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Scalar wave collapse at critical dimension

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The collapse of wave packets governed by the Zakharov equations is investigated at critical dimension. Their classical self-similar solutions are described by a linear time-dependent contraction scale $\xi(t)=V(t_*-t)$ where t_* denotes the collapse time. We study two spherically symmetric versions of self-similar collapses, namely one corresponding to a vectorial electric field and another one relative to a scalar modelization of the Langmuir field. Each of these solutions can be regarded as a function of the collapse velocity $V=-\dot{\xi}$. In the case of a *vectorial* electric field, the solutions are analytically shown to exist in the subsonic regime only, provided that the velocity V is lower than a critical velocity $V_{\rm crit}$, in agreement with previous numerical results. By contrast, the solutions of the *scalar* model are found to exist for every value of the collapse velocity; they exhibit two localized modes that evolve continuously as a function of V from the subsonic to the supersonic regime. These two modes are analytically and numerically shown to merge together in the limit $V\rightarrow\infty$.

I. INTRODUCTION

The Zakharov equations,[1] originally introduced in 1972, describe the nonlinear evolution of plasma waves and are still the subject of intensive investigations in the scope of the strong Langmuir turbulence. Their solutions are shown to collapse toward a point singularity whenever the energy of the electrostatic plasma waves exceeds a threshold value. The latter property is true provided the space dimension d is larger than the so-called critical dimension, denoted as d_c. Classical investigations of this problem[2–5] consist in understanding the behavior of one elementary turbulent cell—one collapsing caviton—assuming that a complete scenario of the collapse process in a strongly turbulent plasma can be correctly described, at the lowest level of approximation, as a set of independent collapsing cavitons.

There is also a large body of numerical work related to this question, which can be subdivided into two main groups.[6–12] On the one hand, Refs. 6–11 are devoted to numerical analysis of the two-dimensional (2D) damped and driven Zakharov equations: in this case one observes a steady-state turbulence that consists of cavitons, the latter being created at random, due to the energy drive, and disappearing by the Landau absorption. On the other hand, the nonlinear evolution of one given caviton is investigated in terms of an initial-value problem in Refs. 6, 7, 9, and 12. In these simulations, three stages are clearly seen during the caviton lifetime: after an initial transient period that depends upon the initial conditions, a collapsing behavior is observed in the so-called inertial regime where the dissipation is negligible; moreover, it has been observed that this collapsing behavior develops in a self-similar manner; finally the caviton burns out, due to Landau damping, when its characteristic size becomes of the order of a few Debye lengths.

A collapsing caviton is thus assumed to be correctly described by the self-similar part of solutions that obey the following set of undamped and undriven Zakharov equations, expressed in convenient units as

$$\nabla^2(i\Phi_t+\nabla^2\Phi)=\mathrm{div}(\delta n\,\nabla\Phi)\ , \tag{1a}$$

$$(\partial_t^2-c^2\Delta)\delta n=c^2\Delta|\nabla\Phi|^{2m}\ , \tag{1b}$$

and, under the supersonic assumption, namely, $c^2\Delta\delta n\ll\partial_t^2\delta n$, are known[2,6] to depend on the scaling factor:

$$\xi(t)\propto(t_*-t)^{d_c/d}\ , \tag{2}$$

where t_* here denotes the moment of collapse and where the critical dimension d_c is given by $d_c=2/m$.

Indeed, defining now the caviton contraction velocity as

$$V\sim|\partial_t\delta n/\nabla\delta n|$$

one easily finds that the velocity of a self-similar contracting caviton is given by

$$V\sim|\dot{\xi}|\sim|(t_*-t)|^{(d/d_c-1)}$$

so that the supersonic approximation can be asymptotically satisfied for $t \rightarrow t_*$ in the case $d_c < d$ only, that is when the space dimension exceeds the critical value d_c, as just said above.

In Eqs. (1), $\Phi(\mathbf{x},t)$ corresponds to the electrostatic potential of a plasmon envelope coupled to the low-frequency density fluctuations represented by $\delta n(\mathbf{x},t)$. In the physical case, one has $m=1$ and the right-hand side (rhs) of Eq. (1b) represents then the ponderomotive force of the plasma waves acting on the electrons; this ponderomotive force has been here extended to an arbitrary degree of nonlinearity characterized by the exponent m; the parameter c in Eq. (1b) corresponds to the ion-sound speed.

In this paper we investigate two scalar versions of Eqs. (1) in the case where the space dimension d is equal to the critical dimension d_c. In this special case, the time dependence of the scaling factor $\xi(t)$ is linear, according to Eq. (2), corresponding thus to a velocity V which is time independent: in this regard, such a case is marginal in the sense that the operators ∂_t^2 and $c^2\Delta$ in Eq. (1b) are of the same order, so that neither the supersonic nor the subsonic approximations can be imposed. As originally introduced for a vectorial two-dimensional collapse by Gol'tsman and Fraiman in Ref. 3, a convenient way to deal with the critical collapse consists in defining the scaling factor (2) in the following form:

$$\xi(t)=V(t_*-t) \ , \tag{3}$$

where the time-independent parameter $V=-d\xi/dt$ denotes the collapse velocity. This quantity V is a free parameter and the domain of existence of localized solutions for the set (1) will be investigated as a function of V, or equivalently as a function of the ratio $\Lambda=c/V$. Since localized solutions will be shown to exist for a continuous and connected domain of V, we have found it necessary to generalize somewhat the usual terminology. Indeed, in the standard references on this subject, the supersonic regime is commonly thought as corresponding to the limit $V\rightarrow\infty$ ($\Lambda=0$), and similarly the subsonic regime to the limit $V=0$ ($\Lambda\rightarrow\infty$), that is to the nonlinear Schrödinger equation, for which one has $\delta n=-|E|^{2m}$. In our paper the supersonic regime relates more generally to the domain $V \gg c$ ($\Lambda \ll 1$) and the subsonic regime to the domain $V \ll c$ ($\Lambda \gg 1$). On the other hand, the terminology "strongly supersonic" will be used when referring to the limit $V\rightarrow\infty$ ($\Lambda=0$); similarly "strongly subsonic" ($V=0, \Lambda=\infty$) will refer to the nonlinear Schrödinger equation.

In order to study a critical self-similar collapse, we first recall in Sec. II the set of self-similar substitutions which reduce the Zakharov equations to a system of nonlinear ordinary differential equations which only contains derivatives with respect to the transformed coordinates $\mathbf{x}'=\mathbf{x}/\xi(t)$. We then discuss the behavior of the solutions at the vicinity of the sonic point, defined as the place where one has $\partial_t^2\delta n=c^2\Delta\delta n$ in the ion-sound equation (1b) and we prove that such substitutions continuously go through this sonic point under certain conditions on the density potential $\delta n(\mathbf{x},t)$. On the other hand, we study within the same framework the so-called vectorial case and scalar model of a self-similar collapse in order to explain the apparent paradox between the Gol'tsman and Fraiman results[2] on the axisymmetric collapse corresponding to a vectorial electric field, and the Zakharov and Shur ones[2] relative to a spherically symmetric collapse in which the vectorial nature of the electrostatic field is disregarded (corresponding to what is referred to in our article as the scalar model): in the case of a vectorial electric field, the collapsing solutions have been found to be always subsonic, whereas in the scalar model of a spherically symmetric collapse there exist localized solutions defined in the supersonic regime. In order to investigate this paradox, we develop a least action principle by using 2D soliton-type trial functions, whose aim consists in anticipating analytically the realization domain of two distinct localized modes, previously observed in numerous works.[2,3,5] At the opposite of the vectorial collapsing solutions that may be self-similarly realized for V below a critical subsonic value $V_{\rm crit}$ only, with $V_{\rm crit}=0.14c$,[3] we show that scalar modes of a two-dimensional collapse can be self-similarly realized with subsonic and supersonic velocities as well, for every value of V from zero to infinity. For increasing V, the second mode converges to the first one until both reach the collapse singularity for the supersonic velocity $V=\infty$, in which case the maximum of fields is located at the center of the caviton.

In Sec. III we perform a numerical integration of the Zakharov equations transformed under self-similar substitutions, which enables us to illustrate and confirm all the theoretical results briefly stated above. As a result, we show in Sec. IV that a one-dimensional scalar collapse simulates with a great accuracy the problem of self-similar solutions of a real 2D Langmuir collapse.

II. SELF-SIMILARITY ANALYSIS

Because it is difficult to analyze asymmetric supersonic self-similar distributions associated with the partial differential problem (1), we discuss the self-similar regimes of a wave collapse which is described by the following scalar equation set:

$$i\partial_t E+\Delta E-\frac{d_c-1}{r^2}\epsilon E=\delta n E \ , \tag{4a}$$

$$(\partial_t^2-c^2\Delta)\delta n=c^2\Delta|E|^{2m} \ . \tag{4b}$$

In the latter set, the Laplacian operator reads as follows:

$$\Delta=\frac{1}{r^{d_c-1}}\frac{\partial}{\partial r}r^{d_c-1}\frac{\partial}{\partial r} \ , \tag{5}$$

for an axisymmetric collapse expressed in spherical geometry, with $r\equiv|\mathbf{x}|$. The additional parameter ϵ has been introduced to distinguish a wave collapse corresponding to a vectorial electric field, from the one corresponding to a scalar modelization of the electric field.

(i) In the case of a centrally symmetric *vectorial* field (see Refs. 1, 3, and 5), one has $\mathbf{E}(\mathbf{x})=-\nabla\Phi(r)\equiv E(r)(\mathbf{x}/r)$ with $r\equiv|\mathbf{x}|$, where Φ depends on r only, in which case one has $\nabla(\mathrm{div}\mathbf{E})=[\Delta-(d_c-1)/r^2]\mathbf{E}$, corresponding to the case $\epsilon=1$ in Eq. (4a).

(ii) In the case of a *scalar* modelization of the electric field, one simply replaces in the Zakharov equation set (1) $-\nabla\Phi$ by a *scalar* function $E(r)$, and similarly $\nabla\cdot\nabla(-\Phi)$ by $\nabla^2 E$, corresponding thus to $\epsilon=0$ in Eq. (4a); this modelization has been widely used in the Soviet literature (see, e.g., Refs. 2–13), although it does not receive any rigorous justification (it will be simply shown further on that the scalar model corresponds to the *same* Lagrangian density as for the exact vectorial case, in which one assumes the electric field to be scalar).

A. General properties of a self-similar collapse

We denote by t_* the finite time at which the amplitudes of E and δn go to infinity. The latter divergence for $t\to t_*$ is included in the rescaling factors in front of the self-similar quantities

$$E(r,t)=\frac{1}{\xi(t)^{d_c/2}}E'(r',V)\exp\left[+\frac{i}{2}\left[2\Lambda^2 t'+\frac{\dot{\xi}}{\xi}r^2\right]\right], \tag{6a}$$

$$\delta n(r,t)=\frac{1}{\xi^2(t)}\delta n'(r',V), \tag{6b}$$

for which we define the new variables r' and t' by

$$t\to t'=\int_0^t\frac{d\tau}{\xi^2(\tau)}, \tag{7a}$$

$$r\to r'=|\mathbf{x}'|,\quad \text{with } \mathbf{x}'\equiv\frac{\mathbf{x}}{\xi(t)}. \tag{7b}$$

It can be checked that the scaling law (3) ensures $t'\to\infty$ as $t\to t_*$. In this limit, self-similar solutions E' and $\delta n'$, defined by (6), correspond to the asymptotic state of the transformed problem (4) expressed in the new system of coordinates (7). Inserting expressions (6) and (7) into (4) leads to the following set of ordinary differential equations for the unknown functions E' and $\delta n'$;

$$-\Lambda^2E'+\Delta'E'-\frac{d_c-1}{(r')^2}\epsilon E'-\delta n'E'=0, \tag{8a}$$

$$V^2\frac{1}{r'}\frac{d^2}{dr'^2}[(r')^3\delta n']-c^2\Delta'\delta n'=c^2\Delta'|E'|^{2m}, \tag{8b}$$

with

$$\Delta'=\frac{1}{(r')^{d_c-1}}\frac{d}{dr'}(r')^{d_c-1}\frac{d}{dr'}. \tag{8c}$$

As already known,[3,4] such a system admits localized solutions under the boundary conditions:

$$E',\delta n'\to 0 \quad \text{as } r'\to\infty, \tag{9a}$$

$$E'(r'=0)=0 \quad \text{for the vectorial case}, \tag{9b}$$

$$\left[\frac{\partial}{\partial r'}E'\right](r'=0)=0 \quad \text{for the scalar case}. \tag{9c}$$

We also note that Eq. (8a) is invariant under the substitutions

$$r'\to\mu r',\quad \delta n'\to\mu^{-2}\delta n',\quad \Lambda\to\mu^{-1}\Lambda, \tag{10}$$

so that, without loss of generality, we can assume $\Lambda^2=1$ in Eq. (8a) by setting $\mu^2\Lambda^2=1$; this single eigenvalue makes the isospectral Schrödinger problem (8a) highly degenerated, which implies the possible existence of several localized solutions defined under the boundary conditions (9). After performing the transformations (10) together with $E'\to\mu^{1/m}E'$ in (8b) and taking $\mu=V/c$, we finally obtain an equivalent form of Eqs. (8):

$$\left[-1+\Delta'-\frac{d_c-1}{(r')^2}\epsilon\right]E'=\delta n'E', \tag{11a}$$

$$\frac{1}{r'}\frac{d^2}{dr'^2}[(r')^3\delta n']-\Lambda^2\Delta'\delta n'=\Delta'|E'|^{2m}, \tag{11b}$$

where Λ^2 is given by the simple relation

$$\Lambda^2=c^2/V^2. \tag{12}$$

The quantity Λ^2 appears thus to be an appropriate parameter to investigate self-similar collapses.

B. Strongly supersonic collapse

The limit $\Lambda^2=0$, that we refer to as "strongly supersonic," has already been intensively discussed by many authors,[1–6,12] so that we will simply recall briefly their results. In this limit $\Lambda^2=0$, Eq. (11b) reduces to

$$\frac{1}{r'}\frac{d^2}{dr'^2}[(r')^3\delta n']=\Delta'|E'|^{2m}. \tag{13}$$

Integrating Eq. (13), the expression for $\delta n'(r')$, which is regular at $r'=0$, can be written as

$$\delta n'(r',\infty)=\frac{\phi'-\phi_0'}{(r')^2}+\frac{d_c-3}{(r')^3}\int_0^{r'}(\phi'-\phi_0')dr', \tag{14a}$$

with $\phi'=|E'(r')|^{2m}$ and $\phi_0'=|E'(r'=0)|^{2m}$. From Eq. (14a) one realizes that in the degenerate limit $\Lambda=0$, ϕ_0' is the only relevant parameter regarding the nonlinear eigenvalue corresponding to the set (11).

1. *Large-r' behavior*

According to the classical procedure originally developed by Zakharov and Shur in Ref. 2, the self-similar solutions (6) become time independent asymptotically as $t'\to\infty$ and the function $\delta n'$ (13) is given by its asymptotic behavior as $r'=r/(t_*-t)$ tends to infinity for any finite point of space r when $t\to t_*$. So, we obtain for large r'

$$\delta n'(r')=-\frac{d_c-2}{(r')^2}\phi_0'+\cdots, \tag{14b}$$

where the remaining terms are exponentially small as $r'\to\infty$, due to the exponential decay to zero of ϕ' for $r'\to\infty$ [see Eq. (11a) with $\delta n'\to 0$ for $r'\to\infty$]. By using Eqs. (6b) and (7), one easily finds that expression (14b) for $\delta n'$ corresponds in real space to a potential well δn given by

$$\delta n(r)=-\frac{d_c-2}{r^2}\phi_0'. \tag{15}$$

From the latter expression, one may see that the self-similar solution for δn is positive for sufficiently large r provided the critical dimension is smaller than 2; such a behavior corresponds to the so-called "lips" of the caviton (see, e.g., Refs. 4 and 5).

2. Small-r' behavior

It will be seen further on that in the general case $\Lambda \neq 0$ the point $r'=\Lambda$ plays a particular role regarding the continuity of $\delta n'$: for reasons detailed later, this point may receive the name of "sonic" point. In the limit $\Lambda=0$ considered here, the sonic point reduces to the origin, and for further comparison we will restrict ourselves to the case $d_c=2$.

Let us consider first the scalar model $\epsilon=0$. Inserting $\delta n'=\delta n'_0+O((r')^2)$ and $E'=E'_0+O((r')^2)$ into Eqs. (11a) and (13), one obtains at the origin[2]

$$\delta n'_0=\left[\frac{3}{\phi'_0}-1\right]^{-1} . \tag{16}$$

A localized solution for $\delta n'$ with a negative minimum at $r'=0$ can therefore exist only if the condition

$$\phi'_0>3$$

is satisfied.

Regarding now the vectorial case $\epsilon=1$, let us show that there is no solution in the strongly supersonic regime $\Lambda=0$ considered here (more generally it will be found later on that solutions for the vectorial case may exist only if the velocity V is smaller than a critical velocity V_{crit}, which lies in the subsonic regime).

We already know that the boundary condition imposes $E'(r'=0)=0$; since the equation of evolution for E' is of second order, let us show that one also has $[(\partial/\partial r')E']_{r'=0}=0$, in which case the function $E'(r')$ is necessarily null everywhere. If we indeed had $[(\partial/\partial r')E']_{r'=0}\neq 0$, the expansion for E' in the vicinity of $r'=0$ would be $E'\cong\beta r'+\cdots$ (see, e.g., Refs. 6 and 12). By inserting this expansion into Eqs. (11b) and (13), $\delta n'(r'=0)$ would be found to be given by

$$\delta n'_0=2\frac{m^2-m+1}{m(2m+1)}\beta^{2m}(r')^{2(m-1)} , \tag{17}$$

which is a positive quantity for $m>0$. In particular, one would have $\delta n'_0=\frac{2}{3}\beta^2$ for the case $d_c=2$, $m=1$ considered here. On the other hand, substituting the same expansion for E' into Eq. (11a) yields $\delta n'_0=-1$, in obvious contradiction with Eq. (17). We therefore conclude that no solution can exist for the vectorial case in the strongly supersonic regime.

C. Trans-sonic problem

Let us now consider the general case $\Lambda^2\neq 0$. By integrating once Eq. (11b), one obtains the first-order differential equation

$$r'[(r')^2-\Lambda^2]\frac{\partial}{\partial r'}\delta n'+[3(r')^2-\Lambda^2(d_c-2)]\delta n' =r'\frac{\partial}{\partial r'}\phi'+(d_c-2)[\phi'-(\phi'_0+\Lambda^2\delta n'_0)] , \tag{18}$$

with $\delta n'_0=\delta n'(r')|_{r'=0}$. One can see on the latter expression that the points $r'=0$ and $r'=\Lambda$ play a particular role because they make the leading order in $(\partial/\partial r')\delta n'$ vanish. The point $r'=\Lambda$ will henceforth be referred to as the "sonic" point since it corresponds to the point where V is equal to the ion-sound velocity c in the prime dimensionless units.

Equation (18) is easily seen to impose a boundary condition for $\delta n'$ at the sonic point $r'=\Lambda$. Assuming the continuity and the derivability of $\delta n'$ everywhere, and taking the limit $r'\to\Lambda$, one indeed obtains

$$\delta n'(r'=\Lambda) =\left[\Lambda\left[\frac{\partial}{\partial r'}\phi'(r')\right]_{r'=\Lambda}+\chi\right]\Big/\Lambda^2(5-d_c) , \tag{19}$$

with

$$\chi=(d_c-2)(\phi'_s-\phi'_0-\Lambda^2\delta n'_0) ,$$

where ϕ'_s denotes the quantity $\phi'(r')$ at the sonic point,

$$\phi'_s\equiv\phi'(r'=\Lambda) .$$

One of the main difficulties when solving numerically the set (11) is therefore to determine self-consistently on the first hand the proper boundary value $|E'_0|\equiv|E'(r'=0)|$ so as to obtain localized solutions, and on the second hand the potential well $\delta n'_0\equiv\delta n'(r'=0)$ so as to fulfill the constraint (19).

D. Two mode solutions

Now, we use a variational method in order to estimate the range of collapse velocities in which localized solutions may exist and thus describe a self-similar critical collapse. To investigate this problem, as primarily exposed in Ref. 3, we confine ourselves to the physical case $m=1$ and $d_c=2$ and we first define the action integral for the set (4) as follows:

$$S=\int dt\int_0^\infty L r^{d_c-1}dr . \tag{20}$$

Here the Lagrangian density is written as

$$L=i\left[\mathbf{E}^*\frac{\partial}{\partial t}\mathbf{E}-\mathbf{E}\frac{\partial}{\partial t}\mathbf{E}^*\right]+|\operatorname{div}\mathbf{E}|^2+c^2\tfrac{1}{2}|\nabla U|^2 -\tfrac{1}{2}\left[|\mathbf{E}|^2+\frac{\partial U}{\partial t}\right]^2 , \tag{21}$$

where $U(r,t)$ is the hydrodynamical potential relative to the sound wave flow, which satisfies the following relation:

$$\frac{\partial U}{\partial t}=-|\mathbf{E}|^2-\delta n . \tag{22}$$

In the expression (21), $\mathbf{E}$ should read $\mathbf{E}=E(r)(\mathbf{x}/r)=-\nabla\Phi$ in the vectorial case, and the quantity $\operatorname{div}\mathbf{E}$ is then given by $\operatorname{div}\mathbf{E}=(1/r^{d_c-1})(\partial/\partial r)r^{d_c-1}E(r)$. By contrast, in the scalar modelization of the electric field, $\mathbf{E}$ is simply to be considered as a scalar function, and accordingly $\operatorname{div}\mathbf{E}$ reduces to $(\partial/\partial r)E(r)$. It can be checked by varying the action with respect to E^* that these two definitions of divE correctly reproduce Eq. (4a) with $\epsilon=0$ in the scalar model and $\epsilon=1$ in the vectorial case.

After writing the self-similar substitutions (6) and (7) into (21) and taking the self-similar transformation on $U(r,t)$ into account, by setting namely

$$U(r,t)=-\frac{\dot{\xi}}{\xi(t)}U'(r',t') , \tag{23}$$

the Lagrangian density (21) becomes, assuming a steady-state asymptotic limit as t' goes to infinity

$$L'=\left[\frac{dE'}{dr'}+\frac{E'}{r'}\right]^2+\left[1+\frac{\epsilon-1}{(r')^2}\right]|E'|^2+\tfrac{1}{2}c^2(\nabla'_r U')^2 -\frac{1}{2}\left[|E'|^2-V\left[r'\frac{\partial}{\partial r'}U'+U'\right]\right]^2 . \tag{24}$$

For the sake of clarity, we will use system (8) and the original Lagrangian density (24) to analyze the realization of localized solutions according to the collapse velocity, i.e., the value of V which is chosen here, as being the only relevant parameter. The interested reader will find in the Appendix a brief explanation of the connection between the Lagrangian density (24) and Eqs. (8).

By analogy with Langmuir 1D solitons,[3,14] we found it appropriate to use the following trial function:

$$E'(r',V)=\left[2\left[1-\frac{(R'_0)^2}{\Lambda^2}\right]\right]^{1/2}\operatorname{sech}(r'-R'_0) = \left[1-\frac{(R'_0)^2}{\Lambda^2}\right]^{1/2}f(r'-R'_0) , \tag{25}$$

which is moreover supposed to satisfy approximately system (8) under the following assumption:

$$(R'_0/\Lambda)<1 \Longrightarrow VR'_0<c . \tag{26}$$

Such a function describes a symmetrical cylindrical solitonlike solution characterized by a typical radius $r_0(t)=\xi(t)R'_0(V)$; here $R'_0(V)$ is a function of V which represents the coordinate of the maximum of the localized solution (25). Since the trial solution (25) for E' models a two-dimensional soliton rolled up around its maximum R'_0, a trial function for the density can be evaluated from Eq. (8b) by simply developing its different contributions around $r'=R'_0$; namely, we took the following trial function for $\delta n'$:

$$\delta n'(r',V)=-\frac{(E')^2}{[1-(R'_0)^2/\Lambda^2]} . \tag{27}$$

Using the latter expression for $\delta n'$, one finds that the last contribution of the Lagrangian density (24) is given by

$$\left[(E')^2-V\left[r'\frac{\partial}{\partial r'}U'+U'\right]\right]^2 \equiv(\delta n')^2=\frac{(E')^4}{[1-(R'_0)^2/\Lambda^2]^2} . \tag{28}$$

On the other hand, the quantity $[(\partial/\partial r')U']^2$ may be evaluated as follows [see the Appendix, Eq. (A2)]:

$$(\nabla'_r U')^2\cong\frac{1}{c^2}\left[\frac{R'_0}{\Lambda}\right]^2(\delta n')^2 . \tag{29}$$

Inserting then the expressions (25), (27), (28), and (29) into L' defined by Eq. (24) leads to the following transformed action integral $S'\equiv R'_0\int_{-\infty}^{+\infty}L'dr'$:

$$S'=\tfrac{8}{3}R'_0\left[1-\frac{V^2}{c^2}(R'_0)^2\right]+\frac{4\epsilon}{R'_0}\left[1-\frac{V^2}{c^2}(R'_0)^2\right] . \tag{30}$$

Since the form of the solution is known, it is possible to get the dependence of the maximum coordinate R'_0 on the parameter V: taking the derivative of (30) with respect to the only relevant variable R'_0 leads to a simple quadratic equation for $(R'_0)^2$, whose roots for positive R'_0 are given by

$$(R'_0)^{\pm}=\left[\frac{\alpha\pm(\alpha^2-4\beta)^{1/2}}{2}\right]^{1/2} , \tag{31a}$$

with

$$\alpha=\frac{c^2}{3V^2}-\frac{\epsilon}{2} \tag{31b}$$

and

$$\beta=c^2\frac{\epsilon}{2V^2} . \tag{31c}$$

From the expression (31a) which predicts the existence of two distinct localized modes whose maxima are located at $(R'_0)^{\pm}$ one realizes that there is a critical velocity V_{crit} such that for $V>V_{\text{crit}}$ there is no self-similar solution: V_{crit} corresponds to the vanishing of the discriminant $(\alpha^2-4\beta)^{1/2}$ in Eq. (31a). Under the basic condition $VR'_0 \ll c$, V_{crit} reads as follows:

$$V_{\text{crit}}=0.219c \quad \text{for } \epsilon=1 \tag{32a}$$

and

$$V_{\text{crit}}=\infty \quad \text{for } \epsilon=0 . \tag{32b}$$

As R'_0 (44) is a decreasing function of V, the two modes tend to converge together as V is increased to V_{crit} until the second mode is superposed with the first one at the same coordinate $(R'_0)^+=(R'_0)^-$. A self-similar collapse therefore may be regarded as a mode merging mechanism and V_{crit} as a bifurcation parameter as far as the self-contraction velocities are concerned.

(i) In the vectorial case ($\epsilon=1$), the above results are in agreement with those of Gol'tsman and Fraiman[3] that were deduced from similar investigations. The latter authors have indeed numerically evaluated the critical subsonic value of $V\lesssim V_{\text{crit}}$ as $V_{\text{crit}}\cong 0.14c$, the theoretical prediction (32a) $V_{\text{crit}}=0.22c$ could thus be considered as reasonably a good approximation. Moreover, although the existence of a critical value for V exhibits a velocity limit, the latter value of V_{crit} can be improved by surrendering the supersonic assumption in Eq. (28). Let us indeed show that the numerical result $V_{\text{crit}}=0.14c$

could be deduced from a calculation analogous to ours under the assumption $V^2/c^2 \ll 1$ and $R_0' = O(1)$ characterizing the vectorial case of a wave collapse that is typically subsonic. Indeed, in the limit $V^2 \to 0$, expression (28) can be simplified into $(\delta n')^2 = (E')^4 / \{1 - [V(R_0')/c]^2\}$ which leads to modify the action S' (30) in the following way:

$$S' \to \tilde{S}' = S' - 4\frac{V^2}{c^2}(R_0')^3 \ .$$

Then taking the derivative of $\tilde{S}'$ again with respect to R_0' gives the critical value $V_{\mathrm{crit}} = 0.144c$, which is thus very close to the numerical estimation $V_{\mathrm{crit}} \cong 0.14c$. We can thus understand *a posteriori* why a supersonic collapse with $\Lambda = 0$ does not take place in the vectorial case, as previously announced.

(ii) Regarding now the scalar model ($\epsilon = 0$), the most surprising result which can be deduced from these considerations concerns the existence of self-similar scalar modes whatever the values of V may be, in particular for $V = \infty$, as shown by Eq. (32b). This result is in agreement with the properties of strongly supersonic scalar modes, as given above. Taking the derivative of (30) with respect to R_0' after setting $\epsilon = 0$ enables us to display a first mode located at

$$(R_0')^- = 0 \tag{33a}$$

and a second one located at $(R_0')^+$ given by expressions (31), namely,

$$(R_0')^+ = \frac{\Lambda}{\sqrt{3}} \ . \tag{33b}$$

Setting $\Lambda = c/V$ in Eq. (33b) leads us to find again the critical velocity $V_{\mathrm{crit}} = \infty$ previously obtained, and the two localized modes together reach a maximum at the center $R_0' = 0$.

At the opposite of the vectorial case, a self-similar collapse governed by the scalar model can thus be achieved with supersonic velocities. From a general point of view, an infinite set of collapsing solutions exists, corresponding to every value of collapse velocity V. These solutions, which verify the continuity conditions (19) and (26) [as shown by Eq. (33b)], always pass continuously through the sonic point $\Lambda = c/V$ for each value of V until V is increased to infinity (or, equivalently, $\Lambda = 0$). When this sonic point is located at the origin of the transformed coordinates ($\Lambda = 0$), collapse occurs within the so-called "strongly supersonic regime."

Let us now comment on the opposite limit $\Lambda^2 = \infty$; as recalled in the Introduction, this limit corresponds to the nonlinear Schrödinger equation (NSE). It is well known that in the NSE case, there are two distinct contraction rates $\xi(t)$ for a two-dimensional collapse for which one has $d = d_c = 2$.[15–19] These scaling laws are indeed given on the first hand by $\xi_1(t) = V(t_* - t)$, which corresponds to an exact ground state $E'(r', \infty)$ for a 2D NSE self-similar collapse, and on the other hand by

$$\xi_2(t) = \left\{ \frac{t_* - t}{\ln\ln\left[\dfrac{1}{t_* - t}\right]} \right\}^{1/2} ,$$

which takes into account logarithmic corrections onto the main scale $\xi(t) = (t_* - t)^{1/2}$ characterizing a NSE collapsing solution. This latter contraction rate $\xi_2(t)$ has been recently investigated by Papanicalaou and coworkers[17–19] from both analytical and numerical points of view. One of their important results is that the self-similar solutions corresponding to $\xi_2(t)$ are thought to be stable because they are numerically observed; on the other hand, the solutions relative to $\xi_1(t)$ have never been found through numerical computations. Moreover, such linear time-dependent solutions relative to $\xi_1(t)$ have been proved by K. Rypdal *et al.*[15] to be marginally stable only for a 2D NSE collapse, in the sense that there exist infinitesimal perturbations that would prevent the blowup and other ones that give rise to a radically different evolution of the collapse singularity from the one originally predicted with $\xi_1(t)$.

Returning now to the complete Zakharov equation set (4), the exact self-similar solution (6a) is linearly dependent on time, as is the case for the solution in $\xi_1(t)$ of the 2D NSE. The former solution can therefore be considered as the generalization of the latter one in the general case $V \neq 0$ of the Zakharov equations. Thus the unstable nature of the self-similar solution in $\xi_1(t)$ of the 2D NSE raises the question of the stability of the solution (6a). On the other hand, one can hardly find a generalization of the solution in $\xi_2(t)$ of the 2D NSE to the Zakharov equations, since a time dependence in $(t_* - t)^{1/2}$ cannot respect the balance between the operators ∂_t^2 and Δ in the ion-sound wave equation (4b). For this reason, the strongly subsonic limit $V \to 0$ seems to be highly singular; a stability analysis of the solutions (6) would be certainly very instructive about the proper collapsing solutions of Eqs. (4) and about the link between the NSE solutions and the Zakharov ones in the limit $V \to 0$. This problem is, however, beyond the scope of the present paper.

III. NUMERICAL RESULTS

We integrated the set (11) by using a Runge-Kutta method. Because the vectorial case ($\epsilon = 1$) has already been widely investigated in Ref. 3, we concentrated our attention on the scalar model ($\epsilon = 0$). In order to be definite, we choose three typical values of Λ, namely, $\Lambda = 12.5$, $\Lambda = 1$, and 0 corresponding thus, respectively, to subsonic, so-called "trans-sonic," and supersonic velocities (with, respectively, $V = 0.08c$, $V = c$, and $V = \infty$). For every configuration, the fields $E'(r')$ and $\delta n'(r')$ have to satisfy the boundary conditions (9):

$$E'(\infty) = \delta n'(\infty) = 0$$

and

$$\left.\frac{\partial}{\partial r'} E'(r')\right|_{r'=0} = 0 \ .$$

As already said, the actual problem consists in finding the correct value of $\phi_0' = |E_0'|^{2m}$ at the origin, such as to obtain localized solutions, together with fixing the correct boundary condition for $\delta n'$, namely, $\delta n'(r'=0)$, which verifies the relation (19) [in the case of a fully supersonic behavior, i.e., $V=\infty$ or $\Lambda=0$, $\delta n_0'(r'=0)$ is given by (16)]. For a given ϕ_0', the procedure consists in finding the proper value $\delta n_0'|_{(r'=0)}$ which ensures a regular behavior of the potential well at the sonic point. On the other hand, as system (11) admits solutions with a singularity as some arbitrary point[2] r_0':

$$\delta n' \cong \frac{2}{(r'-r_0')^2}, \quad E' \cong \pm \frac{\sqrt{2} r_0'}{r'-r_0'} , \tag{34}$$

it is necessary to iterate this procedure until one gets the proper value ϕ_0' which ensures that solutions will recover a physical meaning—i.e., to be regular and localized as $r' \to \infty$— for which one has $r_0' = \infty$. For the sake of simplicity, we have mainly investigated the case of a two-dimensional collapse ($m=1$), for which we give in Table I the correct boundary values of fields E' and $\delta n'$ at $r'=0$ corresponding to each Λ.

Using the same notations as in Ref. 2, we denote by $(\phi_{01}')^*$ and $(\phi_{02}')^*$ the respective boundary values of the two localized modes, correctly chosen at the origin $r'=0$, and for which E' and $\delta n'$ tend to zero as $r' \to \infty$. It can be shown that for $\Lambda=0$, the latter quantities respectively satisfy the following inequalities (see Ref. 2):

$$(\phi_{01}')^* > 3, \quad (\phi_{02}')^* > 10 .$$

For too large deviations $\Delta\phi_0'$ from $(\phi_0')^*$, the function E' goes to infinity in accordance with expressions (34). Here, the solutions are ensured to be correctly localized with a good accuracy $\Delta\phi_0' = 5\times 10^{-4}$ and the behavior of $\delta n'$ near $r'=\Lambda$ is regular by satisfying the relation (19) with an authorized deviation $\Delta\delta n_0'$ from $(\delta n_0')^*$ equal to $\Delta\delta n_0' = 2\%$ for each mode. This latter estimate implies that the density function $\delta n'(r')$ goes through the critical point $r'=\Lambda$ without diverging with a permitted very small deviation $\Delta\delta n'(\Lambda)$ from the correct $\delta n'|_{(r'=\Lambda)}$ given by (19) of order $\Delta\delta n'(r'=\Lambda) = (\delta n'_{\text{num}} - \delta n'_{\text{theor}})/\delta n'_{\text{theor}} \cong 10^{-5}$. Here, we have denoted by $\delta n'_{\text{theor}}$ and $\delta n'_{\text{num}}$ the integrated values of $\delta n'(r')$ at $r'=\Lambda$, obtained respectively from the analytical expression (19) and from the numerical computation, after first introducing the boundary values $\delta n_0'$ and ϕ_0' and letting the computer calculate until the critical point. In that case, one has the following implication:

$$|\Delta\delta n'(r'=\Lambda)| \gg 10^{-5} \Longrightarrow \lim_{r' \to \Lambda^-} \delta n'(r') = \infty$$

as shown by Fig. 1. This figure exhibits the physical divergences of the potential well (in the case $d_c=2$) in the vicinity of $r'=\Lambda=1$ when the condition (19) is not satisfied (dashed lines). On the contrary, for the accurate values $(\phi_{01}')^*$ and $(\delta n_{01}')^*$ verifying (19), the density function $\delta n'(r')$ regularly reaches the sonic point with a deviation $\Delta\delta n'(\Lambda) \le 10^{-5}$ (solid line). At the locus $r'=\Lambda$, the constraint $(\partial_{r'}')^2 \delta n'(\Lambda)=0$ has been imposed on a test function to avoid a numerical singularity.

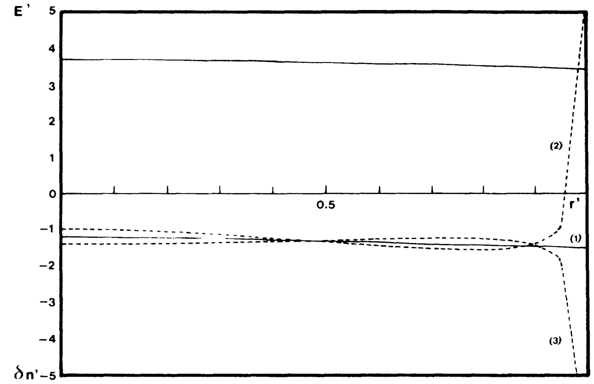

FIG. 1. Diverging curves of functions $\delta n'(r')$ near the critical point $r'=\Lambda=1$. Setting $(\phi_{02}')^* = 13.8901 > 10$, the cases (2) and (3), respectively, refer to $\delta n_0' = -1$ and -1.4 with too large a deviation $\Delta\delta n'(r'=1) = -6.26$ and $\Delta\delta n'(r'=1) = +5.41$. The curve (1) is defined for the correct value of $(\delta n_{02}')^*(r'=0) = -1.208\,975$ which ensures a regular behavior of the potential well at this sonic point with an authorized deviation $\Delta\delta n'(1) = 5\times 10^{-6}$.

We have plotted in Figs. 2(a), 2(b), and 2(c) the localized solutions which respectively correspond to a subsonic ($\Lambda=12.5$), a "trans-sonic" ($\Lambda=1$), and finally a supersonic behavior ($\Lambda=0$) of the two modes. Here, the dashed-line curves refer to the first mode, while solid lines indicate the second one. As predicted by our former theoretical results, the scalar model contains two localized solutions whose first one has always a maximum at the center $r'=0$, whereas the second one tends to be projected onto the first mode as Λ goes to zero ($V \gg 1$). In all cases, the variational principle $\partial S'/\partial R_0' = 0$ gives reasonably good estimations since we have from (31) the

TABLE I. Boundary values of fields $(\phi_{01}')^*, (\phi_{02}')^*$ and $(\delta n_{01}')^*, (\delta n_{02}')^*$ defined at the origin $r'=0$ for various collapse velocities at $d_c=2$ and 1. Such origin conditions correspond to two localized self-similar solutions for each space dimension.

$\Lambda=c/V$	$d=d_c$	$(\phi_{01}')^*$	$(\delta n_{01}')^*$	$(\phi_{02}')^*$	$(\delta n_{02}')^*$
12.5	2	624.5	−4.18	0.15	0.134
1	2	7.196	−3.642 01	13.8301	−1.208 975
0	2	7.279	−3.3455	13.0075	−1.2997
0	1	5.057 17	−2.4654	14.5282	−1.2602

maximum coordinates:

$$(R_0')^+(\Lambda=12.5)\cong 7.2$$

and

$$(R_0')^-(\Lambda=12.5)=(R_0')^-(\Lambda=0)=(R_0')^+(\Lambda=0)=0\ ,$$

which have to be compared with the numerical results $[(R_0')^+(\Lambda=12.5)]_{\mathrm{num}}\cong 6$ and $[(R_0')^-(\Lambda=12.5)]_{\mathrm{num}}=[(R_0')^-(\Lambda=0)]_{\mathrm{num}}=[(R_0')^+(\Lambda=0)]_{\mathrm{num}}=0$. On the

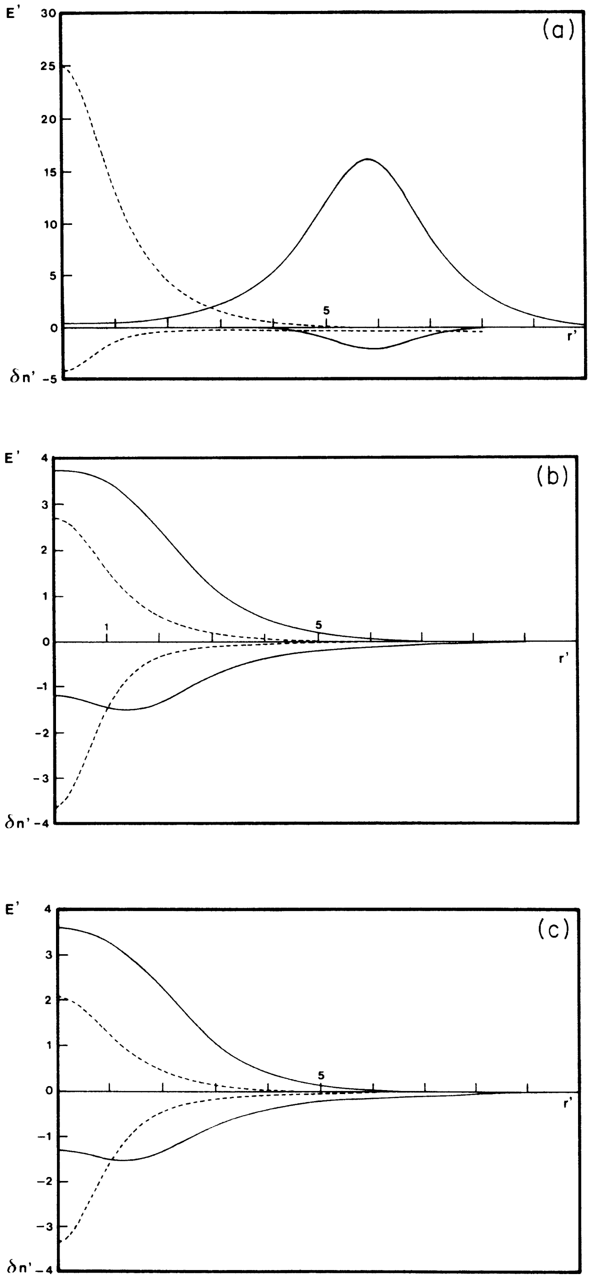

FIG. 2. Two localized modes I and II (respectively plotted in dashed and solid lines) illustrating self-similar solutions $E'(r')$ and $\delta n'(r')$ with (a) subsonic collapse velocity ($\Lambda=12.5$; $V=0.08c$); (b) "trans-sonic" velocity ($\Lambda=1$; $V=c$); (c) supersonic collapse velocity ($\Lambda=0$; $V\gg c$).

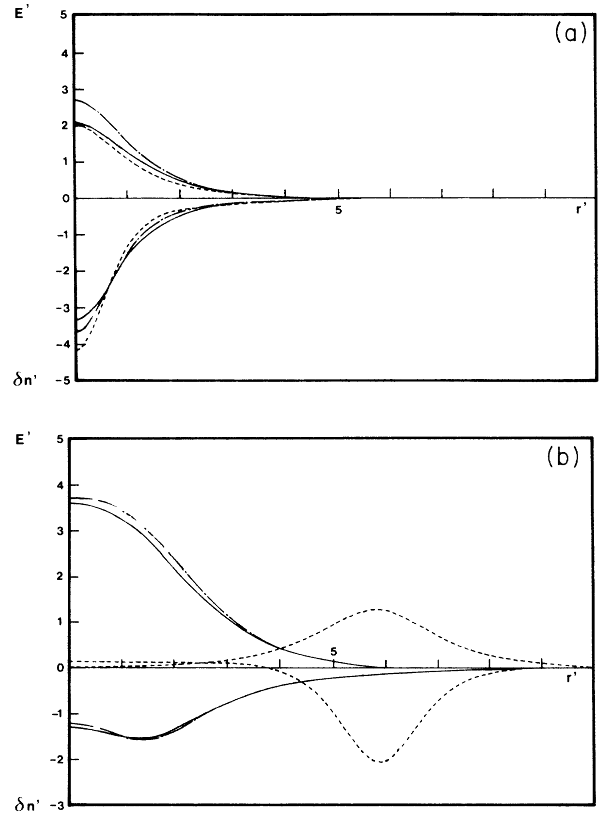

FIG. 3. Three different stages of a collapsing evolution, namely, for subsonic ($\Lambda=12.5$, dashed line), "trans-sonic" ($\Lambda=1$, dash-dotted line), and supersonic ($\Lambda=0$, solid line) collapse velocities describing the regular behavior of (a) mode I; (b) mode II.

other hand, in order to avoid making the rhs of (11b)—the ponderomotive term—too large as compared with $\Lambda^2\Delta'(\delta n')$ in the left-hand side (lhs) of (11b) as Λ^2 tends to infinity (the subsonic limit), it is convenient to introduce the formal rescaling $E'\to\Lambda^{1/m}E'$ in system (11). In that case, this latter substitution makes it possible to deal without any loss of generality with the equation set (8) expressed in reduced units $c^2=1$. Under these conditions, Figs. 3(a) and 3(b) show three stages of self-similar behaviors of the two separated modes defined for different values of V, namely, $V=0.08$ (dashed lines), $V=1$ (dash-dotted lines), and $V\to\infty$ (solid lines).

IV. ANISOTROPIC SOLUTIONS OF A LANGMUIR COLLAPSE

While isotropic axisymmetric solutions of system (4) allow one to describe various phenomena such as self-focusing processes, in particular, in nonlinear optics,[20] the exact solutions of the original Zakharov equations (1) are not spherically symmetric, as shown in Refs. 2 and 12. Though the quantity $|\nabla\Phi|^2$ has a maximum in the center $r=0$ as in the case of a scalar collapse, the Langmuir collapse has necessarily a dipolar character; in cylindrical symmetry $\delta n(r,z)$ is then axially symmetric

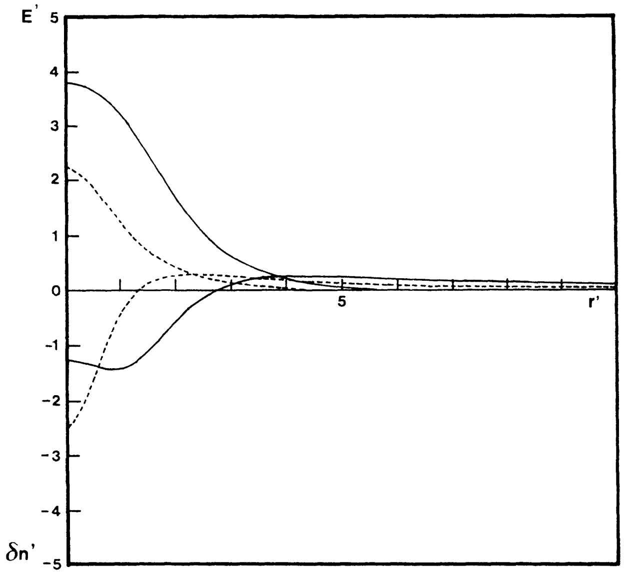

FIG. 4. Supersonic self-similar solutions ($\Lambda=0$) defined for $d_c=1$; this field configuration simulates the two localized modes (dashed line for mode I, solid line for mode II) of a critical Langmuir collapse.

around the z axis and the potential $\Phi(r,z)$ is antisymmetric as follows:

$$\Phi(r,-z)=-\Phi(r,z) .$$

It is thus useful to integrate the peculiar case for which one sets $d=d_c=1$ and $m=1$ (in this case, one no longer has $m=2/d_c$) and $\epsilon=0$ in Eqs. (11), because a two-dimensional *asymmetric* Langmuir collapse can behave in a way that is very similar to a scalar one-dimensional collapse. The coordinates r and z here denote, respectively, the radial and the axial components of an ellipsoidal caviton depicted in plane geometry. As shown by many numerical experiments,[8,12,21] a Langmuir caviton becomes asymptotically flat along the privileged z-axial direction, so that a critical collapse may be modeled by a scalar model by expanding various quantities around $r=0$ in the following way:

$$\Phi=\Phi_0+\Phi_1 r^2+\cdots, \quad \delta n=\delta n_0+\delta n_1 r^2+\cdots , \tag{35}$$

with

$$\Phi_1\cong a^2\frac{\partial^2}{\partial z^2}\Phi_0(z), \quad \delta n_1\cong a^2\frac{\partial^2}{\partial z^2}\delta n_0(z) ,$$

where $a \ll 1$ is here defined as the ratio of the caviton thickness $\xi_z(0)$ over its radial size $\xi_r(0)$.[2,6] Substituting (35) under the limit $a^2\to 0$ together with $E=\partial\Phi_0(z)/\partial z$ into system (1) leads to the one-dimensional problem (11). Thus a two-dimensional Langmuir collapse can be described by a scalar critical collapse with $d_c=1$ at the zeroth order in the a^2 expansions (35). To solve the trans-sonic problem, it is then convenient to take the general relation (19) into account with $d_c=1$ and an anomalous value for m (let us recall that the relation $d_c=2/m$ is here assumed to be no longer satisfied since one has $m=1$ instead of $m=2$). The numerical solutions of such a critical collapse are very similar to those obtained in the case of a two-dimensional one. As an example, we have illustrated the supersonic behavior of these latter in Fig. 4, where, for the two localized modes, one can observe the positive contributions of the density functions (the so-called "lips") at $r'\to\infty$, as predicted by its asymptotic definition (14b), namely, $\lim_{r'\to\infty}\delta n'(r')=\phi_0'/(r')^2>0$.

V. CONCLUSION

We have shown that at critical dimension, the *scalar* modelization of a spherically symmetric collapse, which in several aspects can be regarded as the most appropriate to describe a Langmuir collapse, admits self-similar solutions, with arbitrary supersonic collapse velocities, unlike the *vectorial* case whose self-similar behavior is limited by a critical contraction velocity $V\le V_{crit}=0.14c$, corresponding to the subsonic regime. It has been analytically and numerically proved that the field distributions are continuous at the sonic point $r'=\Lambda$ provided the constraint given by Eq. (19) on the potential well at this point be satisfied. Thus a critical wave collapse described by the scalar model is always self-similarly realized and as the control parameter V is increased from zero to infinity, its two localized scalar modes merge together until reaching a maximum of the fields in the center of the caviton, where collapse can occur for very large values of V.

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APPENDIX

In this appendix, we present the connection between the Lagrangian density (24) and Eqs. (8). By using a variational principle with respect to the various fields $E', U', \delta n'$ appearing in the functional (24), one obtains in particular Eq. (8b) that may be directly deduced from a combination of the two following ion-density equations:

$$V\left[1+r'\frac{\partial}{\partial r'}\right]U'=|E'|^2+\delta n' , \tag{A1}$$

$$V\left[2+r'\frac{\partial}{\partial r'}\right]\delta n'=c^2\frac{1}{r'}\frac{\partial}{\partial r'}r'\frac{\partial}{\partial r'}U' , \tag{A2}$$

where Eq. (A1) corresponds to the transformed Eq. (22) under the substitution (7). We will also notice that one may easily pass from Eqs. (8) to (11) by simply transforming E' in the Lagrangian density (24) according to

$$E'\to\frac{1}{\Lambda}E' , \tag{A3}$$

with $\Lambda=c/V$.

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