

## Stability of particle trajectories and generalized coherent states

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We show that particle trajectories in the Wilson cloud chamber are straight and stable, using perturbation theory and generalized coherent states. The model consists of an incoming particle, many molecules, and the immediate environment of each molecule acting as a detector. Each such detector has many harmonic oscillators whose states are totally described by generalized coherent states, because of which particle trajectories are almost stable. That is, if a molecule is ionized through its interaction with the particle, the ionized excited state does not return to the lower states. Many molecules can be ionized when the line joining them is practically parallel to the direction of motion of the particle. However, when the molecules do not lie on a straight line, all of them cannot be ionized. The particle trajectory that is made up of the ionized molecules, therefore, is straight.

### I. INTRODUCTION

In a quantum-measurement process, we know the state of an object system through its interaction with a detector. The detector may be described by quantum theory, so we can treat the total system quantum mechanically, with one of the characteristic features of quantum theory being that one must use noncommutative observables in a Hilbert space to treat quantum fluctuations, unstable excited states, and quantum jumps between states and other phenomena. However, the detector must register each state of the object system by its pointer. Thus the state of the detector has to be almost stable, i.e., it must behave like a macroscopic (or classical) object. In classical theories, we meet with commutative physical quantities having definite values, whose dynamics are described with the help of a phase space, the Poisson bracket or a symplectic form, and a classical Hamiltonian. There are a large number of differences between these approaches.

Many authors have made efforts to find a link between the microscopic and macroscopic theories. A wide class of quantum theories contain some measure of the number of dynamical variables  $N$  (or  $1/\hbar$ ); these quantum theories have a sensible large- $N$  limit, in which the dynamics become much simpler.<sup>1</sup> Coherence groups and generalized coherent (GC) states play important roles in this approach. In the large- $N$  limit, the expectation value of any product of reasonable operators (called classical operators) satisfies the factorization relation, that is, the expectation value approaches the product of the expectations of each operator. As a result, quantum fluctuations of these operators become irrelevant in this limit. This is due to the fact that any GC state is almost an eigenstate of any classical operator, and as a result standard deviations of these operators are almost negligible when  $N$  is very large. Also, the phase space, Poisson bracket, and classical Hamiltonian from the quantum Hamiltonian

emerge in a natural way in this limit. Moreover, quantum evolutions reduce to classical ones.

We have applied this approach previously to a quantum-measurement process, and presented several models in GC-state formalism.<sup>2,3</sup> The detectors in our formalism are described by GC states, and have many desirable functions as a measuring apparatus. These models indicate that the GC-state approach to quantum measurement can become a powerful method for a unified description of various measuring processes, in particular for a macroscopic description of detectors.

The essential features of particle trajectories in the Wilson cloud chamber have been explained by several authors using the perturbation theory<sup>4-6</sup> These treatments are justified from the Copenhagen point of view. However, the stability of the particle trajectories in the chamber cannot be treated without modifications to the theory.

We have considered previously<sup>3</sup> a model consisting of an incoming particle, *one* molecule, and the environment of the molecule acting as a detector, all treated as quantum systems. The detector was treated as  $N$  harmonic oscillators described by GC states, and exhibits sensible behavior in the large- $N$  limit. We showed that the detector behaves like a macroscopic object when  $N \gg 1$ , and as a result a particle trajectory in the cloud chamber is stable. The macroscopic properties result from the properties of GC states and classical operators. However, some calculations in the model are difficult because the coherence group used there is small, that is it does not contain all bilinear operators.

The model treated in this article consists of an incoming particle, *many* molecules, and the immediate environment of each molecule. When the particle interacts with many molecules simultaneously, the measuring process will be quite different, and analysis of trajectories is performed more precisely. Molecules can be ionized only if they lie on a straight line parallel to the momentum of

the incoming particle. And there is essentially only one trajectory in the cloud chamber if one particle enters it. In a similar way to the previous model, stability of the particle trajectory is a result of the properties of the GC states and classical operators. Furthermore, the coherence group is extended to  $SU(1,1)$ , so that the treatment of this model becomes quite simple, because many calculations can be made group theoretically. The  $SU(1,1)$  coherent states and their applications to physical problems have been studied by many authors.<sup>7</sup> As a model of a detector, we use a harmonic-oscillator model based on the extended coherence group, which is a modified version of the previous one. The interaction between the particle and each molecule is assumed to be weak, and thus the interaction may be treated as a perturbation.

In the next section the modified model is presented, and we also show how the group  $SU(1,1)$  can be derived. Using the new detector constructed in Sec. II, it is shown in Sec. III that particle trajectories in the cloud chamber must be straight and almost stable.

## II. HARMONIC-OSCILLATOR MODEL BASED ON $SU(1,1)$

### A. The coherence group and generalized coherent states

The modified model, which is used in Sec. III as a detector for particle trajectories, consists of one molecule and  $N$  harmonic oscillators, all oscillators have unit mass and frequency  $\omega$ . Suppose that the total Hamiltonian of the model is given by

$$\tilde{H} = H_M + H_A + H_{MA} \quad (2.1)$$

with

$$H_A = \frac{1}{2} \sum_{i=1}^N (\hat{P}_i^2 + \omega^2 \hat{X}_i^2) \equiv N(\hat{C} + \omega^2 \hat{A}), \quad (2.2a)$$

$$H_{MA} = \frac{1}{2} g H_M \sum_{i=1}^N (\hat{X}_i \hat{P}_i + \hat{P}_i \hat{X}_i) \equiv N g H_M \hat{B}, \quad (2.2b)$$

where  $H_M$  is the Hamiltonian of the molecule,  $\hat{X}_i$  and  $\hat{P}_i$  are the position and momentum operators with respect to the  $i$ th oscillator ( $[\hat{X}_i, \hat{P}_j] = i\hbar\delta_{ij}$ ), and

$$\begin{aligned} \hat{A} &= \frac{1}{2N} \sum_i \hat{X}_i^2, \quad \hat{B} = \frac{1}{2N} \sum_i (\hat{X}_i \hat{P}_i + \text{H.c.}), \\ \hat{C} &= \frac{1}{2N} \sum_i \hat{P}_i^2. \end{aligned} \quad (2.3)$$

The interaction  $H_{MA}$  contains the Hamiltonian  $H_M$  of the molecule, which plays an important role in our quantum-measurement process. Owing to this fact, the detectors in the next section can register the states of the molecules (see Sec. III). The total Hamiltonian  $\tilde{H}$  is invariant under the group  $O(N)$  consisting of all orthogonal transformations in  $(\hat{X}_1, \dots, \hat{X}_N)$  or  $(\hat{P}_1, \dots, \hat{P}_N)$ .

Hence we can restrict our attention to the  $O(N)$  invariant sector of the present model. Then the Hilbert space  $\mathcal{H}_N$  consists of  $O(N)$  invariant states.

The operators  $\hat{A}$ ,  $\hat{B}$ , and  $\hat{C}$  constitute a Lie algebra:

$$\begin{aligned} [\hat{A}, \hat{B}] &= \frac{2i\hbar}{N} \hat{A}, \quad [\hat{B}, \hat{C}] = \frac{2i\hbar}{N} \hat{C}, \\ [\hat{C}, \hat{A}] &= -\frac{i\hbar}{N} \hat{B}. \end{aligned} \quad (2.4)$$

Setting

$$\begin{aligned} D_N(\rho_1) &= \frac{N}{i\hbar} \hat{B}, \quad D_N(\rho_2) = \frac{N}{i\hbar\omega} (\hat{C} - \omega^2 \hat{A}), \\ D_N(\rho_3) &= -\frac{N}{i\hbar\omega} (\hat{C} + \omega^2 \hat{A}), \end{aligned} \quad (2.5)$$

we find

$$[D_N(\rho_1), D_N(\rho_2)] = -2D_N(\rho_3), \quad (2.6a)$$

$$[D_N(\rho_2), D_N(\rho_3)] = 2D_N(\rho_1), \quad (2.6b)$$

$$[D_N(\rho_3), D_N(\rho_1)] = 2D_N(\rho_2). \quad (2.6c)$$

They constitute the Lie algebra  $\mathfrak{su}(1,1)$  with generators

$$\rho_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \rho_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \quad (2.7)$$

satisfying the commutation relations

$$[\rho_1, \rho_2] = -2\rho_3, \quad [\rho_2, \rho_3] = 2\rho_1, \quad [\rho_3, \rho_1] = 2\rho_2. \quad (2.8)$$

Equations (2.6) and (2.8) imply that  $D_N$  is a representation of  $\mathfrak{su}(1,1)$  in the space  $\mathcal{H}_N$ . Using the annihilation and creation operators defined by

$$a_j = \frac{1}{\sqrt{2\hbar\omega}} (\omega\hat{X}_j + i\hat{P}_j), \quad a_j^\dagger = \frac{1}{\sqrt{2\hbar\omega}} (\omega\hat{X}_j - i\hat{P}_j), \quad (2.9)$$

the representation for  $\rho_i$  becomes

$$D_N(\rho_1) = \frac{1}{2} \sum_j (a_j^\dagger a_j^\dagger - a_j a_j), \quad (2.10a)$$

$$D_N(\rho_2) = \frac{i}{2} \sum_j (a_j^\dagger a_j^\dagger + a_j a_j), \quad (2.10b)$$

$$D_N(\rho_3) = i \sum_j (a_j^\dagger a_j + \frac{1}{2}), \quad (2.10c)$$

and so the Hamiltonian of the detector is given by

$$H_A = -i\hbar\omega D_N(\rho_3) = \hbar\omega \sum_j (a_j^\dagger a_j + \frac{1}{2}). \quad (2.11)$$

The algebra  $\mathfrak{su}(1,1)$  generates the coherence group  $SU(1,1)$ :

$$SU(1,1) = \left\{ u \mid u^\dagger \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} u = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}, \quad (2.12)$$

all elements  $u$  of  $SU(1,1)$  being parametrized as follows:

$$u = u(\alpha, \beta) \equiv e^{\alpha\rho_-} e^{\beta\sigma_3} e^{\gamma\rho_+} = \begin{pmatrix} e^\beta & \alpha^* e^{\beta^*} \\ \alpha e^\beta & e^{\beta^*} \end{pmatrix}, \quad (2.13)$$

with

$$(1 - |\alpha|^2)e^{\beta + \beta^*} = 1, \quad \gamma = \alpha^* e^{\beta^* - \beta}, \quad (2.14)$$

where  $\rho_\pm = (\rho_1 \pm i\rho_2)/2$ ,  $\sigma_3 = i\rho_3$ , and  $\alpha$ ,  $\beta$ , and  $\gamma$  are complex numbers. Note that  $|\alpha| < 1$ . The independent parameters are, for example,  $\alpha$  and the imaginary part of  $\beta$ . Including  $\hat{A}$ ,  $\hat{B}$ , and  $\hat{C}$  in  $\mathfrak{su}(1,1)$  conveniently allows one to express the action of the coherence group in Fock space, which makes our calculations much simpler.

The GC states are generated by applying elements of the group  $SU(1,1)$  to a "base state"  $|\text{in}\rangle$ ; that is, for any element  $u(\alpha, \beta) \in SU(1,1)$ , we find a GC state

$$\begin{aligned} |\alpha, \beta\rangle &\equiv D_N(u(\alpha, \beta))|\text{in}\rangle = e^{-N\beta/2} D_N(e^{\alpha\rho_-})|\text{in}\rangle \\ &\equiv \exp\left[-i\frac{N}{2}\beta_2\right]|\alpha\rangle, \end{aligned} \quad (2.15)$$

where  $\beta = \beta_1 + i\beta_2$  ( $\beta_i \in \mathbb{R}$ ). It follows from Eqs. (2.13) and (2.15) that  $|\text{in}\rangle = |0, 0\rangle$ . Different coherent states are not orthogonal; their overlaps are given by

$$\begin{aligned} \langle \alpha, \beta | \alpha', \beta' \rangle &= \langle 0, 0 | D_N(u^{-1}(\alpha, \beta)u(\alpha', \beta')) | 0, 0 \rangle \\ &= [e^{\beta^* + \beta'} (1 - \alpha^* \alpha')]^{-N/2}, \end{aligned} \quad (2.16a)$$

$$\langle \alpha | \alpha' \rangle = \left[ \frac{1 - \alpha^* \alpha'}{\sqrt{(1 - |\alpha|^2)(1 - |\alpha'|^2)}} \right]^{-N/2}. \quad (2.16b)$$

When  $N \gg 1$  and  $\alpha \neq \alpha'$ , these two different states are approximately orthogonal, because

$$\begin{aligned} |\langle \alpha, \beta | \alpha', \beta' \rangle|^2 &= |\langle \alpha | \alpha' \rangle|^2 \\ &= \left[ 1 + \frac{|\alpha - \alpha'|^2}{(1 - |\alpha|^2)(1 - |\alpha'|^2)} \right]^{-N/2}. \end{aligned} \quad (2.17)$$

This is a very important property of GC states, because it is used later for deriving the mixed-state statistical operator and for explaining stability of particle trajectories. The reduced coherent states  $|\alpha\rangle$  are very useful in the actual calculations.

### B. Classical operators

Any operator in  $\hat{X}_i$  and  $\hat{P}_i$  need not have a sensible limit as  $N \rightarrow \infty$ . In order to have control over this limit, we have to introduce a restricted set of operators  $K$ , consisting of operators  $A$  whose coherent state matrix elements  $\langle u | A | u' \rangle_N / \langle u | u' \rangle_N$  have finite limits as  $N \rightarrow \infty$ , that is,

$$\lim_{N \rightarrow \infty} \frac{\langle \alpha, \beta | A | \alpha', \beta' \rangle_N}{\langle \alpha, \beta | \alpha', \beta' \rangle_N} < \infty \quad \text{for all } \alpha, \beta, \alpha', \beta' \in \mathbb{C}, \quad (2.18)$$

where  $|\alpha, \beta\rangle_N \in \mathcal{H}_N$  and  $|\alpha|, |\alpha'| < 1$ . Such operators are referred to as classical operators. The expectation value of any pair of operators  $A, B \in K$  satisfies the factorization relation

$$\lim_{N \rightarrow \infty} (\langle u | AB | u \rangle_N - \langle u | A | u \rangle_N \langle u | B | u \rangle_N) = 0. \quad (2.19)$$

Therefore the variance of any such operator vanishes as  $N \rightarrow \infty$ , and then the quantum fluctuations for the operators become irrelevant. All  $N$ -independent polynomials in  $\hat{A}$ ,  $\hat{B}$ , and  $\hat{C}$  given by Eq. (2.3) are classical operators. These facts will be shown below by explicit calculation.

Let us consider the matrix elements of the operators  $D_N(\rho_i)$ ; they are obtained as follows:

$$\begin{aligned} \langle \alpha, \beta | D_N(\rho_i) | \alpha', \beta' \rangle_N &= \frac{\partial}{\partial t} \langle \alpha, \beta | D_N(e^{t\rho_i}) | \alpha', \beta' \rangle_N \Big|_{t=0} \\ &= \frac{\partial}{\partial t} \langle 0, 0 | D_N(e^{\alpha'\rho_-} e^{\beta'\sigma_3} e^{\gamma''\rho_+}) | 0, 0 \rangle_N \Big|_{t=0} \\ &= \frac{\partial}{\partial t} \left[ \exp\left[-\frac{N}{2}\beta''\right] \right] \Big|_{t=0}, \end{aligned} \quad (2.20)$$

where  $\alpha''$ ,  $\beta''$ , and  $\gamma''$  are defined by

$$u^{-1}(\alpha, \beta) e^{t\rho_i} u(\alpha', \beta') = e^{\alpha''\rho_-} e^{\beta''\sigma_3} e^{\gamma''\rho_+}. \quad (2.21)$$

Taking into account the (1,1) element of  $u^{-1}(\alpha, \beta) e^{t\rho_1} u(\alpha', \beta')$ , which is given by  $e^{\beta^* + \beta'} [(1 - \alpha^* \alpha') \cosh t + (\alpha - \alpha^*) \sinh t]$ , we arrive at

$$\begin{aligned} \langle \alpha, \beta | D_N(\rho_1) | \alpha', \beta' \rangle_N &= \frac{N}{2} [e^{\beta^* + \beta'} (1 - \alpha^* \alpha')]^{-N/2-1} \\ &\quad \times e^{\beta^* + \beta'} (\alpha^* - \alpha'). \end{aligned} \quad (2.22)$$

Similarly,

$$\begin{aligned} \langle \alpha, \beta | D_N(\rho_2) | \alpha', \beta' \rangle_N &= \frac{iN}{2} [e^{\beta^* + \beta'} (1 - \alpha^* \alpha')]^{-N/2-1} \\ &\quad \times e^{\beta^* + \beta'} (\alpha^* + \alpha'), \end{aligned} \quad (2.23a)$$

$$\begin{aligned} \langle \alpha, \beta | D_N(\rho_3) | \alpha', \beta' \rangle_N &= \frac{iN}{2} [e^{\beta^* + \beta'} (1 - \alpha^* \alpha')]^{-N/2-1} \\ &\quad \times e^{\beta^* + \beta'} (1 + \alpha^* \alpha'). \end{aligned} \quad (2.23b)$$

The matrix elements for the operators  $\hat{A}$ ,  $\hat{B}$ , and  $\hat{C}$  are thus obtained:

$$\frac{\langle \alpha, \beta | \hat{A} | \alpha', \beta' \rangle_N}{\langle \alpha, \beta | \alpha', \beta' \rangle_N} = \frac{\hbar}{4\omega} \frac{1 + \alpha^* + \alpha' + \alpha^* \alpha'}{1 - \alpha^* \alpha'}, \quad (2.24a)$$

$$\frac{\langle \alpha, \beta | \hat{B} | \alpha', \beta' \rangle_N}{\langle \alpha, \beta | \alpha', \beta' \rangle_N} = \frac{i\hbar}{2} \frac{\alpha^* - \alpha'}{1 - \alpha^* \alpha'} \quad (2.24b)$$

$$\frac{\langle \alpha, \beta | \hat{C} | \alpha', \beta' \rangle_N}{\langle \alpha, \beta | \alpha', \beta' \rangle_N} = \frac{\hbar\omega}{4} \frac{1 - \alpha^* - \alpha' + \alpha^* \alpha'}{1 - \alpha^* \alpha'} \quad (2.24c)$$

The right-hand sides of Eq. (2.24) are all independent of  $N$ , indicating that  $\hat{A}$ ,  $\hat{B}$ , and  $\hat{C}$  are classical operators. In the case of  $\alpha = \alpha'$  and  $\beta = \beta'$ , Eq. (2.24) reduces to the expectation value

$$\langle \alpha, \beta | \hat{A} | \alpha, \beta \rangle_N = \frac{\hbar}{4\omega} \frac{1 + \alpha^* + \alpha + |\alpha|^2}{1 - |\alpha|^2}, \quad (2.25a)$$

$$\langle \alpha, \beta | \hat{B} | \alpha, \beta \rangle_N = \frac{i\hbar}{2} \frac{\alpha^* - \alpha}{1 - |\alpha|^2}, \quad (2.25b)$$

$$\langle \alpha, \beta | \hat{C} | \alpha, \beta \rangle_N = \frac{\hbar\omega}{4} \frac{1 - \alpha^* - \alpha + |\alpha|^2}{1 - |\alpha|^2}. \quad (2.25c)$$

It follows from Eqs. (2.11) and (2.25) that the expectation value of the Hamiltonian  $H_A$  of the oscillators is written in the form

$$\left\langle \alpha, \beta \left| \frac{1}{N} H_A \right| \alpha, \beta \right\rangle_N = \frac{\hbar\omega}{2} \frac{1 + |\alpha|^2}{1 - |\alpha|^2}. \quad (2.26)$$

The corresponding classical Hamiltonian  $h_{cl}$  (defined by

$\lim_{N \rightarrow \infty} \langle \alpha, \beta | H_A | \alpha, \beta \rangle_N / N\hbar$ ) is thus  $(\omega/2)(1 + |\alpha|^2)/(1 - |\alpha|^2)$ , which determines the classical dynamics.

Let us proceed to estimate the matrix elements of any polynomial in  $\hat{A}$ ,  $\hat{B}$ , and  $\hat{C}$ . To this end we need some manipulation to reduce some quantities described in terms of  $N$  to those with  $N = 1$ . For example, we find

$$D_N(\rho_i) = \sum_{k=1}^N D_1^{(k)}(\rho_i), \quad (2.27a)$$

$$\begin{aligned} |\alpha, \beta \rangle_N &= \prod_{k=1}^N D_1^{(k)}(e^{\alpha\rho_-} e^{\beta\sigma_3} e^{\gamma\rho_+}) |0, 0\rangle \\ &\equiv \prod_{k=1}^N |\alpha, \beta(k) \rangle_1, \end{aligned} \quad (2.27b)$$

where  $D_1^{(k)}(\rho_i)$  contains only  $a_k$  or  $a_k^\dagger$ , for example,  $D_1^{(k)}(\rho_3) = i(a_k^\dagger a_k + \frac{1}{2})$ . We thus find easily

$$\begin{aligned} \langle \alpha, \beta | \alpha', \beta' \rangle_N &= \prod_{k=1}^N \langle \alpha, \beta(k) | \alpha', \beta'(k) \rangle_1 \\ &= (\langle \alpha, \beta | \alpha', \beta' \rangle_1)^N = (e^{-\beta'/2})^N, \end{aligned} \quad (2.28)$$

where

$$\langle \alpha, \beta(k) | \alpha', \beta'(k) \rangle_1 \equiv \langle \alpha, \beta | \alpha', \beta' \rangle_1$$

is independent of  $k$  and  $e^{\beta'} = e^{\beta^* + \beta'} (1 - \alpha^* \alpha')$ . The matrix elements of  $D_N(\rho_i)$  also reduce to those of  $D_1^{(k)}(\rho_i)$ :

$$\begin{aligned} \langle \alpha, \beta | D_N(\rho_i) | \alpha', \beta' \rangle_N &= \sum_{k=1}^N \langle \alpha, \beta(k) | D_1^{(k)}(\rho_i) | \alpha', \beta'(k) \rangle_1 \langle \alpha, \beta | \alpha', \beta' \rangle_{N-1} \\ &= N \langle \alpha, \beta | \alpha', \beta' \rangle_N \langle \alpha, \beta | D_1(\rho_i) | \alpha', \beta' \rangle_1 e^{\beta'/2}, \end{aligned} \quad (2.29)$$

where we have used

$$\langle \alpha, \beta | \alpha', \beta' \rangle_{N-k} = \langle \alpha, \beta | \alpha', \beta' \rangle_N e^{(k/2)\beta'}. \quad (2.30)$$

Similarly we obtain

$$\begin{aligned} \langle \alpha, \beta | D_N(\rho_{i_1}) \cdots D_N(\rho_{i_n}) | \alpha', \beta' \rangle_N &= \sum_{k_1, \dots, k_n=1}^N \langle \alpha, \beta | D_1^{(k_1)}(\rho_{i_1}) \cdots D_1^{(k_n)}(\rho_{i_n}) | \alpha', \beta' \rangle_N \\ &= N^n \langle \alpha, \beta | \alpha', \beta' \rangle_N \left[ \prod_{k=1}^n \langle \alpha, \beta | D_1(\rho_{i_k}) | \alpha', \beta' \rangle_1 e^{(n/2)\beta'} + O\left(\frac{1}{N}\right) \right]. \end{aligned} \quad (2.31)$$

From Eqs. (2.29) and (2.31) we arrive at

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\langle \alpha, \beta | (1/N) D_N(\rho_{i_1}) \cdots (1/N) D_N(\rho_{i_n}) | \alpha', \beta' \rangle_N}{\langle \alpha, \beta | \alpha', \beta' \rangle_N} &= \prod_{k=1}^n \langle \alpha, \beta | D_1(\rho_{i_k}) | \alpha', \beta' \rangle_1 e^{(n/2)\beta'} \\ &= \prod_{k=1}^n \frac{\langle \alpha, \beta | (1/N) D_N(\rho_{i_k}) | \alpha', \beta' \rangle_N}{\langle \alpha, \beta | \alpha', \beta' \rangle_N}. \end{aligned} \quad (2.32)$$

Equations (2.5) and (2.32) show that any  $N$ -independent polynomial in  $\hat{A}$ ,  $\hat{B}$ , and  $\hat{C}$  is a classical operator; Eq. (2.32) is nothing more than the factorization relation. Then the variance of such polynomials becomes very small when  $N \gg 1$ , and the quantum fluctuation for these operators vanishes.

In the large- $N$  limit, a phase space and the Poisson bracket can be defined; the quantum equation of motion for a classical operator reduces to a "classical" equation expressed by the Poisson bracket.

### III. PARTICLE TRAJECTORIES IN THE CLOUD CHAMBER

#### A. Perturbation theory

The model discussed in this section consists of an incoming particle (with Hamiltonian  $H_S$  and momentum  $\mathbf{p}$ ) and some molecule-detector systems as described earlier. Each detector is constructed of many harmonic oscillators. The molecule-detector systems are contained in a cloud chamber. The incoming particle enters the cloud chamber, and makes a stable trajectory. In our model the stability of the particle trajectory is a result of macroscopic properties of GC states of the detectors. Suppose that the particle interacts only with the molecules, and that the interaction  $V$  between the particle and the molecules may be treated as a perturbation.

When the particle interacts with  $l$  molecules simultaneously, and each molecule also interacts with its immediate environment (i.e., a detector), the total Hamiltonian of the model becomes

$$H = H_0 + V, \quad (3.1a)$$

$$H_0 = H_S + H_M + H_A + H_{MA} \quad (3.1b)$$

with

$$V = \sum_{i=1}^l V_i, \quad H_M = \sum_{i=1}^l H_{Mi}, \quad H_A = \sum_{i=1}^l H_{Ai}, \quad (3.2)$$

$$H_{MA} = \sum_{i=1}^l H_{MAi} = \sum_{i=1}^l N g H_{Mi} \hat{B}_i,$$

where  $V_i$  is the interaction between the incoming particle and the  $i$ th molecule,  $H_{Ai}$  is the Hamiltonian of the  $i$ th detector which is given by Eq. (2.2a), etc. Although the incoming particle interacts with  $l$  molecules simultaneously, we assume that the particle meets with them consecutively. Using the definition in Eq. (3.2), the calculations below are almost parallel to our previous paper.<sup>3</sup>

The state of the total system at time  $t$  becomes  $\Psi(t) = U(t, t_0)\Psi(t_0)$ , where  $\Psi(t_0)$  is the initial state at time  $t_0$ . Setting  $U(t, t_0) = U_0(t, t_0)U_I(t, t_0)$  (where  $U_0$  is the free evolution operator), we find

$$i\hbar \frac{\partial}{\partial t} U_0(t, t_0) = H_0 U_0(t, t_0), \quad (3.3a)$$

$$i\hbar \frac{\partial}{\partial t} U_I(t, t_0) = H_I(t) U_I(t, t_0), \quad (3.3b)$$

where

$$H_I(t) = U_0^\dagger(t, t_0) V U_0(t, t_0).$$

Hence the perturbation series for  $U_I$  is obtained:

$$\begin{aligned} U_I(t, t_0) &= 1 + \frac{1}{i\hbar} \int_{t_0}^t dt_1 H_I(t_1) \cdots \\ &\equiv 1 + U_I^{(1)}(t, t_0) + U_I^{(2)}(t, t_0) + \cdots \end{aligned} \quad (3.4)$$

If the interaction  $V$  between the particle and the molecules is neglected, the particle and molecule-detector systems can move independently, and so  $U_0(t, t_0) = U_0^S(t, t_0) U_0^{MA}(t, t_0)$ ,  $U_0^S$  and  $U_0^{MA}$  being the time evolution operators with respect to the particle and the molecule-detector systems respectively. The state at time  $t$  is thus written

$$\begin{aligned} \Psi(t) &= U_0^{MA}(t, t_0) U_0^S(t, t_0) U_I(t, t_0) \Psi(t_0) \\ &\equiv U_0^{MA}(t, t_0) \tilde{\Psi}(t). \end{aligned} \quad (3.5)$$

The state  $\tilde{\Psi}(t)$  satisfies

$$i\hbar \frac{\partial}{\partial t} \tilde{\Psi}(t) = (H_S + \tilde{V}) \tilde{\Psi}(t), \quad (3.6)$$

where  $\tilde{V} = U_0^{MA\dagger}(t, t_0) V U_0^{MA}(t, t_0)$ .

The perturbation theory is applied to the wave equation (3.6) for  $\tilde{\Psi}(t)$  rather than that for  $\Psi(t)$ , as then the calculation will be much simplified. Set the perturbation series

$$\tilde{\Psi}(t) = \sum_{k=0}^{\infty} \tilde{\Psi}^{(k)}(t),$$

where

$$\tilde{\Psi}^{(k)}(t) = U_0^S(t, t_0) U_I^{(k)}(t, t_0) \Psi(t_0), \quad (3.7)$$

and  $U_I^{(0)} = 1$ . The perturbation theory then gives us

$$\left[ i\hbar \frac{\partial}{\partial t} - H_S \right] \tilde{\Psi}^{(k)}(t) = \tilde{V} \tilde{\Psi}^{(k-1)}(t), \quad (3.8)$$

where  $\tilde{\Psi}^{(-1)} \equiv 0$ . The perturbation expansion of the original wave function becomes

$$\Psi(t) = \sum_{k=0}^{\infty} \Psi^{(k)}(t)$$

with

$$\Psi^{(k)}(t) = U_0^{MA}(t, t_0) \tilde{\Psi}^{(k)}(t).$$

Suppose that the initial state  $\Psi(t_0)$  has the lowest energy; then the molecules and detectors are all in their ground states. Thus their state is  $|0\rangle|0,0\rangle$ , where  $|0\rangle$  and  $|0,0\rangle$  are, respectively, the ground states of the molecules and detectors. The initial state becomes

$$\Psi(t_0) = \varphi(t_0) |0\rangle |0,0\rangle, \quad (3.9)$$

where  $\varphi(t_0) = \exp(i(\mathbf{p}\mathbf{x} - \mathbf{p}^2 t_0 / 2m) / \hbar)$  is the initial state of the particle ( $\mathbf{x}$  and  $m$  are, respectively, the position and the mass). The state  $\tilde{\Psi}^{(0)}(t)$  at time  $t$  can be written

$$\tilde{\Psi}^{(0)}(t) = \varphi(t) |0\rangle |0,0\rangle, \quad (3.10)$$

then, using Eq. (3.5), the total state at time  $t$  becomes

$$\Psi^{(0)}(t) = \varphi(t)|0\rangle \exp\left[-\frac{i}{\hbar}H_A(t-t_0)\right]|0,0\rangle, \quad (3.11)$$

where  $E_0 \equiv 0$ . Since every eigenstate  $|n\rangle = |n_1\rangle \cdots |n_l\rangle$  of the molecules (where  $|n_i\rangle$  is a state of the  $i$ th molecule) and GC states  $|\alpha, \beta\rangle$  are, respectively, complete in each Hilbert space, the function  $\tilde{\Psi}^{(k)}(t)$  ( $k=1, 2, \dots$ ) for  $k$ th perturbation can be expanded as

$$\tilde{\Psi}^{(k)}(t) = \sum_{n, \alpha, \beta} v_{n\alpha\beta}^{(k)}(\mathbf{x}, t) |n\rangle |\alpha, \beta\rangle = \sum_{n, \alpha} v_{n\alpha}^{(k)}(\mathbf{x}, t) |n\rangle |\alpha\rangle, \quad (3.12)$$

where

$$v_{n\alpha}^{(k)} = \sum_{\beta_2} v_{n\alpha\beta}(\mathbf{x}, t) \exp(-iN\beta_2/2),$$

$$(\beta = \beta_1 + i\beta_2, \beta_i \in \mathbb{R}),$$

and  $|\alpha, \beta\rangle$  and  $|\alpha\rangle$  are GC states given by Eq. (2.15). Although  $\alpha$  and  $\beta$  are continuous complex numbers, they have been treated as discrete. Note that  $|\alpha, \beta\rangle = |\alpha_1, \beta_1\rangle \cdots |\alpha_l, \beta_l\rangle$ , where  $|\alpha_i, \beta_i\rangle$  is a GC state with respect to the  $i$ th detector, and the functions  $v^{(k)}(\mathbf{x}, t_0) = 0$  because  $U_j^{(k)}(t_0, t_0) = 0$ .

### B. Properties of the functions $v^{(k)}$

Let us investigate the conditions that ensure that a nonzero solution  $v^{(1)}(\mathbf{x}, t)$  exists following the treatment of Refs. 4–6. Substituting Eqs. (3.10) and (3.12) into Eq. (3.8), we obtain

$$\sum_{\alpha'} \left[ i\hbar \frac{\partial}{\partial t} - H_S \right] v_{n\alpha'}^{(1)}(\mathbf{x}, t) \langle \alpha | \alpha' \rangle = \langle n | V | 0 \rangle \langle \alpha | \exp\left[\frac{i}{\hbar}(E_n + H_A + H_{MA}^n)(t-t_0)\right] \exp\left[-\frac{i}{\hbar}H_A(t-t_0)\right] | 0, 0 \rangle \varphi(t), \quad (3.13)$$

where  $E_n = \sum_{i=1}^l E_{ni}$  and

$$H_{MA}^n = \sum_{i=1}^l N g E_{ni} \hat{B}_i.$$

$E_{ni}$  and  $\hat{B}_i$  are with respect to the  $i$ th molecule. When  $N$  is large, Eq. (3.13) thus reduces to

$$\left[ i\hbar \frac{\partial}{\partial t} - H_S \right] v_{n\alpha}^{(1)}(\mathbf{x}, t) = \langle n | V | 0 \rangle \langle \alpha | \alpha_n, \beta_n \rangle \varphi(t), \quad (3.14)$$

where

$$|\alpha_n, \beta_n\rangle = \exp\left[\frac{i}{\hbar}(E_n + H_A + H_{MA}^n)(t-t_0)\right] \times \exp\left[-\frac{i}{\hbar}H_A(t-t_0)\right] | 0, 0 \rangle. \quad (3.15)$$

If the right-hand side of Eq. (3.12) is zero, then we have  $v_{n\alpha}^{(1)} = 0$  because of the initial condition. Consequently,

the condition for nonzero  $v_{n\alpha}^{(1)}$  is as follows:

$$|\alpha\rangle \simeq |\alpha_n\rangle, \quad (3.16a)$$

$$\langle n | V | 0 \rangle = \sum_{i=1}^l \delta_{n_i, 0} \cdots \delta_{n_{i-1}, 0} \langle n_i | V_i | 0 \rangle \delta_{n_{i+1}, 0} \cdots \delta_{n_l, 0} \neq 0, \quad (3.16b)$$

where

$$|\alpha_n, \beta_n\rangle = \exp(-iN\beta_{n2}/2) |\alpha_n\rangle$$

$$(\beta_n = \beta_{n1} + i\beta_{n2}, \beta_{ni} \in \mathbb{R}).$$

Since according to Eqs. (3.14) and (3.16),  $v_{n\alpha_n}^{(1)}$  is zero unless at least  $(l-1)$   $n_i$ 's are zero, to the first-order approximation only one of the molecules can be excited. Substituting Eq. (3.16a) into Eq. (3.12) we find

$$\tilde{\Psi}^{(1)}(t) = \sum_n^{(1)} v_{n\alpha_n}^{(1)}(\mathbf{x}, t) |n\rangle |\alpha_n\rangle, \quad (3.17)$$

where  $\sum_n^{(1)}$  is a sum such that at least  $(l-1)$   $n_i$ 's are zero. Equation (3.17) leads us to

$$\Psi^{(1)}(t) = \sum_n^{(1)} v_{n\alpha_n}^{(1)}(\mathbf{x}, t) \exp\left[i\frac{N}{2}\beta_{n2}\right] |n\rangle \exp\left[-\frac{i}{\hbar}H_A(t-t_0)\right] | 0, 0 \rangle. \quad (3.18)$$

It should be noted that  $|\alpha\rangle$  in Eq. (3.12) is independent of time, while  $|\alpha_n\rangle$  in Eq. (3.17) contains a variable  $t$ . If we fix the number  $n$  and time  $t$  in Eq. (3.12), then only  $v_{n\alpha_n}^{(k)}$  is nonzero, and we arrive at Eq. (3.17). Hence all states  $|n\rangle$  of the molecules couple with the same GC state  $\exp[-(i/\hbar)H_A(t-t_0)]|0,0\rangle \propto |0,0\rangle$ , and as a result the

detector does not register each state of the molecules. To the first approximation two molecules cannot be ionized simultaneously, so that the incoming particle cannot make a visible trajectory, because such trajectory is made up of more than one ionized molecule.

Henceforth we consider for simplicity the case where

the particle interacts with two molecules simultaneously, i.e.,  $l=2$ . Then there are three types of  $v^{(1)}$ :  $v_{(0,0)}^{(1)}$ ,  $v_{(n_1^*,0)}^{(1)}$ ,  $v_{(0,n_2^*)}^{(1)}$ , where  $n_i^* \neq 0$ , and we have omitted the subscript  $\alpha$ . For  $v_{(0,0)}^{(1)}$  Eq. (3.14) reduces to

$$\begin{aligned} \left[ i\hbar \frac{\partial}{\partial t} - H_S \right] v_{(0,0)}^{(1)}(\mathbf{x}, t) \\ = (\langle 0|V_1|0\rangle + \langle 0|V_2|0\rangle)\varphi(t)\exp\left[-i\frac{N}{2}\beta_{(0,0)2}\right]. \end{aligned} \quad (3.19)$$

The nonzero region of the term  $\langle 0|V_i|0\rangle\varphi(t)$ , which is the fictitious oscillator producing the wave, is located in the small volume near the  $i$ th molecule. The volume is determined by the function  $\langle 0|V_i|0\rangle$ . The function  $v_{(0,0)}^{(1)}$  is different from zero only in two beams originating in the small volumes near the two molecules and having the direction  $\mathbf{p}$ . Similarly,  $v_{(n_1^*,0)}^{(1)}$  and  $v_{(0,n_2^*)}^{(1)}$  are beams originating, respectively, near the first and second molecules and having the direction  $\mathbf{p}$ .

We now proceed to the second-order perturbation. The equation for  $v^{(2)}$  is obtained in a similar way to the first approximation:

$$\left[ i\hbar \frac{\partial}{\partial t} - H_S \right] v_{n\alpha}^{(2)}(\mathbf{x}, t) = \sum_m^{(1)} v_{m\alpha_m}^{(1)}(\mathbf{x}, t) \langle n|V|m\rangle \langle \alpha|\alpha_n\rangle \exp\left[i\frac{N}{2}(\beta_{m2} - \beta_{n2})\right]. \quad (3.20)$$

The condition for  $v^{(2)} \neq 0$  is thus  $|\alpha\rangle = |\alpha_n\rangle$  and  $\sum_m^{(1)} v_{m\alpha_m}^{(1)}(\mathbf{x}, t) \langle n|V|m\rangle \exp(iN\beta_{m2}/2) \neq 0$ . Note that there is no restriction on the number  $n$  in  $v_{n\alpha_n}^{(2)}$  because  $l=2$ . The function  $v_{(n_1^*,0)}^{(2)}$  satisfies

$$\begin{aligned} \left[ i\hbar \frac{\partial}{\partial t} - H_S \right] v_{(n_1^*,0)}^{(2)}(\mathbf{x}, t) = v_{(0,0)}^{(1)}(\mathbf{x}, t) \langle n_1^*|V_1|0\rangle \exp\left[i\frac{N}{2}(\beta_{(0,0)2} - \beta_{(n_1^*,0)2})\right] \\ + \sum_{m_1^*}^{(1)} v_{(m_1^*,0)}^{(1)}(\mathbf{x}, t) (\langle n_1^*|V_1|m_1^*\rangle + \langle 0|V_2|0\rangle \delta_{n_1^*m_1^*}) \exp\left[i\frac{N}{2}(\beta_{(m_1^*,0)2} - \beta_{(n_1^*,0)2})\right]. \end{aligned} \quad (3.21)$$

By Eq. (3.21)  $v_{(n_1^*,0)}^{(2)}$  is a beam originating near the first molecule, and containing a contribution whose source is near the second molecule if the second one lies in the first beam. The function  $v_{(n_1^*,n_2^*)}^{(2)}$  satisfies

$$\left[ i\hbar \frac{\partial}{\partial t} - H_S \right] v_{(n_1^*,n_2^*)}^{(2)}(\mathbf{x}, t) = v_{(n_1^*,0)}^{(1)}(\mathbf{x}, t) \langle n_2^*|V_2|0\rangle \exp\left[i\frac{N}{2}(\beta_{(n_1^*,0)2} - \beta_{(n_1^*,n_2^*)2})\right], \quad (3.22)$$

from which the function  $v_{(n_1^*,n_2^*)}^{(2)}$  is nonzero only when the two molecules lie nearly on straight lines having direction  $\mathbf{p}$ . It should be noted that the term with the subscript  $(n_1^*,n_2^*)$  represents two ionized molecules. The functions  $v_{(0,0)}^{(2)}$  and  $v_{(0,n_2^*)}^{(2)}$  have similar shapes to those of  $v^{(1)}$ .

In the second approximation, we have

$$\tilde{\Psi}^{(2)}(t) = \sum_n v_{n\alpha_n}^{(2)}(\mathbf{x}, t) |n\rangle |\alpha_n\rangle, \quad (3.23a)$$

$$\begin{aligned} \Psi^{(2)}(t) = \sum_n v_{n\alpha_n}^{(2)}(\mathbf{x}, t) e^{iN/2\beta_{n2}} |n\rangle \\ \times \exp\left[-\frac{i}{\hbar}H_A(t-t_0)\right] |0,0\rangle. \end{aligned} \quad (3.23b)$$

Similarly, in the  $k$ th-order perturbation ( $k \geq 3$ ),

$$\tilde{\Psi}^{(k)}(t) = \sum_n v_{n\alpha_n}^{(k)}(\mathbf{x}, t) |n\rangle |\alpha_n\rangle, \quad (3.24a)$$

$$\begin{aligned} \Psi^{(k)}(t) = \sum_n v_{n\alpha_n}^{(k)}(\mathbf{x}, t) \exp\left[i\frac{N}{2}\beta_{n2}\right] |n\rangle \\ \times \exp\left[-\frac{i}{\hbar}H_A(t-t_0)\right] |0,0\rangle. \end{aligned} \quad (3.24b)$$

The functions  $v_{n\alpha_n}^{(k)}$  have the same properties as those of  $v_{n\alpha_n}^{(2)}$ . Consequently, we arrive at

$$\begin{aligned} \Psi(t) = \sum_n v_{n\alpha_n}(\mathbf{x}, t) \exp\left[i\frac{N}{2}\beta_{n2}\right] |n\rangle \\ \times \exp\left[-\frac{i}{\hbar}H_A(t-t_0)\right] |0,0\rangle, \end{aligned} \quad (3.25)$$

where  $v_{n\alpha_n} = \sum_{k=0}^{\infty} v_{n\alpha_n}^{(k)}$ ,  $v_{n\alpha_n}^{(0)} = \varphi(t)\delta_{n0}$ , and  $v_{(n_1^*,n_2^*)}^{(1)} = 0$ .

In all higher-order perturbations also, two molecules can be ionized only if the line joining them is practically parallel to the direction of motion of the incoming parti-

cle. This effect only appears in the second and higher approximations. Equation (3.24) shows that all states  $|n\rangle$  of the molecules couple with the same GC state (i.e., the initial state  $|0,0\rangle$ ), and as a result the detectors cannot detect each state of the molecules.

### C. Statistical operator

If the interaction  $V$  can never be neglected, then the initial state of the detectors remains unchanged. Thus we assume that the interaction  $V$  is a short-range one so that at a later time  $\tau_0$  the interaction is negligibly small. In this situation, Eq. (3.25) gives us the total state at time  $\tau_0$ :

$$\begin{aligned} \Psi(\tau_0) &= \sum_n v_{n\alpha n}(\mathbf{x}, \tau_0) \exp\left[i\frac{N}{2}\beta_{n2}\right] |n\rangle \\ &\times \exp\left[-\frac{i}{\hbar}H_A(\tau_0 - t_0)\right] |0,0\rangle, \end{aligned} \quad (3.26)$$

where we have suppressed the  $\tau_0$  dependence of the subscript of  $v_{n\alpha n}$ . Since  $V=0$  ( $\tau > \tau_0$ ), the time-evolution operator becomes

$$U(\tau, \tau_0) = U_0^S(\tau, \tau_0) U_0^{MA}(\tau, \tau_0),$$

from which we find

$$\begin{aligned} \Psi(\tau) &= \sum_n v_{n\alpha_n\beta_n}(\mathbf{x}, \tau) |n\rangle \exp\left[i\frac{N}{2}\beta_{n2}\right] \\ &\times \exp\left[-\frac{i}{\hbar}(E_n + H_A + H_{MA}^n)(\tau - \tau_0)\right] \\ &\times \exp\left[-\frac{i}{\hbar}H_A(\tau_0 - t_0)\right] |0,0\rangle, \end{aligned} \quad (3.27)$$

where  $v_{n\alpha_n}(\mathbf{x}, \tau) = U_0^S(\tau, \tau_0) v_{n\alpha_n}(\mathbf{x}, \tau_0)$ . In this case each energy state of the molecules correlates with each GC state labeled by  $n$ .

Let us obtain a GC state  $|\xi_n, \eta_n\rangle$  corresponding to a state  $|n\rangle$  of the molecules. Using Eqs. (2.11) and (3.2), we find

$$\begin{aligned} \exp\left[-\frac{i}{\hbar}H_A(\tau_0 - t_0)\right] |0,0\rangle \\ = \exp\left[-i\frac{N\omega l}{2}(\tau_0 - t_0)\right] |0,0\rangle. \end{aligned}$$

Equations (2.2b) and (2.5) give the GC state

$$\begin{aligned} \exp\left[-\frac{i}{\hbar}(H_A + H_{MA}^n)(\tau - \tau_0)\right] |0,0\rangle \\ = D_N(e^{\theta_1\rho_1 + \theta_3\rho_3}) |0,0\rangle \\ = |\xi_n, \eta_n\rangle, \end{aligned} \quad (3.28)$$

where  $\theta_1 = gE_n(\tau - \tau_0)$  and  $\theta_3 = -\omega l(\tau - \tau_0)$ . Setting  $\theta^2 = \theta_1^2 - \theta_3^2$ , according to  $\theta^2 > 0$  ( $g^2E_n^2 > \omega^2l^2$ ),  $\theta^2 < 0$  ( $g^2E_n^2 < \omega^2l^2$ ),  $\theta^2 = 0$  ( $g^2E_n^2 = \omega^2l^2$ ) we have three types of the corresponding GC states. When  $\theta^2 > 0$  ( $\theta \equiv \sqrt{\theta^2}$ ),

Eq. (3.28) becomes

$$\begin{aligned} \exp\left[-\frac{i}{\hbar}(H_A + H_{MA}^n)(\tau - \tau_0)\right] |0,0\rangle \\ = D_N(e^{\xi_n^1\rho_-} e^{\eta_n^1\sigma_3} e^{\xi_n^1\rho_+}) |0,0\rangle \\ \equiv |\xi_n^1, \eta_n^1\rangle, \end{aligned} \quad (3.29)$$

with

$$e^{\eta_n^1} = \cosh\theta - i\frac{\theta_3}{\theta}\sinh\theta, \quad \xi_n^1 e^{\eta_n^1} = \frac{\theta_1}{\theta}\sinh\theta. \quad (3.30)$$

In Eqs. (3.29) and (3.30), we have used Eq. (2.13) and

$$\begin{aligned} \exp(\theta_1\rho_1 + \theta_3\rho_3) \\ = \begin{bmatrix} \cosh\theta - i(\theta_3/\theta)\sinh\theta & (\theta_1/\theta)\sinh\theta \\ (\theta_1/\theta)\sinh\theta & \cosh\theta + i(\theta_3/\theta)\sinh\theta \end{bmatrix}. \end{aligned} \quad (3.31)$$

When  $\theta^2 < 0$  ( $\phi \equiv \sqrt{-\theta^2}$ ), Eq. (3.28) reduces to  $|\xi_n^2, \eta_n^2\rangle$ , where

$$e^{\eta_n^2} = \cos\phi - i\frac{\theta_3}{\phi}\sin\phi, \quad \xi_n^2 e^{\eta_n^2} = \frac{\theta_1}{\phi}\sin\phi. \quad (3.32)$$

Similarly in the last case  $\theta^2 = 0$ , the corresponding GC state is  $|\xi_n^3, \eta_n^3\rangle$ , where

$$e^{\eta_n^3} = 1 - i\theta_3, \quad \xi_n^3 e^{\eta_n^3} = \theta_1. \quad (3.33)$$

The total state at time  $\tau$  now becomes

$$\Psi(\tau) = \sum_n w_n(\mathbf{x}, \tau) |n\rangle |\xi_n, \eta_n\rangle, \quad (3.34)$$

where

$$\begin{aligned} w_n(\mathbf{x}, \tau) &= v_{n\alpha_n}(\mathbf{x}, \tau) \exp\left[i\frac{N}{2}\beta_{n2}\right] \\ &\times \exp[-iE_n(\tau - \tau_0)/\hbar] \\ &\times \exp[-iN\omega l(\tau_0 - t_0)/2], \end{aligned}$$

and  $\xi_n$  and  $\eta_n$  are given by Eqs. (3.30), (3.32), or (3.33). The correspondence between the states of the molecules and the detectors are thus as follows:

$$|n\rangle \rightarrow |\xi_n, \eta_n\rangle. \quad (3.35)$$

The statistical operator  $\rho$  corresponding to the state  $\Psi(\tau)$  is written in the form

$$\rho = \rho_0 + \sum_{n \neq m} w_n(\mathbf{x}, \tau) w_m^*(\mathbf{x}, \tau) |n\rangle \langle m| \otimes |\xi_n, \eta_n\rangle \langle \xi_m, \eta_m|, \quad (3.36)$$

where  $\rho_0$  is the mixed-state statistical operator given by

$$\rho_0 = \sum_n |w_n|^2 |n\rangle \langle n| \otimes |\xi_n, \eta_n\rangle \langle \xi_n, \eta_n|. \quad (3.37)$$

Consider the expectation value of any observable  $O$  with respect to the molecules and any classical operator  $A$



with respect to the detectors:

$$\begin{aligned} \text{Tr}[\rho(O \otimes A)] &= \text{Tr}[\rho_0(O \otimes A)] \\ &+ \sum_{n \neq m} w_n w_m^* \langle m | O | n \rangle \langle \xi_m, \eta_m | A | \xi_n, \eta_n \rangle. \end{aligned} \quad (3.38)$$

When  $N$  is sufficiently large, using Eqs. (2.17), (2.18), and (2.24), we find

$$\begin{aligned} &|\langle \xi_m, \eta_m | A | \xi_n, \eta_n \rangle|^2 \\ &\simeq \text{const} \times |\langle \xi_m, \eta_m | \xi_n, \eta_n \rangle|^2 \\ &= \text{const} \times \left[ 1 + \frac{|\xi_m - \xi_n|^2}{(1 - |\xi_m|^2)(1 - |\xi_n|^2)} \right]^{-N/2}. \end{aligned} \quad (3.39)$$

Hence for operators such as  $O \otimes A$  ( $A \in K$ ), the difference between the expectation values for  $\rho$  and  $\rho_0$  is very small when  $N \gg 1$ . In this sense, the operator  $\rho$  is approximately the mixed-state statistical operator, so that the probability interpretation may be introduced into our model.

#### D. Stability of particle trajectories

The incoming particle interacts with the molecule-detector system, after which the operator  $\rho$  of the total system reduces, in the above sense, to the mixed-state statistical operator. Then, with a certain probability, the total state  $\Psi(\tau)$  changes, for example, into a state  $w_n(\mathbf{x}, \tau) | n \rangle | \xi_n, \eta_n \rangle$ , where  $w_n$  is a beam originating near the molecules and having the direction  $\mathbf{p}$ . When  $w_n = w_{(n_1^*, n_2^*)}$  ( $n_i^* \neq 0$ ), the two (or in general  $l$ ) molecules are ionized; this occurs if the line joining the two molecules is parallel to the direction  $\mathbf{p}$  of the incoming particle. When the two molecules do not lie on a line parallel to the direction  $\mathbf{p}$ , only one of them can be ionized. It has therefore been proved that the ionized molecules will be practically on a straight line.

Excited states of molecules are in general unstable, so that the molecules revert to the ground state almost instantaneously. However, in GC-state formalism the situation is quite different. Equations (3.30), (3.32), and (3.33) show that if  $E_n \neq E_m$ , then  $\xi_n \neq \xi_m$ , where we assume for simplicity that the energy states of the molecules are non-degenerate. Taking into account Eq. (2.16), one gets  $\langle \xi_m, \eta_m | \xi_n, \eta_n \rangle \simeq 0$  if  $E_n \neq E_m$ . Therefore we find

$$\begin{aligned} &\langle m | O | n \rangle \langle \xi_m, \eta_m | A | \xi_n, \eta_n \rangle \\ &\simeq \text{const} \times \langle m | O | n \rangle \langle \xi_m, \eta_m | \xi_n, \eta_n \rangle \simeq 0, \end{aligned} \quad (3.40)$$

when  $N \gg 1$  and  $E_n \neq E_m$ . Because of this fact the states of all molecule-detector systems are almost stable. Each quantum state of the molecules associated with the corresponding GC state behaves like a macroscopic one. The trajectory of the incoming particle, which is made up of the ionized molecules, is thus also stable.

#### IV. CONCLUSIONS

We have considered a model consisting of an incoming particle, many molecules interacting with the particle simultaneously, and detectors. We have used the coherence group  $SU(1,1)$  to construct the states of the detectors. Because of this our calculations became much simpler. The excited states of the molecules, which are considered to be ionized, are registered by the detectors through their interactions.

Treating the interaction  $V$  between the particle and the molecules as a perturbation, we have shown that all molecules can be ionized only when the line joining them is practically parallel to the motion of the particle. That is, the functions  $v_{n\alpha_n}$  in Eq. (3.25) and  $w_n$  in Eq. (3.34) are beams originating near the molecules and have the above properties. In particular, there is no possibility that two molecules which lie on different parallel lines are ionized.

If the interaction  $V$  is a long-range one, the initial state of the detectors remains unchanged, and thus they cannot register the states of the molecules. We assumed therefore that the interaction is a short-range one. Then the states of the molecules and detectors correlate with each other, as is confirmed in Eqs. (3.34) and (3.35).

The statistical operator  $\rho$  constructed by the state  $\Psi(\tau)$  [see Eq. (3.34)] is pure. However, if we assume that it is very difficult to "detect" nonclassical operators by means of another detector, i.e., we can treat only classical operators, then it follows that

$$\text{Tr}[\rho(O \otimes A)] \simeq \text{Tr}[\rho_0(O \otimes A)]$$

when  $A \in K$  and  $N \gg 1$ . In this situation we may introduce the notion of "approximate" probability into our model, which becomes more correct as  $N \rightarrow \infty$ . The total state  $\Psi$  changes into a state  $w_n | n \rangle | \xi_n, \eta_n \rangle$  with a certain probability, and then we can discuss particle trajectories.

Introducing the notion of equivalent states (or statistical operators) Jauch has studied the problem of quantum measurement.<sup>8</sup> The two states  $\rho_1$  and  $\rho_2$  are called equivalent with respect to the set of observables  $\mathcal{S}$  of a physical system if  $\text{Tr}(\rho_1 A) = \text{Tr}(\rho_2 A)$  for all  $A \in \mathcal{S}$ . Such  $\rho_i$  are written as  $\rho_1 \sim \rho_2$ . The relation  $\rho_1 \sim \rho_2$  means that the two states cannot be distinguished by any measurement whatever with observables from the set  $\mathcal{S}$ . In our GC-state formalism also, we can define such equivalent states:  $\rho_1 \sim \rho_2$  if

$$\text{Tr}[(\rho_1 - \rho_2)(O \otimes A)] \rightarrow 0 \quad (N \rightarrow \infty)$$

for all  $A \in K$ . These equivalent states will be discussed elsewhere.

We have also shown for the case of classical operators that the trajectory of the particle made up of the ionized molecules is stable. Excited states of the molecules associated with GC states are almost stable; a single molecule with the appropriate environment cannot revert to its ground state.

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