N-photon bremsstrahlung in the soft-photon approximation

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An approximation is derived for the amplitude for spontaneous emission of an arbitrary number of low-frequency photons during the collision of a charged particle with a (short-range) center of force. The result correctly reproduces the first four terms in an expansion of the amplitude in powers of the frequency. The evaluation of the first two terms requires a knowledge of the physical (on-shell) amplitude for radiationless scattering. At this level the approximation represents a generalization of Low's theorem, reducing to his result for the case of single-photon bremsstrahlung. The next two terms involve as additional input the physical one-photon emission amplitude. The calculation is based on the Bloch-Nordsieck method for summing those contributions arising from the emission of soft photons in either the initial or final state of the scattering process. The derivation of the soft-photon approximation is facilitated by the introduction of a transformation to the length gauge. The possibility of extending this approach to allow for a more general class of scattering systems is briefly discussed.

I. INTRODUCTION

It was shown some time ago by Low[1] that the amplitude for the spontaneous emission of a single soft photon during a scattering process can be determined from a knowledge of the physical (on-shell) amplitude for radiationless scattering, with an error of first order in the frequency of the emitted photon. Low's derivation was based on general invariance requirements, which gives the result a universal character. He also provided a version derived directly from the Schrödinger equation, applicable to nonrelativistic potential scattering. A generalization of Low's theorem to the process of two-photon emission during a collision of two relativistic particles was later obtained by Brown and Goble[2] and the nonrelativistic, potential-scattering version of that theorem was derived more recently.[3] Our purpose here is to extend the theorem, in its nonrelativistic form, to the general case in which an arbitrary number of soft photons are radiated during the collision. The method used in the derivation is both simpler and more rigorous than that of Ref. 3, and it provides a higher order of accuracy.

The present approach is rather similar to that used previously in a derivation of a low-frequency approximation for scattering in an *external* field.[4] The dominant soft-photon interaction takes place in initial and final states. In the external-field problem these interactions are treated exactly through the use of Volkov functions serving as modified plane waves, with intermediate-state interactions accounted for perturbatively. An important feature of the derivation given in Ref. 4, which ensures that a well-defined expansion in ascending powers of frequency is obtained, is the representation of the particle-field interaction in intermediate states in terms of electric field rather than the vector potential. This is accomplished by a transformation from the momentum gauge to the length gauge. The same strategy is followed in the present treatment of the problem of spontaneous radiation. Here, in place of the Volkov states, coherent states of the type introduced by Bloch and Nordsieck[5] are adopted as modified plane waves to account for the effectively strong initial- and final-state interactions of the projectile (an electron, say) with the low-frequency radiation field. Bloch and Nordsieck were concerned with a solution of the infrared divergence problem and, accordingly, modes of the field of arbitrarily low frequency were retained in the coherent states employed in their theory. It will be sufficient for our purposes to include only those modes corresponding to observable photons. The utility of these coherent states in the present context lies in the fact that they sum up all radiative interactions which introduce inverse powers of the frequency into the perturbation expansion of the transition amplitude. Since, as indicated above, the perturbation terms arising from intermediate-state interactions do in fact enter with increasing powers of the frequency, it is possible to be quite definite in determining the power of the frequency associated with each term in the expansion. This is an important consideration in extending the analysis from the case of one-photon bremsstrahlung to the more complex multiphoton process.

The calculational procedure to be followed is defined in Sec. II. In order to focus on the essential features of the method we confine our attention here to the relatively simple problem of scattering from a short-range potential. The one-photon emission amplitude is discussed, as a first illustration, in Sec. III A. The amplitude is represented as the sum of a low-frequency approximation (in its familiar form[1]) plus an error, correct to the second power of the frequency. This analysis provides the background for treatment for the N-photon emission problem, which follows along very similar lines. It is shown that the first *four* terms in an expansion of the N-photon bremsstrahlung amplitude in ascending powers of the frequency can be calculated from a knowledge of the physical (on-shell) amplitudes for radiationless scattering and for scattering with the emission of a single photon. (More precisely, it is the above-mentioned correction to the low-frequency approximation to the one-photon amplitude which enters.) For the sake of clarity the two-photon process is treated separately, in Sec. III B, with the general case taken up in Sec. III C.

While multiphoton spontaneous bremsstrahlung may not be accessible to experimental study for some time, the analysis of such processes is useful for the insight it gives concerning the nature of the low-frequency approximation and the relationship which exists between spontaneous and stimulated bremsstrahlung. More specifically, the external-field scattering problem may be treated, with greater accuracy than achieved previously,[4] by generalizing the present approach to the case where the initial state of the field is not the vacuum but rather is one with a high photon occupation number. (The Bloch-Nordsieck coherent states are transformed into Volkov states in that limit.) In another very natural generalization the method described here may be applied to the infrared divergence problem, for which the coherent states were originally introduced,[5] in order to obtain higher-order corrections to the Bloch-Nordsieck sum rule for the scattering cross section. We hope to elaborate further on these matters in the future.

II. MODIFIED PERTURBATION THEORY

A. Bloch-Nordsieck states

At low frequencies the dominant radiative interactions are those in which photons are emitted in either initial or final states. To proceed systematically we sum all such contributions at the outset; the remaining terms are then guaranteed to be of higher order in the expansion in powers of the frequency. One knows that in the dipole approximation, which is adopted here, the sums of the initial- and final-state interactions can be put in a very simple product form. Equivalently, and more conveniently for our present purposes, one may employ the Bloch-Nordsieck states as modified plane waves in the formulation of the theory of the potential-scattering process. Interactions are thereby included to all orders in the electric charge. We confine our attention here to the lowest nonvanishing order of perturbation theory and we must ultimately expand in powers of the charge and select the terms contributing to the appropriate order. The amplitude obtained using the Bloch-Nordsieck states thus plays the role of a generating function for the transition amplitudes of physical interest.[6] The assumption of the dipole approximation introduces a significant simplification in the development of this program since one may then omit the A^2 contribution to the particle-field interaction.[7] The interaction is then linear in the field operators, and this enables one to carry out the construction of the Bloch-Nordsieck coherent states.

The Hamiltonian is taken to be of the form (in units with $\hbar=1$)

$$H=\frac{(-i\nabla-e\,\mathbf{A}/c)^2}{2m}+H_F+V-\frac{e^2A^2}{2mc^2}\ . \tag{2.1}$$

The vector potential is represented, in the dipole approximation, as

$$\mathbf{A}=\sum_j\left[\frac{2\pi c^2}{\omega_j L^3}\right]^{1/2}\boldsymbol{\lambda}_j(a_j+a_j^\dagger)\ , \tag{2.2}$$

where L^3 is the quantization volume, the sum is over all modes of the field, and a basis of linearly polarized states has been adopted. In Eq. (2.1), V is the scattering potential, taken here for simplicity to be that associated with a structureless target, and H_F is the field Hamiltonian. In constructing the modified plane waves we look for solutions of the wave equation

$$(H-V-E_\alpha)|\psi_\alpha\rangle=0\ , \tag{2.3}$$

in which the scattering potential has been removed but the particle-field interaction remains. The channel index α specifies the momentum of the incident (or emergent) particle along with the set of photon occupation numbers $n_1,n_2,\ldots$, which characterize the state of the field. The solutions may be put in the form[8]

$$|\psi_\alpha\rangle=W_{\mathbf{p}}|\mathbf{p}\rangle|n_1,n_2,\ldots\rangle\ . \tag{2.4}$$

Here we have

$$W_{\mathbf{p}}=\exp\left[\sum_j\rho_{j\mathbf{p}}(a_j-a_j^\dagger)\right]\ , \tag{2.5}$$

with

$$\rho_{j\mathbf{p}}=-\frac{e}{m}\left[\frac{2\pi}{\omega_j L^3}\right]^{1/2}\frac{\mathbf{p}\cdot\boldsymbol{\lambda}_j}{\omega_j}\ . \tag{2.6}$$

The energy eigenvalue is

$$E_\alpha=\frac{p^2}{2m}+\sum_j n_j\omega_j-\sum_j\omega_j\rho_{j\mathbf{p}}^2\ . \tag{2.7}$$

The second sum in Eq. (2.7) is a level-shift term, representing the effect of absorption and reemission of photons; it is ignored in the following since it contributes to a higher order of perturbation theory than is being accounted for here. The presence of the destruction operator in Eq. (2.5) will be ignored for the same reason. The solution (2.4) corresponds to that given by Bloch and Nordseick. There is a difference in interpretation, however, since here the photons are assumed to be of low frequency, but not so low as to make them unobservable.

We are concerned with free-free transitions in which no photons are present in the initial state and N distinguishable photons are emitted during the collision. A generating function for the transition amplitude may be defined as

$$F_{\alpha';\alpha}=\langle\psi_{\alpha'}|[V+VG(E_\alpha)V]|\psi_\alpha\rangle\ , \tag{2.8}$$

with $E_{\alpha'}=E_\alpha$ and

$$G(E)=(E-H)^{-1}\ . \tag{2.9}$$

(As usual, the energy in the denominator of the Green's function is understood to contain an infinitesimal positive imaginary part.)

B. Gauge transformation

The derivation of the low-frequency approximation is very much simplified by a transformation to the length gauge, in which the interaction is of the $\mathbf{E}\cdot\mathbf{r}$ form, since this interactions is of first order in the frequency. The transformation is brought about by writing the Green's function as

$$G(E)=e^{\chi}\bar{G}(E)e^{-\chi}\,, \tag{2.10}$$

with

$$\chi=i\frac{e}{c}\mathbf{A}\cdot\mathbf{r}\,. \tag{2.11}$$

The transformed Green's function is

$$\bar{G}(E)=\left[E-\left[-\frac{\nabla^2}{2m}+V+\bar{H}_F-\frac{e^2A^2}{2mc^2}\right]\right]^{-1}, \tag{2.12}$$

where

$$\bar{H}_F=e^{-\chi}H_Fe^{\chi}\,. \tag{2.13}$$

This expression for the transformed field energy may be expanded as

$$\bar{H}_F=H_F-[\chi,H_F]+\frac{1}{2!}[\chi,[\chi,H_F]]\,, \tag{2.14}$$

where

$$-[\chi,H_F]=-e\mathbf{E}\cdot\mathbf{r} \tag{2.15}$$

and

$$\mathbf{E}=\sum_j i\left[\frac{2\pi}{\omega_jL^3}\right]^{1/2}\omega_j\boldsymbol{\lambda}_j(a_j-a_j^\dagger) \tag{2.16}$$

is the electric-field operator. The third term in Eq. (2.14) is a c number of second order in the charge; this level-shift contribution is ignored for reasons already discussed. Since higher-order commutators vanish the expansion terminates, as shown, after the third term.

A perturbation expansion for the transformed Green's function may now be developed with

$$H_0=-\frac{\nabla^2}{2m}+H_F+V \tag{2.17}$$

taken as the unperturbed Hamiltonian and with

$$G_0(E)=(E-H_0)^{-1} \tag{2.18}$$

identified as the associated Green's function. The expansion takes the form

$$\bar{G}(E)=G_0(E)+G_0(E)\left[-e\mathbf{E}\cdot\mathbf{r}-\frac{e^2A^2}{2mc^2}\right]G_0(E)+\cdots\,. \tag{2.19}$$

Only the first two terms, shown here explicitly, will be required in the approximation procedure to be introduced below.

It will be convenient to interpret the vector $\mathbf{r}$ appearing in the gauge function χ as the generator of momentum translations. Accordingly, we make the replacement $\mathbf{r}\to i\nabla_{\mathbf{p}}$ and write

$$W_{\mathbf{p}}e^{-\chi}=\exp\left[\sum_j\frac{e}{m}\left[\frac{2\pi}{\omega_jL^3}\right]^{1/2}\left[\frac{\mathbf{p}-m\omega_j\nabla_{\mathbf{p}}}{\omega_j}\right]\cdot\boldsymbol{\lambda}_ja_j^\dagger\right], \tag{2.20}$$

with the momentum-gradient operator understood to act on the initial-state ket $|\mathbf{p}\rangle$. [Since we omit higher-order terms involving photon absorption, no annihilation operators appear in Eq. (2.20).] The "dressing" of the final state is brought about by the operator

$$e^{\chi}W_{\mathbf{p}'}^\dagger=\exp\left[\sum_j\frac{e}{m}\left[\frac{2\pi}{\omega_jL^3}\right]^{1/2}\times\left[-\frac{\mathbf{p}'+m\omega_j\nabla_{\mathbf{p}'}}{\omega_j}\right]\cdot\boldsymbol{\lambda}_ja_j^\dagger\right], \tag{2.21}$$

where we have made the replacement $\mathbf{r}\to i\nabla_{\mathbf{p}'}$; this is understood to be acting on the bra $\langle\mathbf{p}'|$.

C. Low-frequency approximation for the generating function

Somewhat in the spirit of a distorted-wave Born approximation, the low-frequency approximation is defined by breaking off the perturbation expansion of the transformed Green's function after the first two terms—the two shown explicitly in Eq. (2.19). To characterize in the simplest manner the level of accuracy achieved through this procedure, we introduce a reference frequency ω and adopt the convention that, for each low-frequency mode, the frequency is expressed as $\omega_j=c_j\omega$, with the coefficients c_j each of order unity. The expansion (2.19), along with the expansion of the exponentials in Eqs. (2.20) and (2.21), may then be interpreted in terms of a series in ascending powers of ω. The approximation procedure just defined correctly generates the first four terms of this series, of orders ω^{-N}, ω^{-N+1}, ω^{-N+2}, and ω^{-N+3}.

A more explicit form for the approximate generating function is $F\cong F^{(1)}+F^{(2)}$ with

$$F^{(1)}_{\alpha';\alpha}=\langle n_1',n_2',\cdots|e^{\chi}W_{\mathbf{p}'}^\dagger\langle\mathbf{p}'|[V+Vg(E_{\mathbf{p}}-H_F)V]|\mathbf{p}\rangle W_{\mathbf{p}}e^{-\chi}|0\rangle \tag{2.22}$$

and

$$F^{(2)}_{\alpha';\alpha}=\left\langle n_1',n_2',\cdots\left|e^{\chi}W_{\mathbf{p}'}^\dagger\left\langle\mathbf{p}'\left|Vg(E_{\mathbf{p}}-H_F)\left[-e\mathbf{E}\cdot\mathbf{r}-\frac{e^2A^2}{2mc^2}\right]g(E_{\mathbf{p}}-H_F)V\right|\mathbf{p}\right\rangle W_{\mathbf{p}}e^{-\chi}\right|0\right\rangle. \tag{2.23}$$

Here we have defined

$$g(E)=\left[E-\left[-\frac{\nabla^2}{2m}+V\right]\right]^{-1} \tag{2.24}$$

and $E_{\mathbf{p}}=p^2/2m$. The leading term $F^{(1)}$ generates the first two terms in the low-frequency expansion. As shown below these terms may be evaluated provided the physical (on-shell) amplitude for scattering in the absence of the field is known. In this way we obtain a nonrelativistic version of Low's approximation,[1] generalized from $N=1$ to an arbitrary value of N. (The calculation is not, in any essential way, more complex for the multiphoton case than it is for $N=1$.) Evaluation of the correction term $F^{(2)}$, which contains the next two terms in the low-frequency expansion, allows for the possibility of estimating the accuracy of the leading terms and of extending the range of validity of the approximation.

III. SOFT-PHOTON APPROXIMATIONS

A. One-photon amplitude

In this subsection we illustrate, in the simplest context, the approach which was outlined above and introduce notation in preparation for later generalization to the problem of multiphoton bremsstrahlung. The amplitude for the emission of a single photon of frequency ω and polarization $\boldsymbol{\lambda}$ is obtained by selecting terms of first order in the charge in the series expansion of the generating function. [In this case higher-order terms in the perturbation expansion (2.19) are absent.] By expanding the exponentials in Eqs. (2.20) and (2.21) we are led to the transition amplitude

$$T(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda})=T^{(1)}(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda})+T^{(2)}(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda}) . \tag{3.1}$$

With an overall factor $-(e^2 2\pi/m^2\omega L^3)^{1/2}$ omitted to simplify notation, we have

$$T^{(1)}(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda})=-\left[\frac{\mathbf{p}'+m\omega\nabla_{\mathbf{p}'}}{\omega}\right]\cdot\boldsymbol{\lambda}t(\mathbf{p}',\mathbf{p};E_{\mathbf{p}})+\left[\frac{\mathbf{p}-m\omega\nabla_{\mathbf{p}}}{\omega}\right]\cdot\boldsymbol{\lambda}t(\mathbf{p}',\mathbf{p};E_{\mathbf{p}'}) . \tag{3.2}$$

Here $E_{\mathbf{p}'}=E_{\mathbf{p}}-\omega$ and

$$t(\mathbf{p}',\mathbf{p};E)=\langle\mathbf{p}'|[V+Vg(E)V]|\mathbf{p}\rangle \tag{3.3}$$

is the (off-shell) scattering amplitude. The correction term is

$$T^{(2)}(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda})=im\omega\boldsymbol{\lambda}\cdot\langle\mathbf{p}'|Vg(E_{\mathbf{p}'})\mathbf{r}g(E_{\mathbf{p}})V|\mathbf{p}\rangle . \tag{3.4}$$

The identity (3.1) provides a natural starting point for approximations since the correction term in Eq. (3.4) is seen to be of order ω; dropping this term gives an approximation to the transition amplitude which is correct to order unity.

As noted, the field-free t-matrix elements in Eq. (3.2) are off the energy shell. One may expand these amplitudes about their on-shell values and observe that off-shell information is not required if terms of order ω are neglected in the evaluation of the bremsstrahlung amplitude. This is the content of Low's theorem for potential scattering. For later purposes, as part of our study of the multiphoton problem, it will be useful to retain the terms of order ω and ω^2 in this expansion, now to be defined. Let the function $t(\mathbf{p}',\mathbf{p};E)$ be expressed in terms of the scalar variables

$$\xi=\frac{p^2}{2m}-E,\quad \xi'=\frac{p'^2}{2m}-E,\quad \tau=(\mathbf{p}'-\mathbf{p})^2 \tag{3.5}$$

as $t(\xi',\xi,\tau,E)$. The momentum gradients may be transformed as

$$\nabla_{\mathbf{p}}=\frac{\mathbf{p}}{m}\frac{\partial}{\partial\xi}-2(\mathbf{p}'-\mathbf{p})\frac{\partial}{\partial\tau} ,\qquad \nabla_{\mathbf{p}'}=\frac{\mathbf{p}'}{m}\frac{\partial}{\partial\xi'}+2(\mathbf{p}'-\mathbf{p})\frac{\partial}{\partial\tau} . \tag{3.6}$$

Expanding about the on-shell function $t(0,0,\tau,E)\equiv t(E,\tau)$, we have

$$t(\xi',0,\tau,E_{\mathbf{p}})=t(E_{\mathbf{p}},\tau)+\xi'\frac{\partial t(\xi',0,\tau,E_{\mathbf{p}})}{\partial\xi'}+\cdots , \tag{3.7a}$$

$$t(0,\xi,\tau,E_{\mathbf{p}'})=t(E_{\mathbf{p}'},\tau)+\xi\frac{\partial t(0,\xi,\tau,E_{\mathbf{p}'})}{\partial\xi}+\cdots . \tag{3.7b}$$

Here and in the following, derivatives of the t matrix are understood to be evaluated on shell, i.e., for $\xi'=\xi=0$. These Taylor series expansions are truncated after the cubic terms and then substituted in Eq. (3.2). This yields an expression for $T^{(1)}$ which is correct to order ω^2. We absorb the first- and second-order terms into $T^{(2)}$, thereby defining a new correction term R. After this rearrangement we have

$$T=T_{\mathrm{LFA}}+R , \tag{3.8}$$

with the low-frequency approximation, correct to order unity and expressed in terms of the on-shell t matrix, given by

$$T_{\mathrm{LFA}}(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda})=-\frac{\mathbf{p}'\cdot\boldsymbol{\lambda}}{\omega}t(E_{\mathbf{p}},\tau)+\frac{\mathbf{p}\cdot\boldsymbol{\lambda}}{\omega}t(E_{\mathbf{p}'},\tau) . \tag{3.9}$$

This is Low's approximation. With T assumed to be known, as it will be in the following treatment of the multiphoton problem, the remainder R is determined by Eq. (3.8). For later reference, however, we will require the explicit form obtained for this function, correct to order ω^2, by means of the expansion procedure outlined above. We find that

$$R(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda})=T^{(2)}(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda})+2m(\mathbf{p}'-\mathbf{p})\cdot\boldsymbol{\lambda}\frac{\partial}{\partial\tau}\left[t(E_{\mathrm{p}'})-t(E_{\mathrm{p}})+\omega\left[\frac{\partial t(E_{\mathrm{p}})}{\partial\xi'}+\frac{\partial t(E_{\mathrm{p}'})}{\partial\xi}\right]\right]$$

$$+\mathbf{p}'\cdot\boldsymbol{\lambda}\frac{\partial^2}{\partial\xi'^2}\left[\frac{1}{2}\omega t(E_{\mathrm{p}})-\frac{1}{3}\omega^2\frac{\partial t(E_{\mathrm{p}})}{\partial\xi'}\right]-\mathbf{p}\cdot\boldsymbol{\lambda}\frac{\partial^2}{\partial\xi^2}\left[\frac{1}{2}\omega t(E_{\mathrm{p}'})-\frac{1}{3}\omega^2\frac{\partial t(E_{\mathrm{p}'})}{\partial\xi}\right]. \tag{3.10}$$

To simplify notation only the energy variable has been displayed as an argument of the on-shell t matrix. We remark that the difference $t(E_{\mathrm{p}'})-t(E_{\mathrm{p}})$ appearing in the second term on the right-hand side of Eq. (3.10) is considered here to be of first order in the frequency. It should be kept in mind, however, that in the neighborhood of a resonance whose width is comparable to the frequency, this difference may be large enough to warrant the inclusion of the term in which it appears as part of the low-frequency approximation. That is, one would add this term to that shown in Eq. (3.9), leaving an error which is truly of first order. This was the suggestion made by Feshbach and Yennie who modified Low's formula to account for resonant free-free transitions.[9] In the present discussion, however, we shall not be explicitly concerned with resonant scattering processes. In fact, terms of order $\omega^2[t(E_{\mathrm{p}'})-t(E_{\mathrm{p}})]$ have been dropped in arriving at Eq. (3.10). An analogous situation arises in the following treatment of the multiphoton problem and certain terms will be omitted there (without further comment) for the same reason. Such terms, which are effectively of a lower order near a resonance, may be included, without any difficulty in principle, if they turn out to be significant.

B. Two-photon amplitude

The amplitude for a two-photon free-free transition can be expressed, in analogy with Eq. (3.1), as $T=T^{(1)}+T^{(2)}$, with these two components derived from the generating function in the manner outlined above. [A factor $-(e^2 2\pi/m^2\omega_j L^3)^{1/2}$ for each radiated photon is omitted in defining this amplitude.] The first component is

$$T^{(1)}(\mathbf{p}',\mathbf{p},\omega_1\boldsymbol{\lambda}_1,\omega_2\boldsymbol{\lambda}_2)=(\omega_1\omega_2)^{-1}([(\mathbf{p}'+m\omega_1\nabla_{\mathrm{p}'})\cdot\boldsymbol{\lambda}_1(\mathbf{p}'+m\omega_2\nabla_{\mathrm{p}'})\cdot\boldsymbol{\lambda}_2]t(\mathbf{p}',\mathbf{p};E_{\mathrm{p}})$$

$$+[(\mathbf{p}-m\omega_1\nabla_{\mathrm{p}})\cdot\boldsymbol{\lambda}_1(\mathbf{p}-m\omega_2\nabla_{\mathrm{p}})\cdot\boldsymbol{\lambda}_2]t(\mathbf{p}',\mathbf{p};E_{\mathrm{p}}-\omega_1-\omega_2)$$

$$-\{[(\mathbf{p}'+m\omega_1\nabla_{\mathrm{p}'})\cdot\boldsymbol{\lambda}_1(\mathbf{p}-m\omega_2\nabla_{\mathrm{p}})\cdot\boldsymbol{\lambda}_2]t(\mathbf{p}',\mathbf{p};E_{\mathrm{p}}-\omega_2)+(1\leftrightarrow 2)\})\,, \tag{3.11}$$

where $(1\leftrightarrow 2)$ denotes the preceding terms with indices 1 and 2 interchanged. A transformation to the scalar variables introduced in Sec. III A enables us to expand the t matrix, in powers of the frequency, about its on-shell form. The leading terms in this expansion provide us with the low-frequency approximation

$$\begin{aligned}T_{\mathrm{LFA}}=(\omega_1\omega_2)^{-1}[&\mathbf{p}'\cdot\boldsymbol{\lambda}_1\mathbf{p}'\cdot\boldsymbol{\lambda}_2 t(E_{\mathrm{p}},\tau)\\&+\mathbf{p}\cdot\boldsymbol{\lambda}_1\mathbf{p}\cdot\boldsymbol{\lambda}_2 t(E_{\mathrm{p}}-\omega_1-\omega_2,\tau)\\&-\mathbf{p}'\cdot\boldsymbol{\lambda}_1\mathbf{p}\cdot\boldsymbol{\lambda}_2 t(E_{\mathrm{p}}-\omega_2,\tau)\\&-\mathbf{p}'\cdot\boldsymbol{\lambda}_2\mathbf{p}\cdot\boldsymbol{\lambda}_1 t(E_{\mathrm{p}}-\omega_1,\tau)]\,.\end{aligned} \tag{3.12}$$

This result, which involves only on-shell values of the t matrix, represents the two-photon generalization of Low's approximation (3.9). Of the higher-order terms in the expansion of the amplitude $T^{(1)}$ shown in Eq. (3.11), only those of order unity and of first-order in the frequency are retained. These are combined with the correction term $T^{(2)}$, which is thereby converted into a more useful form. The original form of the correction term, as obtained from the generating function, is the sum of two parts

$$T^{(2)}=T_1^{(2)}+T_2^{(2)}\,, \tag{3.13}$$

where the first term on the right arises from presence of the $\mathbf{E}\cdot\mathbf{r}$ interaction in Eq. (2.23) and the second term accounts for the A^2 interaction (the presence of which, recall, was induced by the gauge transformation). The first term is

$$T_1^{(2)}(\mathbf{p}',\mathbf{p};\omega_1\boldsymbol{\lambda}_1,\omega_2\boldsymbol{\lambda}_2)=-\left[\frac{\mathbf{p}'\cdot\boldsymbol{\lambda}_2}{\omega_2}T^{(2)}\left[\mathbf{p}'+\frac{m\omega_2\boldsymbol{\lambda}_2}{\mathbf{p}'\cdot\boldsymbol{\lambda}_2},\mathbf{p};\omega_1\boldsymbol{\lambda}_1\right]+(1\leftrightarrow 2)\right]$$

$$+\left[\frac{\mathbf{p}\cdot\boldsymbol{\lambda}_2}{\omega_2}T^{(2)}\left[\mathbf{p}',\mathbf{p}-\frac{m\omega_2\boldsymbol{\lambda}_2}{\mathbf{p}\cdot\boldsymbol{\lambda}_2};\omega_1\boldsymbol{\lambda}_1\right]+(1\leftrightarrow 2)\right], \tag{3.14}$$

where we have made use of the definition (3.4) for the one-photon matrix element in obtaining this equation. It should be noted that since these one-photon correction terms are of first order in the frequency the insertion of the momentum shifts in their arguments, which is only valid to first order, is legitimate since it leads to second-order errors in the two-photon amplitude and these are ignored here. The induced A^2 term is

$$T_2^{(2)}(\mathbf{p}',\mathbf{p};\omega_1\boldsymbol{\lambda}_1,\omega_2\boldsymbol{\lambda}_2) = -m\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2\langle\mathbf{p}'|Vg(E_{\mathbf{p}'})g(E_{\mathbf{p}})V|\mathbf{p}\rangle \ . \quad (3.15)$$

A more convenient version of this expression is obtained with the aid of the resolvent identity

$$g(E_{\mathbf{p}'})g(E_{\mathbf{p}})=(E_{\mathbf{p}}-E_{\mathbf{p}'})^{-1}[g(E_{\mathbf{p}'})-g(E_{\mathbf{p}})] \ . \quad (3.16)$$

The use of Eq. (3.3) then allows us to convert Eq. (3.15) to a form involving the (off-shell) t matrix. Remarkably, the off-shell contributions cancel when we account for those higher-order terms which were omitted in deriving the low-frequency approximation (3.12) from Eq. (3.11). Moreover, inclusion of these higher-order terms allows us, with the aid of Eq. (3.10), to convert the amplitude shown in Eq. (3.14) to one involving the *on*-shell correction to the one-photon bremsstrahlung matrix element. With algebraic details omitted, we give the expression for the improved low-frequency approximation, correct to first order in the frequency, as

$$T(\mathbf{p}',\mathbf{p};\omega_1\boldsymbol{\lambda}_1,\omega_2\boldsymbol{\lambda}_2)\cong T_{\mathrm{LFA}}+R_1+R_2 \ , \quad (3.17)$$

with

$$R_1(\mathbf{p}',\mathbf{p};\omega_1\boldsymbol{\lambda}_1,\omega_2\boldsymbol{\lambda}_2)=-\left[\frac{\mathbf{p}'\cdot\boldsymbol{\lambda}_2}{\omega_2}R\left[\mathbf{p}'+\frac{m\omega_2\boldsymbol{\lambda}_2}{\mathbf{p}'\cdot\boldsymbol{\lambda}_2},\mathbf{p};\omega_1\boldsymbol{\lambda}_1\right]+(1\leftrightarrow2)\right]+\left[\frac{\mathbf{p}\cdot\boldsymbol{\lambda}_2}{\omega_2}R\left[\mathbf{p}',\mathbf{p}-\frac{m\omega_2\boldsymbol{\lambda}_2}{\mathbf{p}\cdot\boldsymbol{\lambda}_2};\omega_1\boldsymbol{\lambda}_1\right]+(1\leftrightarrow2)\right] \quad (3.18)$$

and

$$R_2=m\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2(\omega_1+\omega_2)^{-1}[t(E_{\mathbf{p}},\tau)-t(E_{\mathbf{p}'},\tau)] \ . \quad (3.19)$$

The method employed earlier to derive a low-frequency approximation for two-photon bremsstrahlung[3] was not sufficiently accurate to correctly reproduce the terms of first-order in the frequency, but with regard to the lower-order terms that result is consistent with the one given here.[10]

C. N-photon amplitude

All of the essential features of the derivation of a low-frequency approximation have appeared in the treatment of the two-photon problem. The emission of additional photons is accounted for by allowing then to be radiated by either the incoming or outgoing electron. That is, it is not necessary, in achieving the stated level of accuracy, to include processes in which more than two photons are emitted in intermediate states. All terms corresponding to a distinct labeling of the emitted photons must be summed, and this requires the introduction of some notation, as follows.

In deriving an expression for the amplitude $T^{(1)}$, the multiphoton generalization of the amplitude shown in Eq. (3.11), one sums a set of terms in which none of the photons are emitted in intermediate states. In a typical term there are n photons which have been radiated before the collision and the remaining n-N emerge in the final state. There is no physical significance to the interchange of photon labels within each of these two groups; only interchanges between the two groups are distinguishable. Accordingly, we introduce the permutation

$$(1,2,\ldots,n|n+1,\ldots,N) \xrightarrow[P(n,q)]{} (i_1,i_2,\ldots,i_n|i_{n+1},\ldots,i_N) \ , \quad (3.20)$$

where the index q labeling each permutation runs from 1 to $N!/n!(N-n)!$. With this notation understood we express the result, obtained from the generating function (2.22), as

$$T^{(1)}(\mathbf{p}',\mathbf{p};\omega_1\boldsymbol{\lambda}_1,\ldots,\omega_N\boldsymbol{\lambda}_N)=\sum_{n=0}^{N}\sum_q\left[\prod_{s=n+1}^{N}\left[\frac{-\mathbf{p}'-m\omega_{i_s}\nabla_{\mathbf{p}'}}{\omega_{i_s}}\right]\cdot\boldsymbol{\lambda}_{i_s}\right]\left[\prod_{s=1}^{n}\left[\frac{\mathbf{p}-m\omega_{i_s}\nabla_{\mathbf{p}}}{\omega_{i_s}}\right]\cdot\boldsymbol{\lambda}_{i_s}\right]t(\mathbf{p}',\mathbf{p};E(n,q)) \ . \quad (3.21)$$

Here we defined

$$E(n,q)=\begin{cases}E_{\mathbf{p}}-\sum_{s=1}^{n}\omega_{i_s}, & 1\le n\le N\\ E_{\mathbf{p}}, & n=0 \ ,\end{cases} \quad (3.22)$$

representing the energy of the electron after it has emitted the first n photons. The products in Eq. (3.21) are interpreted as taking the value unity when the lower limit exceeds the upper limit; this accounts (with a minimum of additional notation) for the terms in which all the photons are emitted in either the initial state ($n=0$) or the final state ($n=N$). If we expand the t matrix in Eq. (3.21) about its on-shell value and retain only those terms which contribute to the first two orders (ω^{-N} and ω^{-N+1}) in the low-frequency expansion we obtain the on-shell approximation

$$T_{\mathrm{LFA}}=\sum_{n=0}^{N}\sum_{q}\prod_{s=n+1}^{N}\left[\frac{-\mathbf{p}'\cdot\boldsymbol{\lambda}_{i_s}}{\omega_{i_s}}\right]\times\prod_{s=1}^{n}\left[\frac{\mathbf{p}\cdot\boldsymbol{\lambda}_{i_s}}{\omega_{i_s}}\right]t(E(n,q),\tau)\ . \tag{3.23}$$

The terms of order ω^{-N+2} and ω^{-N+3} are combined with the amplitude $T^{(2)}$; the similarity of this procedure with that followed for the two-photon case should be clear.

In constructing the N-photon generalization of the amplitude shown in Eq. (3.14) we recognize that since one of the photons is emitted in an intermediate state the permutations of photon indices to be considered are of the form

$$(1,2,\ldots,n|n+1|n+2,\ldots,N)\xrightarrow[P_1(n,q_1)]{}(i_1,i_2,\ldots,i_n|i_{n+1}|i_{n+2},\ldots,i_N)\ , \tag{3.24}$$

with

$$0\le n\le N-1,\quad 1\le q_1\le\frac{N!}{n!(N-n-1)!}\ .$$

We find that the contribution to $T^{(2)}$ arising from the $\mathbf{E}\cdot\mathbf{r}$ interaction is

$$T_1^{(2)}=\sum_{n=0}^{N-1}\sum_{q_1}\prod_{s=n+2}^{N}\left[\frac{-\mathbf{p}'\cdot\boldsymbol{\lambda}_{i_s}}{\omega_{i_s}}\right]\prod_{s=1}^{n}\left[\frac{\mathbf{p}\cdot\boldsymbol{\lambda}_{i_s}}{\omega_{i_s}}\right]T^{(2)}(\mathbf{p}'(q_1),\mathbf{p}(q_1);\omega_{i_{n+1}}\boldsymbol{\lambda}_{i_{n+1}})\ . \tag{3.25}$$

The shifted momenta are

$$\mathbf{p}(q_1)=\begin{cases}\mathbf{p}-\sum_{s=1}^{n}\dfrac{m\omega_{i_s}\boldsymbol{\lambda}_{i_s}}{\mathbf{p}\cdot\boldsymbol{\lambda}_{i_s}}, & 1\le n\le N-1\\ \mathbf{p}, & n=0\end{cases} \tag{3.26a}$$

and

$$\mathbf{p}'(q_1)=\begin{cases}\mathbf{p}'+\sum_{s=n+2}^{N}\dfrac{m\omega_{i_s}\boldsymbol{\lambda}_{i_s}}{\mathbf{p}'\cdot\boldsymbol{\lambda}_{i_s}}, & 0\le n\le N-2\\ \mathbf{p}', & n=N-1\ .\end{cases} \tag{3.26b}$$

The N-photon generalization of Eq. (3.15), arising from the induced A^2 term, involves two intermediate-state photons. Photon labels are assigned from among the permutations

$$(1,\ldots,n|n+1,n+2|n+3,\ldots,N)\xrightarrow[P_2(n,q_2)]{}(i_1,\ldots,i_n|i_{n+1},i_{n+2}|i_{n+3},\ldots,i_N)\ , \tag{3.27}$$

with

$$0\le n\le N-2,\quad 1\le q_2\le\frac{N!}{n!2!(N-n-2)!}\ .$$

The amplitude takes the form

$$T_2^{(2)}=\sum_{n=0}^{N-2}\sum_{q_2}\prod_{s=n+3}^{N}\left[\frac{-\mathbf{p}'\cdot\boldsymbol{\lambda}_{i_s}}{\omega_{i_s}}\right]\prod_{s=1}^{n}\left[\frac{\mathbf{p}\cdot\boldsymbol{\lambda}_{i_s}}{\omega_{i_s}}\right](-m\boldsymbol{\lambda}_{i_{n+1}}\cdot\boldsymbol{\lambda}_{i_{n+2}})\langle\mathbf{p}'(q_2)|Vg(E_{\mathbf{p}'(q_2)})g(E_{\mathbf{p}(q_2)})V|\mathbf{p}(q_2)\rangle\ . \tag{3.28}$$

The shifted momenta are defined as

$$\mathbf{p}(q_2)=\begin{cases}\mathbf{p}-\sum_{s=1}^{n}\dfrac{m\omega_{i_s}\boldsymbol{\lambda}_{i_s}}{\mathbf{p}\cdot\boldsymbol{\lambda}_{i_s}}, & 1\le n\le N-2\\ \mathbf{p}, & n=0\end{cases} \tag{3.29a}$$

and

$$\mathbf{p}'(q_2)=\begin{cases}\mathbf{p}'+\sum_{s=n+3}^{N}\dfrac{m\omega_{i_s}\boldsymbol{\lambda}_{i_s}}{\mathbf{p}'\cdot\boldsymbol{\lambda}_{i_s}}, & 0\le n\le N-3\\ \mathbf{p}', & n=N-2\ .\end{cases} \tag{3.29b}$$

These expression hold for $N>2$; for $N=2$ the momenta are unshifted, as seen earlier in Eq. (3.15).

At this stage we have an approximation for the transition matrix given by the sum of the expressions in Eqs. (3.21), (3.25), and (3.28). When they are combined (with $T^{(1)}$ expanded about its on-shell value in the manner discussed above) we find that off-shell components have canceled and the result can be rearranged in the form

$$T \cong T_{\mathrm{LFA}} + R_1 + R_2 \ , \tag{3.30}$$

with T_{LFA} given by Eq. (3.23). The correction terms take the form

$$R_1 = \sum_{n=0}^{N-1} \sum_{q_1} \prod_{s=n+2}^{N} \left[\frac{-\mathbf{p}'\cdot\boldsymbol{\lambda}_{i_s}}{\omega_{i_s}} \right] \times \prod_{s=1}^{n} \left[\frac{\mathbf{p}\cdot\boldsymbol{\lambda}_{i_s}}{\omega_{i_s}} \right] \times R(\mathbf{p}'(q_1),\mathbf{p}(q_1);\omega_{i_{n+1}}\boldsymbol{\lambda}_{i_{n+1}}) \tag{3.31}$$

and

$$R_2 = \sum_{n=0}^{N-2} \sum_{q_2} \prod_{s=n+3}^{N} \left[\frac{-\mathbf{p}'\cdot\boldsymbol{\lambda}_{i_s}}{\omega_{i_s}} \right] \prod_{s=1}^{n} \left[\frac{\mathbf{p}\cdot\boldsymbol{\lambda}_{i_s}}{\omega_{i_s}} \right] (m\boldsymbol{\lambda}_{i_{n+1}}\cdot\boldsymbol{\lambda}_{i_{n+2}})(\omega_{i_{n+1}}+\omega_{i_{n+2}})^{-1}[t(E_{\mathbf{p}(q_2)},\tau(q_2))-t(E_{\mathbf{p}'(q_2)},\tau(q_2))] \ , \tag{3.32}$$

with

$$\tau(q_2) = [\mathbf{p}'(q_2) - \mathbf{p}(q_2)]^2 \ . \tag{3.33}$$

We recall that the function R on the right-hand side of Eq. (3.31) is obtained from a knowledge of the physical one-photon amplitude according to Eq. (3.8). The leading term in the approximation (3.30) is of order $(\omega_1,\omega_2 \cdots \omega_N)^{-1}$; the approximation includes corrections, relative to the leading term, which are linear, quadratic and cubic in the photon frequencies.

IV. DISCUSSION

Low's theorem for single-photon bremsstrahlung[1] has been generalized here to apply to situations in which an arbitrary number of photons are emitted during the scattering process. This generalized soft-photon approximation is given in its simplest version by Eq. (3.23), which reduces to Low's form for $N=1$. It has been shown that the next two terms in the expansion in powers of the frequency can be obtained, for arbitrary N, provided the physical single-photon amplitude is known. These terms are defined in Eqs. (3.31) and (3.32). A contribution to the first of these correction terms has been identified [it appears in the second term on the right-hand side of Eq. (3.10)] which, under conditions of resonant scattering, can be magnified in importance, in which case it should properly be included as part of the underlying low-frequency approximation. This remark generalizes the observation originally made by Feshbach and Yennie[9] in their treatment of the one-photon process. The present calculation was carried out in the context of a relatively simple model—nonrelativistic scattering by a center of force, but the fact that the approximation can be expressed in terms of physical amplitudes for radiationless and one-photon emission suggests that the theorem is more generally applicable, independent, to a large extent, of the detailed nature of the dynamics of the scattering system. In considering such extensions one must keep in mind the special nature of the Coulomb scattering problem; the presence of the long-range potential tail has the effect of introducing changes in the analytic form of the approximation as a function of the frequency.[11] It seems likely, however, that the theorem can be verified for the scattering of an electron by a neutral atom. Of course the generality of the result for one- and two-photon emission during a *relativistic* collision is clearly seen in the derivations based on gauge invariance, Lorentz invariance, and analyticity assumptions.[1,2] It may be possible to extend these relativistic results to the multiphoton problem. Since the techniques used in the analysis are closely related to the Bloch-Nordsieck approach to the infrared divergence problem,[5] it may be worthwhile to reexamine that problem for the purpose of including higher-order corrections, in analogy with those introduced here for the case where the radiated photons are observable.

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