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# Non-Painlevé reductions of nonlinear Schrödinger equations

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The nonlinear Schrödinger equation $i\psi_t+\Delta\psi=c\psi|\psi|^{2\sigma}$ with boundary conditions imposed on a cylinder or on a sphere is reduced to an ordinary differential equation, namely, the Abel equation. This can be used to investigate particular singular solutions in self-trapping and self-focusing theories.

Two recent articles[1,2] were devoted to exact analytic solutions of the three-dimensional Schrödinger equation with a polynomial (cubic or quintic) nonlinearity. In particular, solutions were found corresponding to a cylindrical[1] or a spherical[2] geometry. Lie group theory was used to reduce the original equation to a variety of different ordinary differential equations. Those that have the Painlevé property[3,4] (no movable singularities other than poles) were identified and integrated in terms of elementary functions, Jacobi elliptic functions, or Painlevé transcendents. The procedure used picks out "well-behaved" solutions and ignores those that may have essential singularities or logarithmic branch points at finite real (and in general complex) values of the independent variables, the position of which depends on the initial, or boundary conditions.

Solutions that have singularities at points, on lines, or even on surfaces, have very important physical applications. In such situations there is no *a priori* reason to require the singularities to be poles, or to require that solutions be single-valued singularity surfaces. Other types of singularities are of equal interest and this provides the motivation for the present Brief Report. Its purpose is to apply group theory to obtain information about non-Painlevé type solutions of the equation

$$i\psi_t+\Delta\psi=c\psi|\psi|^{2\sigma}\quad c,\sigma\in\mathbb{R} \tag{1}$$

in cylindrical and spherical coordinates.

This equation is a basic evolution model for nonlinear waves in various branches of physics. It is a generic equation describing the slowly varying envelope wave train in conservative dispersive systems. For $c>0$ and $\{\sigma<2/(D-2), D>2; \sigma<\infty, D\le 2\}$, where $D$ is the number of spatial dimensions, it has solitary wave solutions that are stable for $\sigma<2/D$. For $c>0$ and $\sigma\ge 2/D$, solutions of Eq. (1) can have absolute value $|\psi|$ that diverges at a finite time $t$. Solutions that do not diverge or diffract are associated with the typical self-trapping phenomenon,[5] while others are associated with the self-focusing one.[6-8] In nonlinear optics, for instance, we have $\sigma=1$ and the self-trapping solution is stable for $D=1$ (soliton solution of the cubic nonlinear Schrödinger equation) but diverges for $D=2$. More information on the physical origin, stability, and local structure of the divergence can be found in Refs. 7 and 8.

Let us first consider a cylindrical geometry, i.e., boundary conditions given on a cylinder. We put[1]

$$\psi=M(\rho)\exp\{i[\chi(\rho)+a\theta-bt]\}\ , \tag{2}$$

where $x=\rho\cos\theta$, $y=\rho\sin\theta$, $a,b,M(\rho),\chi(\rho)\in\mathbb{R}$. Equation (1) then reduces to

$$\chi=S_0\int d\rho\frac{1}{\rho M^2}+\chi_0\ , \tag{3}$$

$$\ddot{M}+\rho^{-1}\dot{M}-S_0^2\rho^{-2}M^{-3}-a^2\rho^{-2}M+bM-cM^{2\sigma+1}=0\ , \tag{4}$$

where $S_0$ and $\chi_0$ are integration constants. [Note that in addition to the cases analyzed in Ref. 1, Eq. (4) is also of the Painlevé type for $\{a^2=\frac{4}{25}, b=S_0=0, \sigma=\frac{1}{2}\}$ which provides the solution

$$M=\frac{6}{c}k^2\rho^{-2/5}\wp(\eta;0;g_2)\ , \tag{5}$$

$$\eta=\tfrac{5}{4}k\rho^{4/5}+\rho_0\ , \tag{6}$$

where $k$, $\rho_0$, and $g_2$ are constants and $\wp(\cdots)$ is the Weierstrass elliptic function.]

Similarly, in the case of a spherical geometry, we put[2]

$$\psi=M(r)\exp\{i[\chi(r)-bt]\}\ , \tag{7}$$

where $r^2=x^2+y^2+z^2$ and obtain

$$\chi=S_0\int dr\frac{1}{r^2M^2}+\chi_0\ , \tag{8}$$

$$\ddot{M}+2r^{-1}\dot{M}-S_0^2r^{-4}M^{-3}+bM-cM^{2\sigma+1}=0\ . \tag{9}$$

Equations (4) and (9) are invariant, whether of the Painlevé type or not, under a simultaneous dilation of the dependent and independent variables if $b=S_0=0$. This symmetry is generated by the differential operator

 

$$D=\xi\partial_\xi-\sigma^{-1}M\partial_M \ , \tag{10}$$

where $\xi=\rho$ or $\xi=r$, respectively.

The restriction to $b=0$ leads to a particular subset of stationary cylindrical and spherical wave solutions for $\psi$. As such, their physical interpretation is more appropriate within self-trapping models (for which $|\psi|$ is independent of $t$). Singularities of $M(\xi)$ for $\xi=\rho=\rho_0$, or $\xi=r=r_0$, correspond to singularity surfaces for the solution $\psi$ of the original equation (1). In optics such singularities at $\rho_0=0$, for instance, are typical for filamentation processes[9] (collapse into a line singularity). On the other hand, one can use symmetries of Eq. (1) to generate other solutions. For example, stationary waves can be transformed into traveling ones by using Galilean symmetries of Eq. (1). A conformal symmetry, which exists[7] only for $\sigma=2/D$, can be used to generate singular solutions at some fixed time $t=t_0$ that indeed correspond, when no other singularities exist in $\rho$ and $r$, to a typical self-focusing phenomenon.[6,10]

Setting $S_0=b=0$ in both cases, we use this residual dilational symmetry in a standard manner[11] to reduce the order of the equations. Indeed, we transform from $(\xi,M(\xi))$, to $(y,w(y))$ given by

$$y=\xi M^\sigma, \quad w=\sigma^{-1}\ln(\xi) \ , \tag{11}$$

and transform Eq. (4), or (9), for $M(\xi)$ into an equation for $w(y)$, namely

$$z_y=[1+\sigma(1-k)-\sigma^2a^2-\sigma^2cy^2]\sigma yz^3 -[2+(1-k)\sigma]z^2+\frac{1-\sigma}{\sigma y}z, \quad z=w_y \tag{12a}$$

$$k=1 \quad \text{for } \xi=\rho \ , \qquad a=0, \quad k=2 \quad \text{for } \xi=r \ . \tag{12b}$$

Equation (12) is an Abel equation of the first kind. A sizable literature exists on this equation; see Kamke[12] or Murphy[13] for brief reviews and references to the original literature. Being a first-order equation, Eq. (12) is convenient for qualitative studies of the behavior of solutions. Once a solution $z=z(y,C_1)$ is obtained, $w(y,C_1,C_2)$ is calculated by a quadrature and $M=M(\xi)$ is found by solving an algebraic equation

$$w(M^\sigma,C_1,C_2)=\sigma^{-1}\ln(\xi) \ , \tag{13}$$

where $C_i$ are integration constants.

We use this opportunity to correct an oversight in Ref. 1, where an Abel equation equivalent to (9) was misidentified as a Riccati equation and treated as such. Solutions of Abel equations, as opposed to Riccati ones, may have movable critical points, not only movable poles, so the difference is an important one.

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