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Zener tunneling in systems without level crossing

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We compare the standard Zener tunneling problem with one in which the unperturbed states do not intersect. We find that even when their adiabatic energy spectra are identical, the two cases have dramatically different asymptotic tunneling probabilities. The time of Zener tunneling, a measure of the duration of the transition, is also different for the two. The latter is determined by calculating the transition probability as a function of time, for both the sudden and adiabatic limits. We also develop analytic asymptotic expansions for the tunneling probability for these limits. The results are discussed in terms suitable for comparison with experiment.

I. INTRODUCTION

The phenomenon of Zener-Landau tunneling,[1-6] or "nonadiabatic transitions," is a canonical example of a quantum-mechanical effect without a classical analog. First addressed in the 1930's , it has been applied in such diverse contexts as atom-surface scattering, molecular physics,[7] and molecular biology.[8-10] More recently this problem has been discussed in the growing field of submicrometer physics, in attempts to describe the dynamics of electronic nanostructures such as small normal metal rings driven by an Aharonov-Bohm flux[11,12] and ultrasmall capacitance tunnel junctions driven by an external current source.[13-16] Although over 50 years old, there are still interesting aspects to the problem that appear in these new fields. One example of this is the time of Zener tunneling.[17] Closely akin to the time of quantum-mechanical tunneling in real space,[18] the time of Zener tunneling is a measure of the duration of the quantum-mechanical transition itself, and can have different answers depending upon whether the system changes adiabatically or suddenly.

In this paper we discuss several other aspects of Zener tunneling, often referred to as the "level-crossing problem."[7] We will show that in some cases this term can be misleading. In particular, we will consider two separate model systems, each with its own time-dependent Hamiltonian, each with different concrete physical realizations, but with *identical* energy spectra (although the labeling of energy states will differ). We will show that, despite the overwhelming similarity of the problems, the two cases have quite different asymptotic tunneling probabilities, and Zener tunneling times. We will also show that Zener tunneling is possible in systems where the uncoupled levels do *not* cross.

In Sec. II below we develop a description of the canonical case of Zener tunneling, in which the unperturbed energy levels intersect [see Fig. 1(a)], and restate results for the asymptotic tunneling probability and Zener time. We also introduce a second, similar Hamiltonian, with an identical energy spectrum, but in which the unperturbed states do not intersect [as in Fig. 1(b)]. In Sec. III we calculate the asymptotic Zener tunneling probability for this case, both numerically and analytically, in the adiabatic limit using the standard adiabatic approximation.[19] We will show that the tunneling probability is quite different. In the adiabatic limit the tunneling probability can be exponentially or algebraically small in a dimensionless "adiabaticity parameter" depending upon the parameters. In Sec. IV we calculate the asymptotic value for the tunneling probability in the sudden limit, using a power-series expansion to calculate the amplitudes. We then use this expansion to calculate the width of the transition in the sudden limit, and to obtain an expression for the time of Zener tunneling in the sudden limit. In Sec. V we use a WKB expansion to calculate the width of the transition in the adiabatic limit. We summarize our results in Sec. VI and consider their potential application to experiments. Subsequent appendixes (A–C) deal with related mathematical subtleties.

II. STATEMENT OF THE PROBLEM

It is simplest, and indeed standard, to consider the Zener tunneling problem in the context of a simple two-level system whose energies cross as a function of some external parameter, which is assumed to vary linearly in time. We further assume a constant, off-diagonal matrix element connects these two states, so that the Hamiltonian for the system can be written

$$\mathcal{H}_X(t)=\alpha t\sigma_z+\Delta\sigma_x \ , \tag{2.1}$$

where α and Δ are real, positive constants and the σ_i are the Pauli spin matrices,

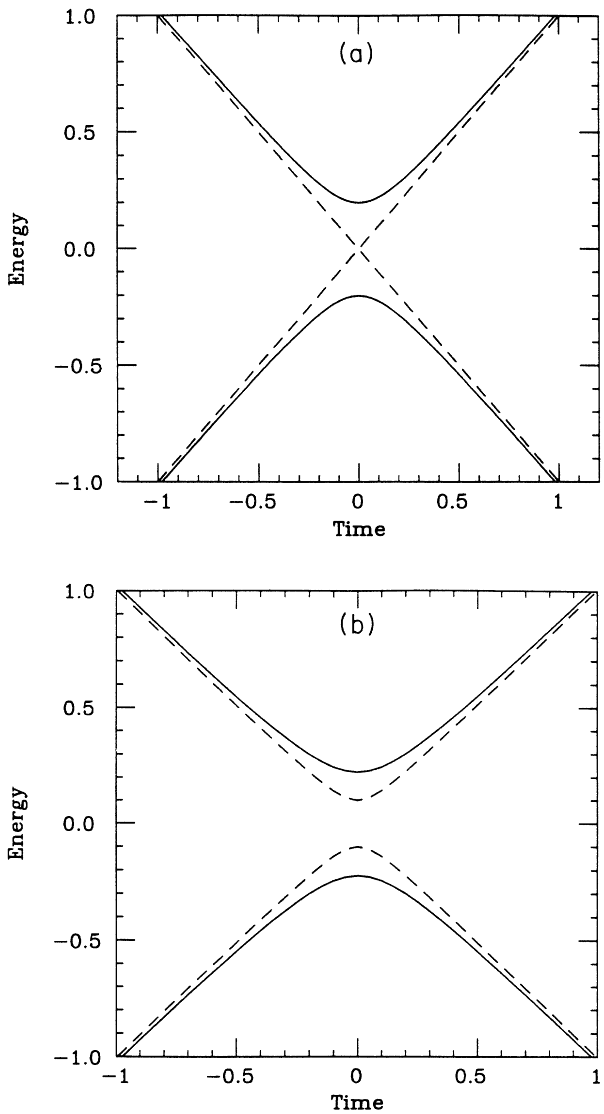

FIG. 1. (a) Adiabatic energy levels of the $\mathcal{H}_X$ Hamiltonian, as given in Eq. (2.5), for $\alpha=1$. The dashed lines are the uncoupled ($\Delta=0$) states, that are degenerate at $t=0$. The solid lines are the adiabatic energy levels for $\Delta=0.2$. (b) Adiabatic energy levels of the $\mathcal{H}_Y$ Hamiltonian, as given in Eq. (2.14), for $\alpha=1$. The dashed lines are the uncoupled ($\Delta=0$) states with $\delta=0.1$. Note that they are *not* degenerate at $t=0$. The solid lines are the adiabatic energy levels for $\Delta=0.2$.

$$\sigma_x=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \sigma_y=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\quad \sigma_z=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \tag{2.2}$$

so that the Hamiltonian in Eq. (2.1) has the simple matrix representation

$$\mathcal{H}_X(t)=\begin{bmatrix}\alpha t & \Delta\\ \Delta & -\alpha t\end{bmatrix}. \tag{2.3}$$

As a pedagogical example used throughout this paper, we will consider the system of an isolated spin-$\frac{1}{2}$ particle, say an electron, in the presence of a large uniform magnetic field, $\mathcal{B}=B_0t\mathbf{k}$ initially pointing in the $-\mathbf{k}$ direction, at time $t=-\infty$, and set $\alpha=\mu B_0$, where μ is the magnetic moment of the electron. We assume the existence of a small transverse magnetic field $\mathcal{B}'=B'\mathbf{i}$ that allows mixing between the states of "spin up" (denoted by $v_\uparrow$) and "spin down" (denoted by $v_\downarrow$) and set $\Delta=\mu B'$.

Let $v_\pm(t)$ be the eigenstates of $\mathcal{H}_X$ when time is treated as a parameter. By this we mean that we replace time in the Hamiltonian by a dummy variable s, and then solve the "time-independent Schrödinger equation"

$$\mathcal{H}_X(s)v_\pm(s)=E_\pm(s)v_\pm(s) \ , \tag{2.4}$$

where s is a time-independent constant, yielding a family of equations and solutions as a function of s. The energy spectrum is

$$E_\pm(t)=\pm(\alpha^2t^2+\Delta^2)^{1/2} \ , \tag{2.5}$$

where $v_+(t)$ and $v_-(t)$ are combinations of $v_\uparrow$ and $v_\downarrow$:

$$v_+(t)\equiv N\begin{bmatrix}\dfrac{\alpha t+(\alpha^2t^2+\Delta^2)^{1/2}}{\Delta}\\ 1\end{bmatrix},\quad v_-(t)\equiv N\begin{bmatrix}1\\ \dfrac{-\alpha t-(\alpha^2t^2+\Delta^2)^{1/2}}{\Delta}\end{bmatrix}, \tag{2.6}$$

where the normalization constant is given by

$$N=\frac{\Delta}{\sqrt{2}(\alpha^2t^2+\Delta^2)^{1/4}}[(\alpha t+\alpha^2t^2+\Delta^2)^{1/2}]^{1/2} \ . \tag{2.7}$$

At time $t=0$ these are just the symmetric and antisymmetric combinations of $v_\uparrow$ and $v_\downarrow$. The states $v_+(t)$ and $v_-(t)$ are sometimes referred to as "adiabatic eigenstates." The energy spectrum is plotted in Fig. 1(a).

We initially prepare the system in the upper of these two adiabatic eigenstates (spin down) and then allow the Hamiltonian to vary with time. At any particular time t_1 the eigenstates of the system, $v_+(t_1)$ and $v_-(t_1)$, are orthogonal. However, the state $v_-(t_1)$ at t_1 *does* have an overlap with the state $v_+(t_2)$ at a later time t_2. This is the underlying mechanism for Zener tunneling. If we prepare the system in state "down" at $t=-\infty$ then there is a finite probability P_Z that we will find it in the state "down" at $t=\infty$, given by

$$P_Z=\exp\left[\frac{-\pi\Delta^2}{\hbar\alpha}\right]=\exp\left[\frac{-\pi}{\gamma}\right], \tag{2.8}$$

where the parameter $\gamma\equiv\hbar\alpha/\Delta^2$. Details of the calculation can be found in Refs. 1–5.

The time of Zener tunneling[17] is a measure of the duration of the actual transition itself. This information is particularly useful in the event that there are other time-dependent interactions (i.e., dephasing events) or multiple level crossings. There are two suggested approaches to defining the Zener time. The first is to measure the width of the transition itself by examining the probability of staying in the initial state, as a function of time. The second is to examine the sensitivity of the asymptotic transition probability to an oscillatory perturbation. If there is a characteristic frequency at which the system is insensitive to the perturbation then this frequency can be taken as the reciprocal of the characteristic time. In both cases the Zener tunneling time τ_z for the $\mathcal{H}_X$ Hamiltonian, for $\gamma \ll 1$, is

$$\tau_z\approx\frac{\Delta}{\alpha} . \tag{2.9}$$

Physically, this is the time that it takes the external driving force to transfer an energy Δ (half the gap) to the system. In the sudden limit ($\gamma\gg 1$) we find

$$\tau_z\approx\left[\frac{\hbar}{\alpha}\right]^{1/2} . \tag{2.10}$$

This quantity is approximately the time it takes for the phase of the wave function to change by order unity near $t=0$. In Fig. 2(a) we plot the transition probability as a function of time for the $\mathcal{H}_X$ Hamiltonian.

We compare these results to those we will derive for a second, nonanalytic Hamiltonian,

$$\mathcal{H}_Y(t)=(\alpha^2t^2+\delta^2)^{1/2}\sigma_z+\Delta\sigma_z \tag{2.11}$$

which, in matrix form, can be written

$$\mathcal{H}_Y(t)=\begin{bmatrix}(\alpha^2t^2+\delta^2)^{1/2} & \Delta\\ \Delta & -(\alpha^2t^2+\delta^2)^{1/2}\end{bmatrix} . \tag{2.12}$$

In the limit $\delta\to 0$ this reduces to

$$\lim_{\delta\to 0}\mathcal{H}_Y(t)=\begin{bmatrix}\alpha|t| & \Delta\\ \Delta & -\alpha|t|\end{bmatrix} . \tag{2.13}$$

As above, this Hamiltonian can be considered as describing a spin-$\frac{1}{2}$ particle, say an electron, in a magnetic field, where the field is reduced to zero and then increased again *in the same direction.* The parameter δ can by interpreted as allowing for an offset in the field strength at the point where the rate of change of the field is reversed.

The eigenvalues of $\mathcal{H}_Y$, with time treated as a parameter, are

$$E_\pm(t)=\pm(\alpha^2t^2+\delta^2+\Delta^2)^{1/2} \tag{2.14}$$

[plotted in Fig. 1(b)] with eigenstates $u_\pm$, where the states $u_\pm$ are combinations of $u_\uparrow$ and $u_\downarrow$. Note that for $\delta=0$ the eigenenergies are *identical* to those of Eq. (2.5). The eigenvectors that correspond to the eigenvalues (2.14) are

$$u_+(t)\equiv\frac{1}{(1+r^2)^{1/2}}\begin{bmatrix}1\\ r\end{bmatrix},\qquad u_-(t)\equiv\frac{1}{(1+r^2)^{1/2}}\begin{bmatrix}-r\\ 1\end{bmatrix}, \tag{2.15}$$

where

$$r=\frac{1}{\Delta}[(\alpha^2t^2+\delta^2+\Delta^2)^{1/2}-(\alpha^2t^2+\delta^2)^{1/2}] . \tag{2.16}$$

As above, it is possible for the driven system, prepared initially in the state "spin down," $u_\downarrow(-\infty)=u_-(-\infty)$, to Zener tunnel. However, in this case the uncoupled ($\Delta=0$) states do not intersect and this is not, strictly speaking, a level-crossing problem. In Figs. 2(b)–2(d) we plot the transition probability as a function of time for the $\mathcal{H}_Y$ Hamiltonian. In this case the transition probability is *not* a monotonic function of the rate of change of the energy.

III. ADIABATIC APPROXIMATION

Consider $\mathcal{H}(t)$, a general time-dependent Hamiltonian, with eigenstates $u_n(t)$ when time is treated as a parameter. By this we mean that if we replace time in the Hamiltonian by a dummy variable s, then $u_n(t)$ is an eigenstate of the time-independent Schrödinger equation

$$\mathcal{H}(s)u_n(s)=E_n(s)u_n(s) , \tag{3.1}$$

where s is a time-independent constant, yielding a family of equations and solutions as a function of s. The solution of the time- *dependent* Schrödinger equation

$$i\hbar\frac{\partial\psi(t)}{\partial t}=\mathcal{H}(t)\psi(t) \tag{3.2}$$

can then be written in the form

$$\psi(t)=\sum_n a_n(t)u_n(t)\exp\left[\frac{1}{i\hbar}\int_{-\infty}^{t}E_n(t')dt'\right] . \tag{3.3}$$

The expansion coefficients $a_n(t)$ can be obtained by the direct substitution of Eq. (3.3) into Eq. (3.2), which yields

$$\dot a_k(t)=\sum_{n\neq k}-a_n(t)\frac{\langle u_k(t)|\dot{\mathcal{H}}(t)|u_n(t)\rangle}{E_n(t)-E_k(t)}\times\exp\left[\frac{1}{i\hbar}\int_{-\infty}^{t}[E_n(t')-E_k(t')]dt'\right] . \tag{3.4}$$

In the lowest-order adiabatic approximation[19] it is assumed that in Eq. (3.4) the coefficients $a_n(t)$ on the right-hand side (RHS) can be treated as constant in time. In what follows it will be assumed that initially ($t=-\infty$) the system is in an eigenstate of $\mathcal{H}(-\infty)$, so that all of the $a_n(-\infty)$ are zero save for one, say $a_0(-\infty)$, which is of unit magnitude. Equation (3.4) can be used to calculate $a_k(\infty)$ for $k \neq 0$, with the approximation that $a_0(t)=1$ at all times.

We consider the Hamiltonian $\mathcal{H}_Y$ given by Eq. (2.11) and described in Sec. I above, with eigenvalues and eigenvectors given by Eqs. (2.14) and (2.15). If initially the system is in the lower "spin down" eigenstate, [that is, $|a_\downarrow(-\infty)|=1$ and $|a_\uparrow(-\infty)|=0$], then in the adiabatic limit Eq. (3.4) reduces to

$$\dot{a}_\uparrow(t)=\frac{-\Delta\alpha^2 t}{2(\alpha^2t^2+\delta^2+\Delta^2)}\frac{1}{(\alpha^2t^2+\delta^2)^{1/2}} \times\exp\left[\frac{2i}{\hbar}\int_{-\infty}^{t'}(\alpha^2t'^2+\delta^2+\Delta^2)^{1/2}dt'+i\varphi_0\right], \tag{3.5}$$

where φ_0 is an arbitrary overall phase. The transition amplitude between the states $u_\downarrow$ and $u_\uparrow$ in the infinite time limit is then given by

$$a_\uparrow \equiv a_\uparrow(\infty)=\int_{-\infty}^{\infty}\dot{a}_\uparrow(t')dt' . \tag{3.6}$$

Motivated by work on the $\mathcal{H}_X$ Hamiltonian (se Appendix A or Ref. 17), we introduce the dimensionless time variable

$$\eta \equiv \frac{\alpha t}{(\delta^2+\Delta^2)^{1/2}} \tag{3.7}$$

so that Eq. (3.5) reduces to

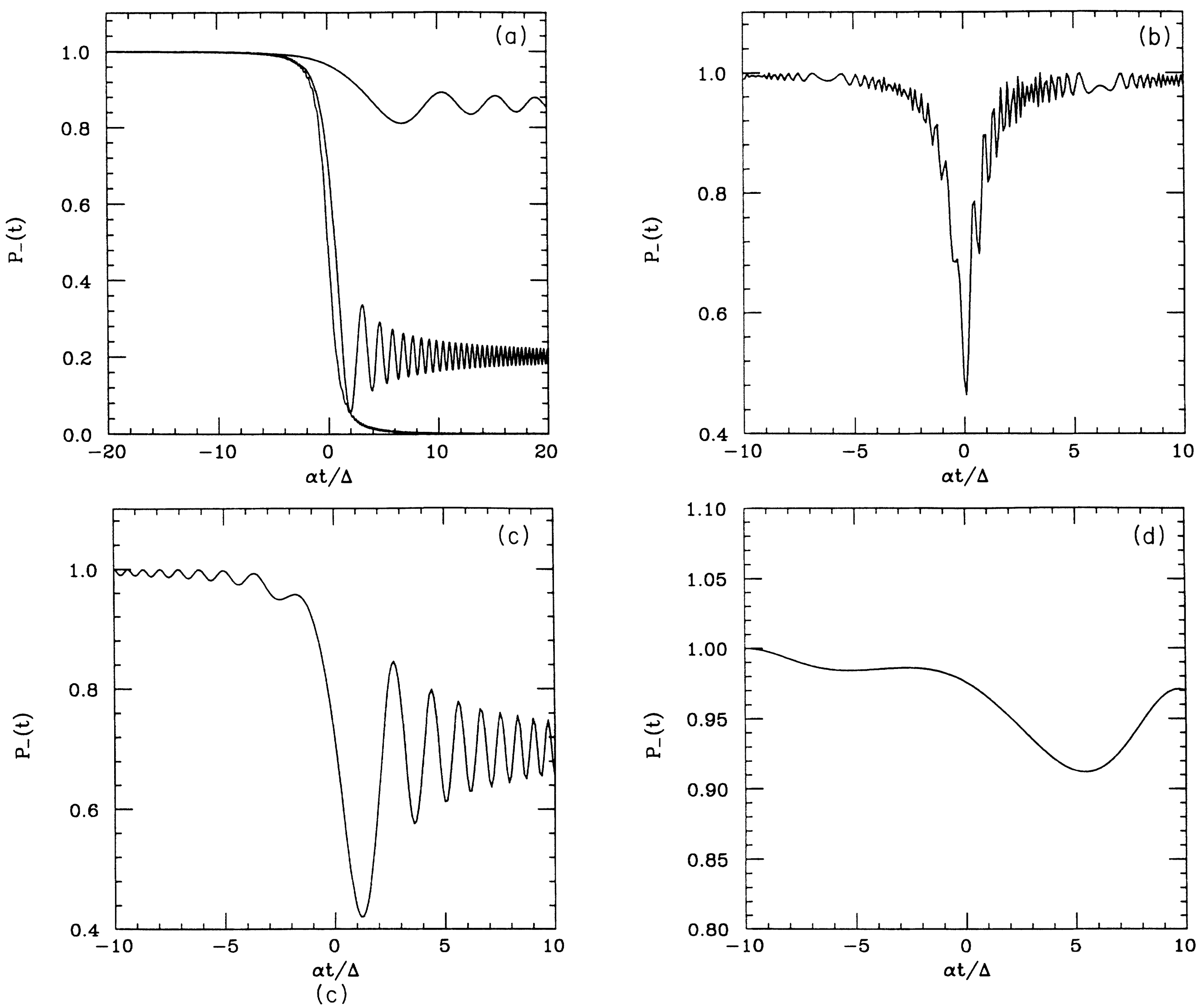

FIG. 2. (a) Plot of the "Zener tunneling" probability $P_\downarrow(t)$ as a function of time for the $\mathcal{H}_X$ Hamiltonian. The parameters are $\gamma=20$ (top), $\gamma=2$ (middle), and $\gamma=0.2$ (bottom). One definition of Zener time is the width of the transition from $P_\downarrow \approx 1$ to its asymptotic value, $P_\downarrow=\exp(-\pi/\gamma)$. (b–d) plots of the Zener tunneling probability $P_\downarrow(t)$ as a function of time for the $\mathcal{H}_Y$ Hamiltonian. The parameters are (b) $\Gamma=0.2$, (c) $\Gamma=2.0$, and (d) $\Gamma=20.0$, where $\Gamma=\hbar\alpha/(\Delta^2+\delta^2)$. Note that the asymptotic "tunneling probability" does *not* increase uniformly with Γ.

$$a_\uparrow = -D_0 \int_{-\infty}^{\infty} \frac{\eta}{(\eta^2+1)(\eta^2+D)^{1/2}} d\eta \times \exp\left[2i\beta \int_{-\infty}^{\eta} (\eta'^2+1)^{1/2} d\eta' + i\varphi_0 \right] , \tag{3.8}$$

where

$$\beta \equiv \frac{1}{\Gamma} \equiv \frac{\delta^2+\Delta^2}{\hbar\alpha} , \quad D \equiv \frac{\delta^2}{(\delta^2+\Delta^2)} , \quad D_0 \equiv \frac{1}{2}\sqrt{1-D} . \tag{3.9}$$

The behavior of this system depends upon the two dimensionless parameters Γ, which determines the adiabaticity of the transition; and D which determines the separation of the uncoupled ($\Delta=0$) states. In the limit $\delta \to 0$, we have that $\Gamma \to \gamma$, as defined in Sec. II; the approximation (3.8) improves as Γ decreases.

The integral in the phase of the exponential is straightforward,

$$\int (\eta^2+1)^{1/2} d\eta = \tfrac{1}{2}\{ \eta[(\eta^2+1)]^{1/2} + \ln[\eta + (\eta^2+1)^{1/2}] \} . \tag{3.10}$$

This result suggests the further change of variable to $\eta \equiv \sinh x$. If we choose the arbitrary phase φ_0 such that the argument of the exponential in Eq. (3.8) vanishes for $x=0$, then we can cast it in the form

$$a_\uparrow = -D_0 \int_{-\infty}^{\infty} f(x) e^{\frac{i\varphi(x)}{\Gamma}} dx , \tag{3.11}$$

with

$$\varphi(x) \equiv \tfrac{1}{2}\sinh 2x + x , \quad f(x) \equiv \frac{\sinh x}{\cosh x (\sinh^2 x + D)^{1/2}} . \tag{3.12}$$

In Appendix A we evaluate this integral numerically, and develop analytic asymptotic expansions in a variety of limits.

We find that in the regime $\Gamma \ll 1$ the transition amplitude can be approximated by

$$a_\uparrow \approx \begin{cases} \dfrac{-i}{4}\Gamma \quad \text{for} \sqrt{D} \ll \sqrt{D_l} & (3.13a) \\ \dfrac{-i}{2}\left[\dfrac{\pi\Gamma}{1-D}\right]^{1/4}\left[\dfrac{D}{1-D}\right]^{1/2} \exp\{-[\sqrt{D(1-D)} + \arcsin\sqrt{D}\,]/\Gamma\} \quad \text{for} \sqrt{D} \gg \sqrt{D_l} , & (3.13b) \end{cases}$$

where one can choose a value of D_l that divides the two regimes of validity satisfying

$$\frac{\Gamma}{4} < \sqrt{D_l} < \frac{\Gamma}{2} . \tag{3.14}$$

Note that this interval shrinks with decreasing Γ. A reasonable approximation for both regimes is

$$a_\uparrow \approx \frac{-i\Gamma}{2\sqrt{1-D}} e^{-\varphi_A/\Gamma} , \tag{3.15}$$

which is identical to Eq. (A18) derived in Appendix A.

In Fig. 3. these approximations are compared with the exact (numerical) value of the integral [Eq. (3.11)] and with the results obtained from numerical integration of the time-dependent Schrödinger equations. The arrows mark the bounds on the limiting value of D_l. Note that the approximants are extremely close to the exact result nearly for all the range of D, for $\Gamma=0.2$. Their quality improves when Γ is decreased, as expected.

It is striking that such a small change in the Hamiltonian, a change that leaves the eigenenergy spectrum for the problem unchanged, results in tunneling rates or order Γ^2 rather than of order $\exp(-1/\Gamma)$ as found in the case of true level crossing.

IV. ZENER TUNNELING PROBABILITY AND ZENER TIME FOR $\mathcal{H}_Y$ IN THE SUDDEN LIMIT

In the sudden limit, $\Gamma \gg 1$, the above contour integral approach is no longer valid, because the tunneling probability is large and therefore the amplitudes $a_k(t)$ in Eq. (3.4) can no longer be considered as constants. In this case we return to the original time-dependent Schrödinger equations

$$i\hbar\dot{a}_\uparrow(t) = (\alpha^2 t^2 + \delta^2)^{1/2} a_\uparrow(t) + \Delta a_\downarrow(t) ,$$
$$i\hbar\dot{a}_\downarrow(t) = -(\alpha^2 t^2 + \delta^2)^{1/2} a_\downarrow(t) + \Delta a_\uparrow(t) . \tag{4.1}$$

Motivated by the results of Ref. 17, we change variables to $\tau = t\sqrt{\hbar/\alpha}$. These equations can then be written as

$$i\dot{a}_\uparrow(\tau) = (\tau^2 + \tilde{\delta}^2)^{1/2} a_\uparrow(\tau) + \tilde{\Delta} a_\downarrow(\tau) ,$$
$$i\dot{a}_\downarrow(\tau) = (\tau^2 + \tilde{\delta}^2)^{1/2} a_\downarrow(\tau) + \tilde{\Delta} a_\uparrow(\tau) , \tag{4.2}$$

where $\tilde{\Delta} \equiv \Delta/\sqrt{\hbar\alpha}$ and $\tilde{\delta} \equiv \delta/\sqrt{\hbar\alpha}$. These equations can be simplified by factoring off the overall oscillatory part due to instantaneous energy. We define

$$b_\uparrow(\tau) = \exp\left[i \int_{-\infty}^{\tau} (\tau'^2 + \tilde{\delta}^2)^{1/2} d\tau' \right] a_\uparrow(\tau) ,$$
$$b_\downarrow(\tau) = \exp\left[-i \int_{-\infty}^{\tau} (\tau'^2 + \tilde{\delta}^2)^{1/2} d\tau' \right] a_\downarrow(\tau) . \tag{4.3}$$

In terms of $b_\uparrow$ and $b_\downarrow$ Eq. (4.2) becomes

$$i\dot{b}_\uparrow(\tau)=\tilde{\Delta} b_\downarrow(\tau)\exp\left[2i\int_{-\infty}^{\tau}(\tau'^2+\tilde{\delta}^2)^{1/2}d\tau'\right],$$
$$i\dot{b}_\downarrow(\tau)=\tilde{\Delta} b_\uparrow(\tau)\exp\left[-2i\int_{-\infty}^{\tau}(\tau'^2+\tilde{\delta}^2)^{1/2}d\tau'\right]. \quad (4.4)$$

Typical numerical solutions of the above differential equations in the sudden limit are shown in Fig. 2d. Unlike the case of the Hamiltonian $\mathcal{H}_X$, the probability of making a Zener transition from the lower to the upper band *decreases* as the transition becomes more sudden. This paradoxical result is easier to understand when one considers that in the sudden limit the tendency for both the $\mathcal{H}_X$ and $\mathcal{H}_Y$ Hamiltonians is to stay in the same, unperturbed ($\Delta=0$), state. In the $\mathcal{H}_X$ case this means passing to the upper band, in the $\mathcal{H}_Y$ case this corresponds to remaining in the lower band.

It is possible to develop analytical approximations to the Zener tunneling formula. For simplicity we first limit ourselves to the case $\delta=\tilde{\delta}=0$, which amounts to solving the Hamiltonian of Eq. (2.13). Since the probability of staying in the same state is almost unity, we can make the assumption that $|a_\downarrow(t)|=1$ for all times, but that its phase varies as the exponential in Eq. (4.3). Thus in Eq. (4.4) we can assume that $b_\downarrow=1$, so that

$$i\dot{b}_\uparrow(\tau)=\tilde{\Delta}\exp\left[2i\int_{-\infty}^{\tau}|\tau'|d\tau'\right], \quad (4.5)$$

which can be integrated easily to find the asymptotic probability

$$b_\uparrow(\infty)=-i\tilde{\Delta}\left[\int_{-\infty}^{0}-d\tau\, e^{-i\tau^2}+\int_{0}^{\infty}d\tau\, e^{i\tau^2}\right]e^{i\varphi_0}, \quad (4.6)$$

where we have introduced the arbitrary phase φ_0 so that at $t=0$ the overall phase of Eq. (4.6) is zero. (Note that we have fixed only the modulus of the initial conditions, so that up to this point the overall phase is arbitrary.) These definite integrals can be evaluated[20] to obtain

$$b_\uparrow(\infty)=-i\tilde{\Delta}\left[\frac{\pi}{2}\right]^{1/2} \quad (4.7)$$

so that the Zener tunneling probability in the sudden limit is given by

$$P_Z\equiv|b_\uparrow(\infty)|^2\approx\frac{\pi\tilde{\Delta}^2}{2}=\frac{\pi}{2\gamma}. \quad (4.8)$$

Unlike the case of the Hamiltonian $\mathcal{H}_X$, the probability of making a Zener transition from the lower to the upper band decreases as the transition becomes more sudden. As discussed above, Zener tunneling corresponds to following the uncoupled ($\Delta=0$) states, so that in the $\mathcal{H}_Y$ case the faster the system is driven, the more it tends to stay in the lower band. We note that the result of Eq. (4.8) is half the probability of *not* Zener tunneling in the $\mathcal{H}_X$ case.

One reasonable definition of the time for Zener tunneling is the temporal width of the transition itself.[17] The transition width cannot be ascertained from the asymp-

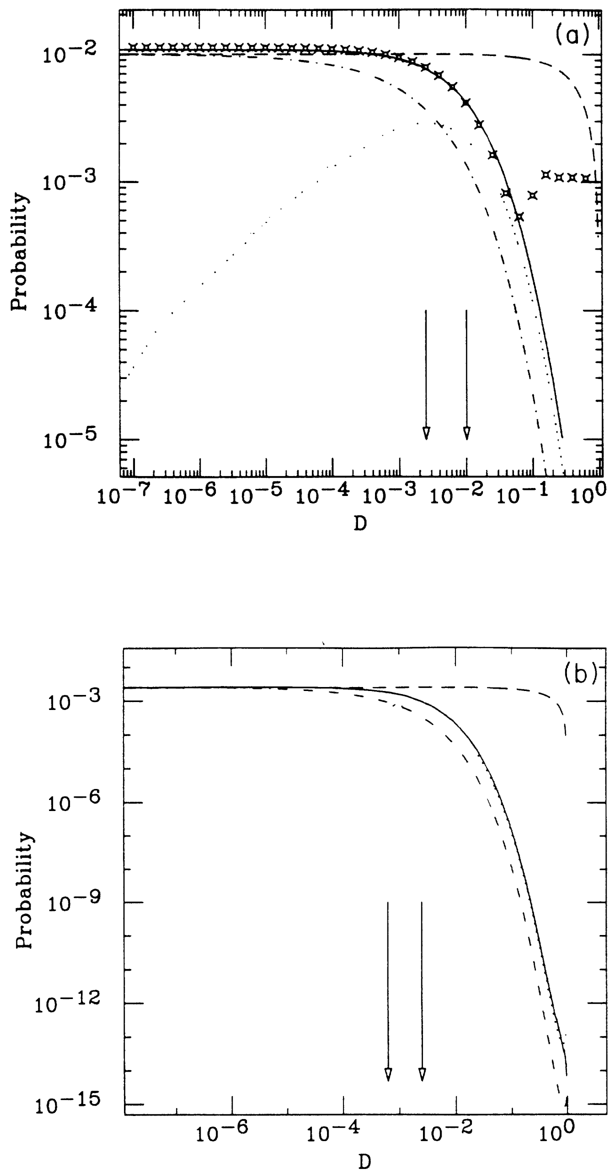

FIG. 3. Comparison of various methods for calculating the asymptotic Zener tunneling probability $|a_\uparrow|^2$, for the $\mathcal{H}_Y$ Hamiltonian as a function of D for fixed Γ. The solid line is the result from integrating Eq. (3.11) along the contour given in (A10). The dashed line is an approximation valid for small D given by Eq. (3.13a) [or Eq. (A19)]. The dotted line is an analytic approximation for large D, given by Eq. (3.13b) [or (A17)]. The dot-dashed line is a reasonable approximation over the entire range, as given by Eq. (3.15) [or (A18)] again valid for small D. The arrows mark the transition region from small to large D, as given by Eq. (3.14). (a) The tunneling probability for $\Gamma=0.2$. The stars are the results of direct numerical solutions of the time-dependent Schrödinger equation for the $\mathcal{H}_Y$ Hamiltonian of Eq. (2.12). Note that these solutions deviate from the correct results when $D\rightarrow 1$, when the asymptotic probability becomes smaller than the numerical accuracy of the calculation. (b) The tunneling probability for $\Gamma=0.1$. The tunneling probability is too small for numerical solutions of Eq. (4.1) to be accurate. Note the improved overlap of the asymptotic solutions as Γ decreases.

totic probability, but only by examining the time-dependent amplitudes near $t=0$. In this paper, we will define the Zener tunneling time by the width of the transition profile. Cursory inspection of the coupled equations [Eqs. (4.4)] shows that for constant $\tilde{\delta}$, when the time is scaled by $\sqrt{\hbar/\alpha}$ the width of the profile is an invariant, although the magnitude of the change will of course depend upon $\tilde{\Delta}$. This is not a "universal" definition in that the time still depends upon an additional external parameter, but it is consistent with the results of the $\mathcal{H}_X$ Hamiltonian. Thus in the sudden limit we have that

$$\tau_z = \left[\frac{\hbar}{\alpha}\right]^{1/2} . \tag{4.9}$$

In Fig. 4 we plot the function $|b_\downarrow|^2$ as a function of time. When plotted with time measured in units of $\sqrt{\hbar/\alpha}$, these numerical solutions all have the same width.

For $\tilde{\delta}\neq 0$ the integral of Eq. (4.6) must be replaced by

$$b_\uparrow(\infty)=\int_{-\infty}^{\infty}\tilde{\delta}\tilde{\Delta}\exp(i\tilde{\delta}^2\{\tau(\tau^2+1)^{1/2}+\ln[\tau+(\tau^2+1)^{1/2}]\})d\tau . \tag{4.10}$$

However, for $\delta/\alpha \ll \tau_Z$ we expect that the result of Eq. (4.8) is still valid, since the δ can be neglected for almost the entire realm of integration. This is satisfied when

$$\frac{\delta}{\sqrt{\alpha\hbar}} \ll 1 , \tag{4.11}$$

which can always be realized if the driving force varies at a rapid enough rate (i.e., for large enough α). Thus, whenever Eq. (4.11) is satisfied, the Zener tunneling probability and time can be approximated by Eqs. (4.8) and (4.9), as shown in Fig. 4, and verified numerically.

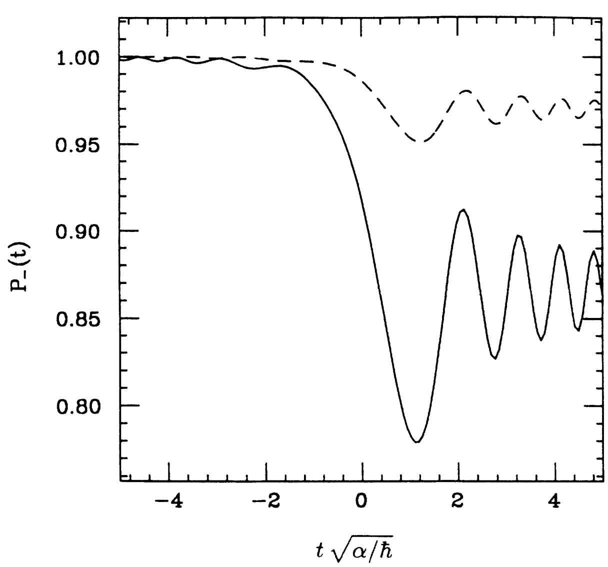

FIG. 4. Plot of the "Zener tunneling" probability $P_-(t)$ as a function of time for the $\mathcal{H}_Y$ Hamiltonian in the sudden limit. The solid line is for $\Gamma=10$, the dashed line for $\Gamma=50$. In both cases $\delta=D=0$. Note that the oscillations line up for $t>0$ when time is measured in units of $\sqrt{\hbar/\alpha}$. The oscillations for $t<0$ are due to the finite interval of integration.

V. WKB EXPANSION FOR THE $\mathcal{H}_Y$ HAMILTONIAN AND THE TIME OF ZENER TUNNELING

As mentioned in Sec. IV, the transition width cannot be ascertained from the asymptotic probability, but only by solving for the time-dependent amplitudes near $t=0$. To this end we consider the time-dependent Schrödinger equation for $\mathcal{H}_Y$, of Eq. (4.1). In the simpler level-crossing Hamiltonian $\mathcal{H}_X$, (see Appendix A or Ref. 17) it is possible to scale time by a factor Δ/α and obtain a parameter-independent transition profile. In the spirit of this result, we change variables to

$$\eta \equiv \frac{\alpha t}{(\delta^2+\Delta^2)^{1/2}} \tag{5.1}$$

and divide through by $(\delta^2+\Delta^2)^{1/2}$ to obtain

$$\begin{aligned} i\Gamma\dot{a}_\uparrow(\eta)&=(\eta^2+D)^{1/2}a_\uparrow(\eta)+(1-D)^{1/2}a_\downarrow(\eta) ,\\ i\Gamma\dot{a}_\downarrow(\eta)&=-(\eta^2+D)^{1/2}a_\downarrow(\eta)+(1-D)^{1/2}a_\uparrow(\eta) , \end{aligned} \tag{5.2}$$

where Γ and D are described in Eq. (3.9).

We can combine these two first-order equations to get a single, second-order equation for $a_\uparrow$:

$$\Gamma^2\ddot{a}_\uparrow(\eta)+i\Gamma\dot{G}(\eta)a_\uparrow(\eta)+[G^2(\eta)+1-D]a_\uparrow(\eta)=0 , \tag{5.3}$$

where $G(\eta)=(\eta^2+D)^{1/2}$. We assume a solution of the form $a_\uparrow \approx K(\eta)\exp[iS(\eta)/\Gamma]$. Substituting in the Schrödinger equation and equating different orders of Γ we obtain to order Γ^0

$$\dot{S}_\pm=\pm(\eta^2+1)^{1/2} , \tag{5.4}$$

where the signs ($\pm$) refer to the choice of sign for the square root. This can be integrated using Eq. (3.10) to

$$S_\pm(\eta)=\pm\tfrac{1}{2}\{\eta(\eta^2+1)^{1/2}+\ln[\eta+(\eta^2+1)^{1/2}]\} . \tag{5.5}$$

Since $S(\eta)$ is real, it contributes only to the phase of the wave function, not its magnitude. In order to determine the shape of the profile we must go to the next order in the expansion and calculate the prefactor, $K(\eta)$. Equating terms of order Γ in the WKB expansion, we have

$$2i\dot{K}(\eta)\dot{S}(\eta)+iK(\eta)\ddot{S}(\eta)+i\dot{G}(\eta)K(\eta)=0 . \tag{5.6}$$

Solving for $K(\eta)$ we obtain

$$\begin{aligned}\frac{\dot{K}(\eta)}{K(\eta)}&=-\frac{1}{2}\frac{\ddot{S}(\eta)+\dot{G}(\eta)}{\dot{S}(\eta)}\\ &=-\frac{1}{2}\left[\frac{\eta}{\eta^2+1}\pm\frac{\eta}{(\eta^2+D)^{1/2}(\eta^2+1)^{1/2}}\right] ,\end{aligned} \tag{5.7}$$

which can be integrated to yield

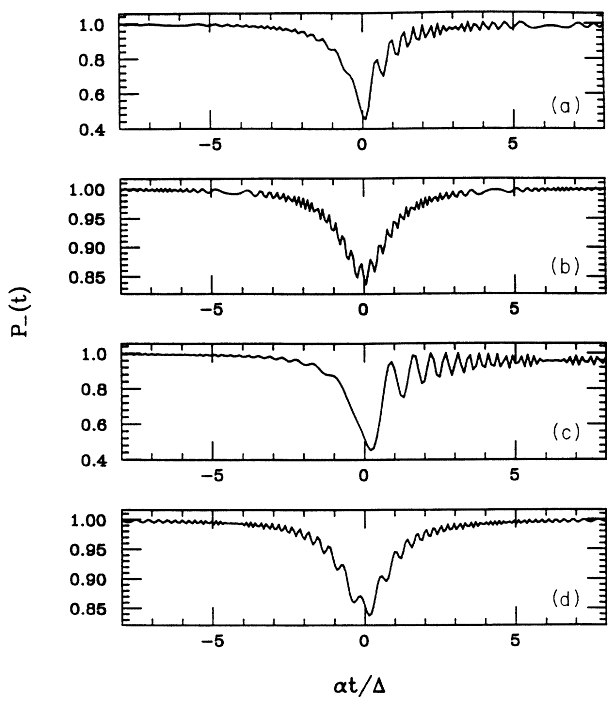

FIG. 5. Plot of the "Zener tunneling" probability as a function of time for the $\mathcal{H}_Y$ Hamiltonian in the adiabatic limit. The parameters are (a) $\Gamma=0.2$ and $\delta=0$, (b) $\Gamma=0.2$ and $\delta=0.1$, (c) $\Gamma=0.4$ and $\delta=0$, and (d) $\Gamma=0.4$ and $\delta=0.1$. Note that the transition widths are all approximately the same, when the time is measured in units of Δ/α.

$$K_{\pm}(\eta)=\left[\frac{1}{1+\eta^2}\right]^{1/4}\{2[\eta^4+(D+1)\eta^2+D]^{1/2}+2\eta^2+D+1\}^{\mp 1/4}. \qquad (5.8)$$

These solutions must then be multiplied by an appropriate constant so as to satisfy the boundary conditions, $|a_\uparrow|\to 1$ at either $\pm\infty$. In our case only the solution K_- is consistent with these boundary conditions.

Inspection of Eq. (5.8) shows that simply scaling the time does not yield a parameter-independent profile. Again, in this sense there is no universal form for the Zener time. However, for fixed D the profile is independent of Γ, i.e., how fast we drive the system, so that the characteristic time

$$\tau_z=\frac{(\Delta^2+\delta^2)^{1/2}}{\alpha} \qquad (5.9)$$

can be viewed as the Zener tunneling time. Moreover, in the limit that $\Delta\gg\delta$ or $\Delta\ll\delta$ it is simple to show that τ_z gives the correct scaling. For example, for $\Delta\gg\delta$ we can change variables to $y\equiv\alpha t/\Delta$ and obtain

$$\begin{aligned} i\gamma\dot{a}_\uparrow(y)&=(y^2+\zeta^2)^{1/2}a_\uparrow+a_\downarrow\ ,\\ i\gamma\dot{a}_\downarrow(y)&=-(y^2+\zeta^2)^{1/2}a_\downarrow+a_\uparrow\ , \end{aligned} \qquad (5.10)$$

where $\gamma=\hbar\alpha/\Delta^2$ as in Sec. I and $\zeta\equiv\delta/\Delta$. As above, using a WKB ansatz $a_\uparrow=K'e^{iS'/\gamma}$ we find

$$\begin{aligned} \dot{S}'_\pm&=\pm(y^2+\zeta^2+1)^{1/2}\ ,\\ \frac{\dot{K}'(y)}{K'(y)}&=-\frac{1}{2}\left[\frac{y}{y^2+\zeta^2+1}\pm\frac{y}{(y^2+\zeta^2+1)^{1/2}(y^2+\zeta^2)^{1/2}}\right]. \end{aligned} \qquad (5.11)$$

In the limit that $\delta/\Delta\to 0$ these equations are independent of any parameters of the problem, so that a plot of the transition probability as a function of time is invariant when time is measured in units of Δ/α. In a similar fashion it is possible to show that in the limit $\Delta\gg\delta$, the profile is invariant when measured in units of δ/α. Thus the expression for the Zener time is correct in the two limits. These results are born out in Fig. 5, which shows different transition profiles for different parameters have the same width when scaled by τ_z of Eq. (5.9).

VI. DISCUSSION AND CONCLUSION

In this paper we have endeavored to show that Zener tunneling can be more subtle than one might suspect. First, it does not require that the uncoupled states have degeneracy. Second, systems that *appear* very similar [as in Figs. 1(a) and 1(b)] can in fact have quite dissimilar asymptotic tunneling probabilities. These issues are quite open to experimentation; any standard Zener tunneling problem with a time-varying driving force can be altered so that the derivative of the driving force changes when the uncoupled states are nearly degenerate. We note parenthetically, that we are dealing with time-varying forces, and thus these issues must be treated differently in the case of Zener tunneling in momentum space.[21]

We have plotted in Fig. 6 a comparison of the asymp-

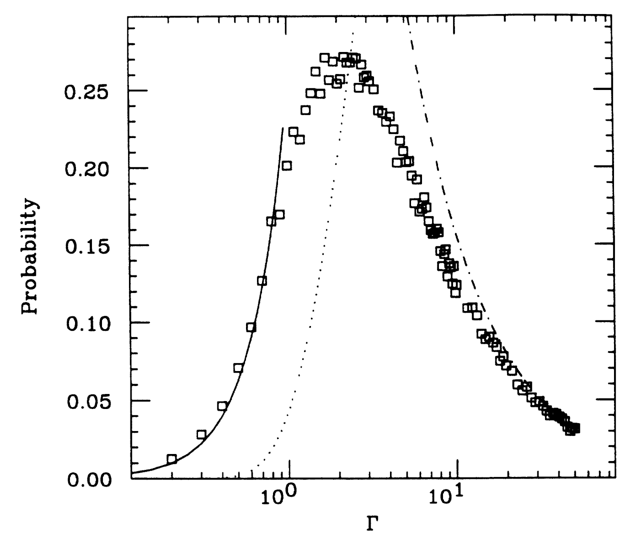

FIG. 6. The asymptotic Zener tunneling probability for $\delta=0.1$ as a function of Γ. The boxes represent numerical results obtained by directly solving Eq. (4.1). The solid and dot-dashed lines are the analytic approximations of Eq. (3.13a) and of Eq. (4.8) limits, respectively. Note that all are distinct from the tunneling probability of the $\mathcal{H}_X$ Hamiltonian, the dotted line, given by Eq. (2.8).

totic Zener tunneling probability for the $\mathcal{H}_Y$ case as given by direct numerical solution of the differential equations [Eq. (4.1)], by our analytic approximations, and also the exact analytic result [Eq. (2.8)] for the $\mathcal{H}_X$ case.

In the ordinary Zener tunneling problem that is defined by the Hamiltonian $\mathcal{H}_X$ [Eq. (2.1)] the solution in the adiabatic limit $\gamma \rightarrow 0$ well approximates that of the regime $\gamma \ll 1$. For the Zener tunneling problem studied in the present work, defined by the Hamiltonian $\mathcal{H}_Y$ [Eq. (2.11)] the parameter Γ introduces an additional energy scale, leading to additional scale of "slowness." The adiabatic limit $\Gamma \rightarrow 0$ does not give a good approximation for the tunneling probability throughout the entire regime $\Gamma \ll 1$. It is a good approximation only in *part* of the regime, namely, where Eq. (3.13b) holds and the tunneling is exponentially small in $1/\Gamma$. There is another regime, where Eq. (3.13a) holds, and the tunneling probability is proportional to Γ^2, in spite of the fact that $\Gamma \ll 1$. In the limit $\delta \rightarrow 0$, the Hamiltonian $\mathcal{H}_Y$ is nonanalytic. This behavior produces an unusual dependence of the tunneling probability on the "adiabaticity" parameter Γ, specifically, it is proportional to Γ^2, in this limit. In other words, the order of the limits $\Gamma \rightarrow 0$ and $\delta \rightarrow 0$ is crucial. This is somewhat reminiscent of the more subtle examples of the asymptotics of wave reflections.[22]

The results discussed in this paper are not limited to linear Hamiltonians. One of the most common cases, in which the energy depends quadratically upon the controlled coordinate (e.g., charge and the electrostatic energy, or flux and the inductive energy) can be mapped directly on to equations like that of Eq. (2.1). This is discussed in detail in Appendix C. This means that calculations involving Zener transitions in systems driven by an oscillatory potential may involve transitions of *both* the $\mathcal{H}_Y$ and $\mathcal{H}_X$ type.

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APPENDIX A: EVALUATION OF THE CONTOUR INTEGRAL

In this appendix we develop the integral of Eq. (3.11) into a contour integral in the complex plane. We then evaluate it numerically, and develop analytic, asymptotic expressions for the Zener tunneling probability.

The main difficulty in evaluating this integral is the rapid oscillation of the integrand due to the rapid variations of the phase of the exponential. This problem is particularly severe in the adiabatic limit since β, a prefactor in the exponent, is large. However, it is possible to evaluate this integral by means of the method of stationary phase.[23] In order to do so, we must first find a contour of integration in the complex $(z=x+iy)$ plane, where the imaginary part of $i\varphi(z)$ is a constant, which we choose to be zero. We may write this as

$$i\varphi(z)=i(\tfrac{1}{2}\sinh 2z+z)$$
$$=i(\tfrac{1}{2}\sinh 2x\,\cos 2y+x)-(\tfrac{1}{2}\cosh 2x\,\sin 2y+y)\ . \tag{A1}$$

The imaginary part vanishes along the imaginary $(x=0)$ axis and along the lines which satisfy the relationship

$$\cos 2y=\frac{-2x}{\sinh 2x}\ , \tag{A2}$$

such as curves C_1 and C_{-1} in Fig. 7. We note for future reference that since $\varphi(x)$ and $f(x)$ of Eq. (3.12) are antisymmetric in x, the quantity $a_\uparrow$ is pure imaginary.

The function $f(z)$ has a branch point (the cross in Fig. 7) at the point $(x=0, y=y_A=\arcsin\sqrt{D}\,)$ and a branch cut BC for $y>y_A$. It has a pole P on the branch cut at $(0,\pi/2)$. The integral [Eq. (3.11)] along the real axis can be replaced by the integral along the contour C consisting of C_{-3}, C_{-1}, C_{-2}, C_-, C_0, C_+, C_2, C_1, and C_3, which circumvents this cut. The contribution from the integrals along C_{-3} and C_3, vertical line segments than join the contour to the real axis, vanishes as their point of intersection recedes to $\mp\infty$, respectively, and thus can be neglected. The contours C_2 and C_{-2} are arcs of radius ϵ around the pole P, parametrized by

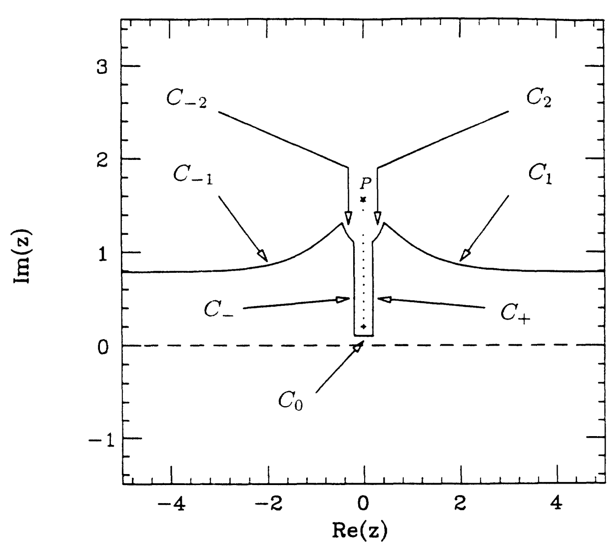

FIG. 7. The contour chosen for the method of stationary phase, circumventing the branch cut (dotted line) that connects the pole P (marked by the X) and the point $y_A=\arcsin D$ (marked by the small cross). The vertical lines $C_{\pm 3}$ which connect the contour to the real axis at $\pm\infty$ are not shown.

$$z=i\frac{\pi}{2}+\epsilon e^{i\theta} . \tag{A3}$$

For small ϵ we may approximate $f(z)$ of Eqs. (3.11) and (3.12) in this region as

$$f(z)\approx\frac{-i\,\mathrm{sgn}(x)}{z\sqrt{1-D}} , \tag{A4}$$

and set $\varphi(z)\approx\varphi(i\pi/2)$. Using Eq. (A2) we can show that C_2 encompansses $-\pi/2\le\theta\le-\pi/6$, while C_{-2} covers $-\pi+(\pi/6)\le\theta\le-\pi/2$. The contour integrals along the curves C_2 and C_{-2} are of opposite sign and thus cancel. As ϵ shrinks to zero, the integral along C_0 vanishes. We are left with only the contributions from integrating along the curves $C_{\pm 1}$ and $C_{\pm}$. On sections $C_{\pm}$ and $C_{\pm 1}$ of the contour $i\varphi(x)$ is real and therefore $e^{i\beta\varphi(x)}$ does not oscillate.

Let I_{-1} and I_1 denote the contour integrals along curves C_{-1} and C_1 respectively, and J_1 be their sum. It is useful to note the symmetry property of $f(z)$ in the region of integration, namely,

$$f(-x,y)=-f^*(x,y) . \tag{A5}$$

This property is a consequence of the fact that $-if(-iz)$ is real in the interval $0<y<y_A$ on the y axis (i.e., it is real analytic with respect to the y axis).

The contribution of the integration along C_{-1} and C_1 reduces to

$$J_1\equiv-D_0\left[\int_{C_{-1}}dz\,f(z)e^{i\beta\varphi(z)}+\int_{C_1}dz\,f(z)e^{i\beta\varphi(z)}\right]$$
$$=-2iD_0\int_{C_1}[dy\,\mathrm{Re}f(z)+dx\,\mathrm{Im}f(z)]e^{i\beta\varphi(z)} \tag{A6}$$

and J_1 is imaginary as expected. Note that $i\varphi(z)$ is real on C_1 and C_{-1}, and we have used Eq. (A5) above.

Finally the contribution from the branch cut comes from integrating along the contours C_- and C_+. The sum of these two integrals, which we denote by J, is

$$J\equiv-2iD_0\int_{y_A}^{\pi/2}dy\,f_c(y)e^{-\beta\varphi_c(y)} , \tag{A7}$$

where

$$f_c(y)=\frac{\sin y}{\cos y(\sin^2 y-D)^{1/2}} \tag{A8}$$

and

$$\varphi_c(y)=-i\varphi(iy)=\tfrac{1}{2}\sin 2y+y . \tag{A9}$$

The total amplitude is then

$$a_\uparrow=J+J_1 . \tag{A10}$$

The integrals J and J_1 are simple to calculate numerically since the integrands do not oscillate. Some care is required since both J and J_1 diverge at the pole P. However, these divergences cancel in the final result. Typical numerical results for the absolute value of the transition probability $P_Z\equiv|a_\uparrow|^2$ calculated by this method are plotted in Fig. 3, as a function of D for various values of β. For relatively small values of β these are compared with the values found by direct numerical solution of the differential equations in Eq. (4.1). It is clear that the agreement improves with increasing β. Note that for large values of β the asymptotic amplitude is very small and consequently the direct numerical solutions of Eq. (4.1) are unreliable, while numerical calculations of the contour integral are still accurate.

We turn now to find two analytical approximations for $a_\uparrow$ for different regimes of D. The pole P at $(0,\pi/2)$ is a saddle point of $i\varphi(z)$. In the vicinity of this point it takes the form

$$i\varphi(z)\approx\frac{-\pi}{2}-\frac{2}{3}i\left[z-i\frac{\pi}{2}\right]^3 , \tag{A11}$$

and its magnitude decreases when one moves away from the pole along the lines C_{-1} and C_1. Since there are no other saddle points on these lines, J_1 is dominated by contributions near the pole. Along the branch cut the function $\varphi_c(y)=-i\varphi(z)$ monotonically increases with y so the integrand decreases exponentially with increasing y save for the contribution from the pole which cancels. For large β the integrand is largest in the vicinity of the branch point y_A and $|J|\gg|J_1|$. In order to find an approximation of J we expand the various functions involved in the calculation of J around y_A. One finds

$$\varphi_c(y)\approx(\tfrac{1}{2}\sin 2y_A+y_A)+(\cos 2y_A+1)\Delta y$$
$$=[\sqrt{D(1-D)}+y_A]+2(1-D)\Delta y , \tag{A12}$$

where $\Delta y=y-y_A$.

From Eq. (A12) we find that near y_A, the function $\epsilon^{-\beta\varphi_c(y)}$ decays exponentially with a rate $2\beta(1-D)$, and consequently the integral J is determined by a region of the size $[2\beta(1-D)]^{-1}$ near y_A. For small D this interval is approximately $1/2\beta$. Near y_A, we expand the argument of the square root in Eq. (A8),

$$\sin^2 y-D\approx(\sin 2y_A)\Delta y+\cos 2y_A(\Delta y)^2$$
$$=2\sqrt{D(1-D)}\Delta y+(1-2D)(\Delta y)^2 . \tag{A13}$$

If D is small, then the second term can be neglected for $\Delta y<2\sqrt{D}$. Moreover, it can be ignored throughout the whole domain of integration if the decay length $1/2\beta$ is smaller than $2\sqrt{D}$ or,

$$\frac{1}{4\beta}\ll\sqrt{D} \tag{A14}$$

since $e^{-\beta\varphi_c}$ is exponentially small when the second term begins to dominate. In the vicinity of y_A the integral (A7) takes the form

$$J\approx\frac{-i}{\sqrt{2}}\left[\frac{D}{1-D}\right]^{1/4}e^{-\beta\varphi_A}\int_0^\infty\frac{e^{-2\beta(1-D)\Delta y}}{\sqrt{\Delta y}}d(\Delta y) , \tag{A15}$$

with

$$\varphi_A=\sqrt{D(1-D)}+\arcsin\sqrt{D} . \tag{A16}$$

Because of the fast decay of the exponential for large β the upper limit of the integral can be extended to infinity. The integral is elementary leading to the result

$$J \approx -i\frac{1}{2(1-D)}\left[\frac{\pi}{\beta}\right]^{1/2}[(1-D)D]^{1/4}e^{-\beta\varphi_A} , \tag{A17}$$

which leads to Eq. (3.13b).

However, the above asymptotic expansion is not valid for $\sqrt{D} \ll 1/4\beta$, and in particular, the important case of $\delta=D=0$. In order to examine this limit we note that for $\sqrt{D} \ll 1/2\beta$, from the definition of y_A we have that $y_A \ll 1/2\beta$. We can assume therefore that $y_A=0$ for these values of D and extend the branch cut down to the origin. With this approximation the integral (A7) reduces to the form

$$\begin{aligned} J &\approx -2iD_0 e^{-\beta\varphi_A}\int_0^\infty e^{-2\beta(1-D)\Delta y}d(\Delta y) \\ &= -\frac{i}{2\beta\sqrt{1-D}}e^{-\beta\varphi_A} , \end{aligned} \tag{A18}$$

which is similar to Eq. (3.15). Since in this limit $D \ll 1$, we can further simplify the above to obtain

$$J \approx \frac{-i}{2\beta} , \tag{A19}$$

which leads to Eq. (3.13a).

APPENDIX B: THE WKB SOLUTION OF $\mathcal{H}_X$ NEAR $t=0$

The Hamiltonian $\mathcal{H}_X$ has been solved elsewhere for the asymptotic probability[1-22] and the Zener time.[17] Here we briefly develop the WKB solution in order to demonstrate the meaning of the Zener time. The Schrödinger equation for this case is

$$\begin{aligned} i\hbar\,\dot{c}_\uparrow(t) &= \alpha t\; c_\uparrow(t)+\Delta c_\downarrow(t) , \\ i\hbar\,\dot{c}_\downarrow(t) &= -\alpha t\; c_\downarrow(t)+\Delta c_\uparrow(t) , \end{aligned} \tag{B1}$$

where $c_\uparrow$ and $c_\downarrow$ are the time-dependent amplitudes to be in the spin-up and spin-down states. Note that because the field has a different time dependence in the $\mathcal{H}_X$ and $\mathcal{H}_Y$ cases, the amplitudes will be different functions, although they correspond to the same physical states, i.e., spin up or spin down. If we change to the variable $y \equiv \alpha t/\Delta$ then Eq. (B1) becomes

$$\begin{aligned} i\gamma\;\dot{c}_\uparrow(y) &= y\; c_\uparrow(y)+c_\downarrow(y) , \\ i\gamma\dot{c}_\downarrow(y) &= -y\; c_\downarrow(y)+c_\uparrow(y) , \end{aligned} \tag{B2}$$

where $\gamma \equiv \hbar\alpha/\Delta^2$ as in Sec. I. These equations can be reduced to a single second-order differential equation

$$\gamma^2\ddot{c}_\uparrow(y)+i\gamma c_\uparrow(y)+(1+y^2)c_\uparrow(y)=0 , \tag{B3}$$

which depends only upon the parameter γ.

In the adiabatic limit ($\gamma \ll 1$) we assume a WKB (Ref. 23) form for $c_\uparrow(y)$

$$c_\uparrow(y) \approx K''(y)e^{\frac{iS''(y)}{\gamma}} . \tag{B4}$$

Substituting Eq. (B4) into Eq. (B5) and equating terms of like order in γ, we obtain, to order γ^0,

$$\dot{S}''_\pm = \pm(y^2+1)^{1/2} , \tag{B5}$$

which has the solution

$$S''_\pm = \pm\tfrac{1}{2}\{y[y^2+1]^{1/2}+\ln[y+(y^2+1)^{1/2}]\} . \tag{B6}$$

To order γ we have

$$2\dot{K}''\dot{S}''+K''\ddot{S}''+K''=0 , \tag{B7}$$

yielding

$$\left[\frac{\dot{K}''}{K''}\right]_\pm = \frac{\mp 1}{2(1+y^2)^{1/2}}-\frac{y}{2(1+y^2)} . \tag{B8}$$

Integrating these equations produces

$$K''_\pm(y)=C_\pm\frac{1}{(y^2+1)^{1/4}}[y+(y^2+1)^{1/2}]^{\mp 1/2} , \tag{B9}$$

where $C_\pm$ are integration constants. The asymptotic values of $K_\pm$ are

$$\begin{aligned} K_+(y) &= \begin{cases} 0 & \text{for } y\to\infty \\ \sqrt{2}C_+ & \text{for } y\to-\infty , \end{cases} \\ K_-(y) &= \begin{cases} \sqrt{2}C_- & \text{for } y\to\infty \\ 0 & \text{for } y\to-\infty . \end{cases} \end{aligned} \tag{B10}$$

The WKB solution must satisfy the boundary conditions, namely, $|c_\uparrow(-\infty)|=1$, so that $C_+=1/\sqrt{2}$. Note that the WKB solution cannot produce the correct solution for $y\to\infty$ which is exponentially small in $1/\gamma$, since it is only valid with a regime $|y|<1/\gamma$.[23] As is discussed in Ref. 17, since the solution has a universal profile when time is measured in units of Δ/α, this identifies Δ/α as the Zener time in the adiabatic limit.

APPENDIX C: ZENER TUNNELING WITH INTERSECTING PARABOLAS

The energy of a variety of mesoscopic systems depends quadratically upon an external parameter (e.g., mesoscopic small normal metal rings in a magnetic field,[11,12] periodic superlattices in an electric field, and ultrasmall capacitance Josephson junctions[13] driven by a current source). One can attempt a description of such systems by a phenomenological Hamiltonian[24] of the form

$$H=A(\hat{n}+Bt)^2+C\cos\hat{\theta} , \tag{C1}$$

where $\hat{n}$ and $\hat{\theta}$ are conjugate operators. This Hamiltonian is analogous to that of the nearly free electron model.[21] For $C=0$ the energy spectrum consists of a series of intersecting parabolas; for $C/A \ll 1$ the points of degeneracy are resolved, and the parabolas are broken into a series of bands.

If we are interested in the time development of such a system, then we have the possibility of multiple Zener tunneling events. In this appendix, we confine ourselves to a much simpler issue: proving that the Hamiltonian of Eq. (2.1) can be used to analyze Zener tunneling at a given intersection of two parabolas.

Consider the case of a two-state system, with the energy spectrum consisting of parabolas, offset from the origin, described by the Hamiltonian

$$H=\begin{bmatrix} \frac{\alpha}{2t_0}(t+t_0)^2 & \Delta \\ \Delta & \frac{\alpha}{2t_0}(t-t_0)^2 \end{bmatrix}, \tag{C2}$$

where t_0 is half the separation between the minima of the parabolas. It is then simple to show that this case is identical to that of Eq. (2.1). We write the time-dependent Schrödinger equation:

$$\begin{aligned} i\hbar\dot{\psi}_1 &= \frac{\alpha}{2t_0}(t+t_0)^2\psi_1+\Delta\psi_2 , \\ i\hbar\dot{\psi}_2 &= \frac{\alpha}{2t_0}(t-t_0)^2\psi_2+\Delta\psi_1 . \end{aligned} \tag{C3}$$

If we make the substitutions

$$\begin{aligned} \psi_1 &= \exp\left[\frac{\alpha}{6i\hbar t_0}t^3+\frac{\alpha t_0}{2i\hbar}t\right]\chi_1 , \\ \psi_2 &= \exp\left[\frac{\alpha}{6i\hbar t_0}t^3+\frac{\alpha t_0}{2i\hbar}t\right]\chi_2 , \end{aligned} \tag{C4}$$

in Eq. (C3), differentiate, and cancel like terms, we obtain

$$\begin{aligned} i\hbar\dot{\chi}_1 &= \alpha t\chi_1+\Delta\chi_2 , \\ i\hbar\dot{\chi}_2 &= -\alpha t\chi_2+\Delta\chi_1 , \end{aligned} \tag{C5}$$

which is identical to Eqs. (2.3) and (B1). The key issue here is the rate of change of the difference of the eigenenergies of the two states. In both the case of the $\mathcal{H}_X$ Hamiltonian and Eq. (C2) this rate of change is linear.

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