac{1}{2}(g_2+g_1e^{-i\phi})\ , \tag{10b}$$

and high-frequency terms, going as $e^{i2\Delta t}$, have been dropped. This interaction Hamiltonian Eq. (9b) is our basic starting point for further deduction.

First, we investigate the deterministic time evolution of the coupled three-level atom and one mode system which is controlled by the interaction Hamiltonian Eq. (9b). The state vector can be written as

$$\begin{aligned}|\tilde{\psi}(t)\rangle=\sum_n a_n(t)|a,n\rangle+b_{n+1}(t)|b,n+1\rangle\\+c_{n+1}(t)|c,n+1\rangle\ .\end{aligned} \tag{11}$$

From the equation of motion Eq. (7) and the interaction Hamiltonian Eq. (9b), we find[15]

$$\begin{aligned}\dot{a}_n(t)=-\frac{\Gamma}{2}a_n(t)-i[G_2^*\sqrt{n+1}b_{n+1}(t)\\+G_1^*\sqrt{n+1}c_{n+1}(t)]\ ,\end{aligned} \tag{12a}$$

$$\dot{b}_{n+1}(t)=-\frac{\Gamma}{2}b_{n+1}(t)-iG_2\sqrt{n+1}a_n(t)\ , \tag{12b}$$

$$\dot{c}_{n+1}(t)=-\frac{\Gamma}{2}c_{n+1}(t)-iG_1\sqrt{n+1}a_n(t)\ , \tag{12c}$$

where we have included the atomic decay and, for simplicity, taken the same decay rate Γ for the three levels. Assume that the atom is in a mixed state at initial time t_0, i.e.,

$$|\psi_{\text{atom}}(t_0)\rangle=\alpha_a|a\rangle+\alpha_b|b\rangle+\alpha_c|c\rangle\ . \tag{13}$$

The initial state vector for the system at time t_0 is

$$\begin{aligned}|\tilde{\psi}(t_0)\rangle&=|\psi_f(t_0)\rangle\otimes|\psi_{\text{atom}}(t_0)\rangle\\&=\sum_n \alpha_a F_n(t_0)|a,n\rangle+\alpha_b F_n(t_0)|b,n\rangle\\&\quad+\alpha_c F_n(t_0)|c,n\rangle\ .\end{aligned} \tag{14}$$

With the initial condition Eq. (14), we solve Eqs. (12) and obtain

$$a_n(t)=e^{-(\Gamma/2)(t-t_0)}\left[\alpha_a F_n\cos x(t-t_0)-\left[\frac{i(G_1^*\alpha_c+G_2^*\alpha_b)}{(|G_1|^2+|G_2|^2)^{1/2}}\sin x(t-t_0)\right]F_{n+1}(t_0)\right]\ , \tag{15a}$$

$$\begin{aligned}b_{n+1}(t)=e^{-(\Gamma/2)(t-t_0)}\Bigg[&\frac{(|G_2|^2\alpha_b+G_1^*G_2\alpha_c)F_{n+1}(t_0)}{|G_1|^2+|G_2|^2}\cos x(t-t_0)\\&-\frac{iG_2\alpha_aF_n(t_0)}{(|G_1|^2+|G_2|^2)^{1/2}}\sin x(t-t_0)+\frac{(|G_1|^2\alpha_b-G_2G_1^*\alpha_c)F_n(t_0)}{|G_1|^2+|G_2|^2}\Bigg]\ ,\end{aligned} \tag{15b}$$

$$\begin{aligned}c_{n+1}(t)=e^{-(\Gamma/2)(t-t_0)}\Bigg[&\frac{(|G_1|^2\alpha_c+G_1G_2^*\alpha_b)F_{n+1}(t_0)}{|G_1|^2+|G_2|^2}\cos x(t-t_0)\\&-\frac{iG_1\alpha_aF_n(t_0)}{(|G_1|^2+|G_2|^2)^{1/2}}\sin x(t-t_0)+\frac{(|G_2|^2\alpha_c-G_1G_2^*\alpha_b)F_n(t_0)}{|G_1|^2+|G_2|^2}\Bigg]\ ,\end{aligned} \tag{15c}$$

where

$$x=[(G_1G_1^*+G_2G_2^*)(n+1)]^{1/2} .$$

III. MASTER EQUATION

The density operator of the system satisfies the equation of motion Eq. (8). The reduced density operator for the light field $\tilde{\rho}^f$ is obtained by tracing over atomic variables,

$$\tilde{\rho}^f=\mathrm{Tr}_{\mathrm{atom}}\tilde{\rho}=\sum_{\tilde{\alpha}=a,b,c}\langle\tilde{\alpha}|\tilde{\rho}|\tilde{\alpha}\rangle , \tag{16a}$$

where $|\tilde{\alpha}\rangle=\tilde{U}^\dagger|\alpha\rangle$. Using Eq. (6), it is easy to prove that

$$\sum_{\tilde{\alpha}=a,b,c}\langle\tilde{\alpha}|\tilde{\rho}|\tilde{\alpha}\rangle=\sum_{\alpha=a,b,c}\langle\alpha|\tilde{\rho}|\alpha\rangle . \tag{16b}$$

Using the interaction Hamiltonian Eq. (9b) and the equation of motion Eq. (8) and carrying out the trace, we find the equation of motion for $\tilde{\rho}^f$

$$\dot{\tilde{\rho}}^f=-i(G_1^*[a,\rho_{ca}]+G_2^*[a,\rho_{ba}]+G_1[a^\dagger,\tilde{\rho}_{ac}]+G_2[a^\dagger,\rho_{ab}]) , \tag{17}$$

where $\tilde{\rho}_{\alpha\beta}=\langle\alpha|\tilde{\rho}|\beta\rangle$ $(\alpha,\beta=a,b,c)$. This is the contribution of one atom, which is pumped into the cavity at time t_0, on the light field. To find the total change of the light field due to many atoms, which is pumped into the cavity regularly, we sum the contributions of all atoms which are pumped into the cavity at times $t_0<t$ with a rate r.[11,16] Then we find the equation of motion for the density operator of the light field,

$$\begin{aligned}\dot{\rho}^f&=r\int_{-\infty}^{t}dt_0\dot{\tilde{\rho}}^f+\mathcal{L}\rho^f\\&=-i\int_{-\infty}^{t}dt_0 r([a,G_1^*\rho_{ca}+G_2^*\rho_{ba}]\\&\qquad+[a^\dagger,G_1\tilde{\rho}_{ac}+G_2\tilde{\rho}_{ab}])+\mathcal{L}\rho^f ,\end{aligned} \tag{18}$$

where we have used an integral to replace the sum. Here $\mathcal{L}\rho^f$ stands for the cavity loss, which has the usual form[12]

$$\mathcal{L}\rho^f=-\frac{\gamma_c}{2}(a^\dagger a\rho^f+\rho^f a^\dagger a-2a\rho^f a^\dagger) . \tag{19}$$

Here γ_c is the cavity loss rate.

The equation of motion of the elements of the density operator of the light field $\rho^f_{n,m}$ is

$$\begin{aligned}\dot{\rho}^f_{n,m}(t)=-i\int_{-\infty}^{t}dt_0 r[&G_1^*\sqrt{n+1}a_m^*(t)c_{n+1}(t)+G_2^*\sqrt{n+1}a_m^*(t)b_{n+1}(t)-G_1\sqrt{m+1}a_n(t)c_{m+1}^*(t)\\&-G_2\sqrt{m+1}a_n(t)b_{m+1}^*(t)+G_2\sqrt{n}\,a_{n-1}(t)b_m^*(t)+G_1\sqrt{n}\,a_{n-1}(t)c_m^*(t)\\&-G_2^*\sqrt{m}\,a_{m-1}^*b_n(t)-G_1^*\sqrt{m}\,a_{m-1}^*(t)c_n(t)]-\frac{\gamma_c}{2}(n+m)\rho_{nm}+\gamma_c\sqrt{(n+1)(m+1)}\rho_{n+1,m+1} .\end{aligned} \tag{20}$$

Before making a further deduction, let us consider the atomic phases, i.e., the phases of α_a, α_b, and α_c. We assume that α_b and α_c have a fixed phase relation between themselves, while they have no fixed phase relation with α_a. Therefore terms contain $\alpha_a\alpha_b^*,\alpha_a\alpha_c^*$ or their Hermitian conjugates will be zero after performing the integral. Now, we substitute Eqs. (15) into Eq. (20) and carry out the integral over time t_0 and find the master equation,

$$\begin{aligned}\dot{\rho}^f_{n,m}(t)=&-\frac{rG^2|\alpha_a|^2[\Gamma^2(n+m+2)+G^2(n-m)]}{\Gamma^4+2\Gamma^2G^2(n+m+2)+G^4(n-m)^2}\rho^f_{n,m}(t)+\frac{2r|G_1\alpha_c^*+G_2\alpha_b^*|^2\Gamma^2\sqrt{(n+1)(m+1)}}{\Gamma^4+2\Gamma^2G^2(n+m+2)+G^4(n-m)^2}\rho^f_{n+1,m+1}(t)\\&+\frac{2rG^2|\alpha_a|^2\sqrt{nm}\,\Gamma^2}{\Gamma^4+2\Gamma^2G^2(n+m)+G^4(n-m)^2}\rho^f_{n-1,m-1}(t)-\frac{r|G_1\alpha_c^*+G_2\alpha_b^*|^2[\Gamma^2(n+m)+G^2(n-m)^2]}{\Gamma^4+2\Gamma^2G^2(n+m)+G^4(n-m)^2}\rho^f_{n,m}(t)\\&-\frac{\gamma_c}{2}(n+m)\rho^f_{n,m}(t)+\gamma_c\sqrt{(n+1)(m+1)}\rho^f_{n+1,m+1}(t) ,\end{aligned} \tag{21}$$

where $G^2=|G_1|^2+|G_2|^2$, and r is the pumping rate. In the deduction we have made the approximate $\rho^f(t_0)=\rho^f(t)$, since the dynamics of the light field is governed by the cavity lifetime γ_c^{-1}, which is much longer than the atomic lifetime Γ^{-1}, and thus within the main integration time ($t-2/\Gamma$ to t) ρ^f does not change appreciably, while the value of the integration from $-\infty$ to $t-2/\Gamma$ is negligible due to the exponential decay factor. For the detail of the deduction see the Appendix. Letting

$$A=\frac{2rG^2}{\Gamma^2} , \tag{22a}$$

$$\frac{B}{A}=\frac{4G^2}{\Gamma^2} , \tag{22b}$$

$$\alpha_{cb}=\frac{1}{G}(G_1^*\alpha_c+G_2^*\alpha_b) , \tag{22c}$$

then the master equation becomes

$$\dot{\rho}^f_{n,m} = -\frac{\frac{1}{2}A|\alpha_a|^2[n+m+2+(B/4A)(n-m)^2]}{1+(B/2A)(n+m+2)+(B^2/16A^2)(n-m)^2}\rho^f_{n,m} + \frac{A|\alpha_{cb}|^2\sqrt{(n+1)(m+1)}}{1+(B/2A)(n+m+2)+(B^2/16A^2)(n-m)^2}\rho^f_{n+1,m+1}$$
$$+\frac{A|\alpha_a|^2\sqrt{nm}}{1+(B/2A)(n+m)+(B^2/16A^2)(n-m)^2}\rho^f_{n-1,m-1} - \frac{\frac{1}{2}A|\alpha_{cb}|^2[n+m+(B/4A)(n-m)]}{1+(B/2A)(n+m)+(B^2/16A^2)(n-m)^2}\rho^f_{n,m}$$
$$-\frac{\gamma_c}{2}(n+m)\rho_{n,m}+\gamma_c\sqrt{(n+1)(m+1)}\rho_{n+1,m+1} \, . \tag{21'}$$

This master equation is the basic equation of our following discussion, which is similar to the master equation of an original single-mode laser except terms containing $|\alpha_{cb}|^2$. The role of $|\alpha_a|^2$ is a gain, which corresponds to the spontaneous and stimulated emission due to the population in the upper level. The role of $|\alpha_{cb}|^2$ is similar to the loss, which corresponds to the stimulated absorption due to the populations in the two lower levels. It is from these terms that we can find the possibility of lasing without population inversion.

IV. LASING WITHOUT POPULATION INVERSION

The equation of motion for the diagonal elements can be obtained from Eq. (21′) by putting $m=n$

$$\dot{\rho}^f_{n,n} = -\frac{A(n+1)|\alpha_a|^2}{1+\frac{B}{A}(n+1)}\rho^f_{n,n} + \frac{A(n+1)|\alpha_{cb}|^2}{1+\frac{B}{A}(n+1)}\rho^f_{n+1,n+1}$$
$$+\frac{An|\alpha_a|^2}{1+\frac{B}{A}n}\rho^f_{n-1,n-1} - \frac{An|\alpha_{cb}|^2}{1+\frac{B}{A}n}\rho^f_{n,n}$$
$$-\gamma_c n\rho^f_{n,n}+\gamma_c(n+1)\rho^f_{n+1,n+1} \, . \tag{23}$$

On the right-hand side of Eq. (23) there are six terms, and they can be interpreted as probability flows which can be expressed by arrows in a probability flow diagram, as shown in Fig. 2. The number attached to each arrow indicates which term in Eq. (23) it represents. The first and third terms stand for the spontaneous and stimulated emission; the second and fourth terms for stimulated absorption, and the final two terms for the cavity loss. Using the principle of detailed balance, we find the steady-state photon statistics[16]

$$\rho_{n,n} = \frac{\left[\dfrac{A^2|\alpha_a|^2}{\gamma_c B}\right]^{n+(A/B)(1+\gamma_c A|\alpha_{cb}|^2)}}{Z[n+(A/B)(1+\gamma_c A|\alpha_{cb}|^2)]} \, ,$$

where Z is a normalization constant.

Consider the average photon number $\langle n\rangle$ which satisfies the following equation of motion:

$$\frac{d}{dt}\langle n\rangle = \sum_n n\dot{\rho}_{n,n} = \frac{A|\alpha_a|^2\langle n+1\rangle}{1+\frac{B}{A}\langle n+1\rangle} - \frac{A|\alpha_{cb}|^2\langle n\rangle}{1+\frac{B}{A}\langle n\rangle} - \gamma_c\langle n\rangle \, , \tag{24}$$

where Eq. (23) has been used. In steady state, the average photon number is

$$\langle n\rangle = \frac{A(A|\alpha_a|^2 - A|\alpha_{cb}|^2 - \gamma_c)}{\gamma_c B} \, , \tag{25}$$

where $\langle n\rangle \gg 1$ has been assumed, and $\langle n+1\rangle \simeq \langle n\rangle$ used.

It is quite clear from Eqs. (24) and (25) that lasing without population inversion is reached when the following two conditions are met:

$$|\alpha_a|^2 > |\alpha_{cb}|^2 \, , \tag{26a}$$

$$|\alpha_a|^2 < |\alpha_b|^2, |\alpha_c|^2 \, , \tag{26b}$$

for lasing and noninversion, respectively. From the definition of α_{cb}, Eq. (22c), we have

$$|\alpha_{cb}|^2 = \frac{1}{G^2}[|G_1\alpha_c|^2 + |G_2\alpha_b|^2 + 2|G_1G_2\alpha_c\alpha_b|\cos(\theta+\theta_{bc})] \, , \tag{27}$$

where $G_1G_2^* = |G_1G_2|e^{i\theta}$ and $\alpha_c^*\alpha_b = |\alpha_c\alpha_b|e^{i\theta_{bc}}$. The last term in the right-hand side of Eq. (27) is the interference term. Assuming $|\alpha_c| < |\alpha_b|$, we find that when the following condition:

$$|G_1|^2 + |G_2|^2 > |G_1|^2 + \left|G_2\frac{\alpha_b}{\alpha_c}\right|^2 + 2\left|G_1G_2\frac{\alpha_b}{\alpha_c}\right|\cos(\theta+\theta_{bc}) \tag{28}$$

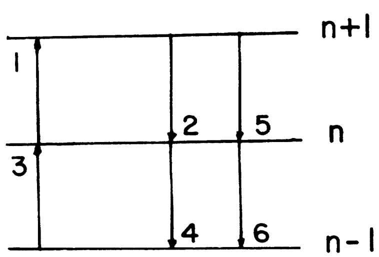

FIG. 2. Diagram of probability flow for the degenerate Λ-type quantum-beat laser.

is valid, we may have lasing without population inversion. It is very clear that this condition can be met if and only if there is coherence between the two lower levels, i.e., θ_{bc} is fixed.

Furthermore, if $|\alpha_c/\alpha_b|=|G_2/G_1|$, we have $\alpha_{cb}=0$. In this case ($\alpha_{cb}=0$), any small amount of population in the upper level, which can be much smaller than the populations in the two lower levels, will lead to lasing if the cavity loss is small.

Noting the definitions of α_{cb}, G_1, and G_2, we have $\alpha_{cb}=0$ when

$$\frac{\alpha_b}{\alpha_c}=\frac{g_2^* e^{-i\phi}-g_1^*}{g_2^*+g_1^* e^{-i\phi}} . \tag{29}$$

In Table I we list four groups of values for the ratios g_2/g_1, α_b/α_c and the phase of the microwave as examples where $\alpha_{cb}=0$. Let us have a close look at the fourth example. The phase of the external microwave makes the effective coupling constants the same, $G_1=G_2$. The two lower levels have the same population and a phase difference π. Because of the interference the total probability of the absorptive transition becomes zero. Thus any small amount of population in the upper level will result in lasing without population inversion.

Physically, lasing without population inversion is a phenomena of quantum interference. When an atom makes a transition from the upper level to the two lower levels, the total transition probability is the sum of the $a\to b$ and $a\to c$ probabilities. However, transition probabilities from the two lower levels to the single upper level are obtained by squaring the sum of the two probability amplitudes. When there is coherence between the two lower levels this can lead to interference terms yielding a null in the transition probability corresponding to photon absorption. From Eq. (23), we see that the contribution of the transition from the upper level to the two lower levels comes from the first and third terms on the right-hand side; and the vice versa transition comes from the second and fourth terms. In the first and third terms there is the factor $G^2|\alpha_a|^2=|G_1^*\alpha_a|^2+|G_2^*\alpha_a|^2$, which is just the statement that the total transition probability from the upper to the lower levels is the sum of the individual transition probabilities. In the second and fourth terms, the corresponding factor is $|G_1^*\alpha_c+G_2^*\alpha_b|^2$. This is just the statement that the total transition probability from the two lower levels to the upper one is the square of the sum of the two individual probability amplitudes. Therefore there are interference terms for the absorption. The interference terms lead to reduction, and even cancellation of the stimulated absorption due to the two lower levels and consequently lead to lasing without population inversion.

TABLE I. Four groups of values that lead to $\alpha_{cb}=0$.

g_2/g_1	α_b/α_c	ϕ
2	$\frac{1}{3}$	0
2	-3	π
$\frac{1}{2}$	$+3$	π
1	-1	$\pi/2$

V. FOKKER-PLANCK EQUATION

In this section, we transform the master equation (21′) into a Fokker-Planck equation for the Glauber P function by expanding the field density operator ρ^f in terms of the diagonal P representation. For the density-matrix elements $\rho^f_{n,m}$, the expansion is

$$\rho^f_{n,n}=\int d^2\alpha\, P(\alpha,\alpha^*,t)e^{-|\alpha|^2}\frac{\alpha^n\alpha^{*m}}{\sqrt{n!m!}} , \tag{30}$$

where $P(\alpha,\alpha^*,t)$ is the Glauber P function. Neglecting 1 compared with $|\alpha|^2$ as large mean photon number is assumed, we obtain the following Fokker-Planck equation for Glauber P function:[17,18]

$$\frac{\partial}{\partial t}P(\alpha,\alpha^*,t)=\left\{\left[-\frac{A}{2}(|\alpha_a|^2-|\alpha_{cb}|^2)\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]+A|\alpha_a|^2\frac{\partial}{\partial\alpha\,\partial\alpha^*}\right.\right.$$
$$\left.\left.-\frac{B}{8}(|\alpha_a|^2+|\alpha_b|^2)\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]\right]M(\alpha)+\frac{\gamma_c}{2}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]\right\}P(\alpha,\alpha^*,t) , \tag{31}$$

where

$$M(\alpha)=\left[1+\frac{B}{A}|\alpha|^2-\frac{B}{2A}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]+\frac{B^2}{16A^2}\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]\right]^{-1} . \tag{32}$$

Equation (31) contains all orders of derivatives with respect to α or α^* because of the inverse operators $M(\alpha)$. Expanding Eq. (31) up to second-order derivatives, which is enough for a discussion on laser intensity and fluctuations, the Fokker-Planck equation becomes

$$\frac{\partial}{\partial t}P(\alpha,\alpha^*,t)=\left[-\frac{\partial}{\partial\alpha}d_\alpha-\frac{\partial}{\partial\alpha}d_{\alpha^*}+2\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}D_{\alpha\alpha^*}+\frac{\partial^2}{\partial\alpha^2}D_{\alpha\alpha}+\frac{\partial^2}{\partial\alpha^{*2}}D_{\alpha^*\alpha^*}\right]P(\alpha,\alpha^*,t) , \tag{33}$$

where

$$d_\alpha = d^*_{\alpha^*} = \frac{\alpha}{2}\left[\frac{A(|\alpha_a|^2-|\alpha_{cb}|^2)}{1+\frac{B}{A}\alpha\alpha^*}-\gamma_c\right], \tag{34a}$$

$$D_{\alpha\alpha}=D^*_{\alpha^*\alpha^*}=-\frac{B(|\alpha_a|^2-|\alpha_{cb}|^2)\alpha\alpha}{4[1+(B/A)\alpha\alpha^*]^2}-\frac{B(|\alpha_a|^2+|\alpha_{cb}|^2)\alpha\alpha}{8[1+(B/A)\alpha\alpha^*]}, \tag{34b}$$

$$D_{\alpha\alpha^*}=\frac{4A|\alpha_a|^2+B(|\alpha_a|^2+|\alpha_{cb}|^2)\alpha\alpha^*}{8[1+(B/A)\alpha\alpha^*]}-\frac{B(|\alpha_a|^2-|\alpha_{cb}|^2)\alpha\alpha^*}{4[1+(B/A)\alpha\alpha^*]^2}. \tag{34c}$$

Next, we express the Fokker-Planck equation in terms of intensity and phase, I and ϕ, through the relation $\alpha=\sqrt{I}e^{i\phi}$,

$$\frac{\partial}{\partial t}P(I,\phi,t)=\left[-\frac{\partial}{\partial I}d_I-\frac{\partial}{\partial \phi}d_\phi+\frac{\partial^2}{\partial I^2}D_{II}+\frac{\partial^2}{\partial \phi^2}D_{\phi\phi}\right]\times P(I,\phi,t), \tag{35}$$

where

$$d_I=I\left[\frac{A(|\alpha_a|^2-|\alpha_{cb}|^2)}{1+(B/A)I}-\gamma_c\right], \tag{36a}$$

$$d_\phi=0, \tag{36b}$$

$$D_{II}=\frac{AI}{[1+(B/A)I]^2}[|\alpha_a|^2+(B/A)I|\alpha_{cb}|^2], \tag{36c}$$

$$D_{\phi\phi}=\frac{A}{4I[1+(B/A)I]}\times[|\alpha_a|^2+(B/2A)I(|\alpha_a|^2+|\alpha_{cb}|^2)]. \tag{36d}$$

From Eq. (36b), it is clear that the phase of the laser field is not locked. This is because we only have atomic coherence between the two lower levels but no atomic coherence between the upper level and the two lower levels. The steady-state laser intensity can be found from Eq. (36a) by $\dot{I}=d_I=0$

$$I=\frac{A(A|\alpha_a|^2-A|\alpha_{cb}|^2-\gamma_c)}{\gamma_c B}. \tag{37}$$

This is the same as Eq. (25). We can see from Eqs. (36c) and (36d) that the phase and intensity diffusion coefficients will be reduced, when the laser operates without population inversion, compared with the situation of no atomic coherence.

Let us consider the ratio of two diffusion coefficients in noncoherent pumping ($|\alpha_{cb}|^2=\frac{1}{2}|\alpha_b|^2+\frac{1}{2}|\alpha_c|^2$) and perfect coherent pumping ($|\alpha_{bc}|=0$) situations,

$$T=D^{(c)}_{\phi\phi}/D^{(n)}_{\phi\phi}, \tag{38}$$

where $D^{(c)}_{\phi\phi}$ and $D^{(n)}_{\phi\phi}$ are the diffusion coefficients for noncoherent pumping and perfect coherent pumping, respectively. For a given intensity, we have

$$A(|\alpha_a|^2-|\alpha_{cb}|^2)=\gamma_c[1+(B/A)I], \tag{39}$$

$$D_{\phi\phi}=\frac{\gamma_c[|\alpha_a|^2+(B/2A)I(|\alpha_a|^2+|\alpha_{cb}|^2)]}{4I(|\alpha_a|^2-|\alpha_{cb}|^2)}. \tag{40}$$

In order to have the same intensity in the two situations, the pumping rate or the atomic populations must be different. We can easily obtain the ratio

$$T=\frac{1-\frac{1}{2}\left[\left|\frac{\alpha_b}{\alpha_a}\right|^2+\left|\frac{\alpha_c}{\alpha_a}\right|^2\right]}{1+\frac{BI}{4A}\left[\left|\frac{\alpha_b}{\alpha_a}\right|^2+\left|\frac{\alpha_c}{\alpha_a}\right|^2\right]\Big/\left[1+\frac{B}{2A}I\right]}, \tag{41}$$

where $|\alpha_a|^2$, $|\alpha_b|^2$, and $|\alpha_c|^2$ are the populations needed to obtain the given intensity in the noncoherence pumping situation. The ratio is the same no matter what is changed, the pumping rate or the atomic populations. From Eq. (41), we see that the ratio can be very small, when $|\alpha_b|^2$ and $|\alpha_c|^2$ are very near $|\alpha_a|^2$, i.e., the phase coefficient is reduced greatly. It is due to the depression of the stimulated absorption, which leads to the reduction of the population of the upper level, and consequently, the reduction of the spontaneous emission. However, the reduction is not large enough to reach noise quenching.

VI. CONCLUSION

In this paper, we use the density-operator method to present the nonlinear theory for the degenerate quantum-beat laser which can be operated without populated inversion. From Eqs. (24), (25), and (28), we can conclude that the lasing without population inversion is independent of the intensity of the laser field. In other words, the conditions for lasing without population inversion are the same in the linear and nonlinear theories. We also worked out the Fokker-Plank equation in the Glauber P representation and pointed out the reduction of diffusion coefficients when the laser operates in the situation of no population inversion. The lasing without inversion is due to the atomic coherence between the two lower levels. For perfect coherent pumping we can have perfect lasing without inversion.

ACKNOWLEDGMENTS

The support of the U.S. Office of Naval Research is gratefully acknowledged.

APPENDIX

We write Eq. (20) in the following form:

$$\dot{\rho}^f_{n,m}(t)=r[G_1^*\sqrt{n+1}I_1(m,n)+G_2^*\sqrt{n+1}I_2(m,n)-G_1\sqrt{m+1}I_1^*(n,m)-G_2\sqrt{m+1}I_2^*(n,m)+G_1\sqrt{n}I_1^*(n-1,m-1)+G_2\sqrt{n}I_2(n-1,m-1)-G_1^*\sqrt{m}I_1(m-1,n-1)-G_2^*\sqrt{m}I_2(m-1,n-1)], \tag{A1}$$

where

$$I_1(m,n)=-i\int_{-\infty}^{t}dt_0 a_m^*(t)c_{n+1}(t)\ , \tag{A2}$$

$$I_2(m,n)=-i\int_{-\infty}^{t}dt_0 a_m^*(t)b_{n+1}(t)\ . \tag{A3}$$

First we calculate $I_1(m,n)$. Substituting Eqs. (15a) and (15c) into (A2), and letting $\tau=t-t_0$, we obtain

$$\begin{aligned}I_1(m,n)=-\int_0^{\infty}d\tau e^{-\Gamma\tau}\Bigg[&\left[\frac{G_1|\alpha_a|^2}{G}\cos G\sqrt{m+1}\tau\sin G\sqrt{n+1}\tau\right]\rho^f_{n,m}(t)\\&-\left[\frac{(G_1\alpha_c^*+G_2\alpha_b^*)(|G_1|^2\alpha_c+G_1G_2^*\alpha_b)}{G^3}\sin G\sqrt{m+1}\tau\cos G\sqrt{n+1}\tau\right]\rho^f_{n+1,m+1}(t)\\&-\left[\frac{(G_1\alpha_c^*+G_2\alpha_b^*)(|G_2|^2\alpha_c-G_1G_2^*\alpha_b)}{G^3}\sin G\sqrt{m+1}\tau\right]\rho^f_{n+1,m+1}(t)\Bigg]\ ,\end{aligned} \tag{A4}$$

where the approximation $\rho^f_{n,m}(t_0)=\rho^f_{n,m}(t)$ has been used. Here, we have noted the fact that the terms containing $\alpha_a^*\alpha_c$, $\alpha_a^*\alpha_b$ and their Hermitian conjugates are equal to zero because of no coherence between $|a\rangle$ and $|b\rangle$ (or $|c\rangle$). Performing the integration we obtain

$$\begin{aligned}I_1(m,n)=&-\frac{G_1|\alpha_a|^2\sqrt{n+1}[\Gamma^2+G^2(n-m)]}{\Gamma^4+2\Gamma^2G^2(n+m+2)+G^4(n-m)^2}\rho^f_{n,m}(t)+\frac{|G_1\alpha_c^*+G_2\alpha_b^*|^2G_1\sqrt{n+1}[\Gamma^2+G^2(n-m)]}{[\Gamma^4+2\Gamma^2G^2(n+m+2)+G^4(n-m)^2]G^2}\rho^f_{n+1,m+1}(t)\\&+\frac{(G_1\alpha_c^*+G_2\alpha_b^*)(|G_2|^2\alpha_c-G_1G_2^*\alpha_b)\sqrt{m+1}}{[\Gamma^2+G^2(m+1)]G^2}\rho^f_{n+1,m+1}(t)\ .\end{aligned} \tag{A5}$$

Similarly, $I_2(m,n)$ is calculated

$$\begin{aligned}I_2(m,n)=&-\frac{G_2|\alpha_a|^2\sqrt{n+1}[\Gamma^2+G^2(n-m)]}{\Gamma^4+2\Gamma^2G^2(n+m+2)+G^4(n-m)^2}\rho^f_{n,m}+\frac{|G_1\alpha_c^*+G_2\alpha_b^*|^2G_2\sqrt{n+1}[\Gamma^2+G^2(n-m)]}{[\Gamma^4+2\Gamma^2G^2(n+m+2)+G^4(n-m)^2]G^2}\rho^f_{n+1,m+1}(t)\\&+\frac{(G_1\alpha_c^*+G_2\alpha_b^*)(|G_1|^2\alpha_b-G_2G_1^*\alpha_c)\sqrt{m+1}}{[\Gamma^2+G^2(m+1)]G^2}\rho^f_{n+1,m+1}(t)\ .\end{aligned} \tag{A6}$$

Substituting Eq. (A5) into (A1) it is easy to obtain Eq. (21).

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