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Homodyne statistics

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It is usually assumed that a balanced homodyne detector measures a variable called the quadrature phase amplitude of a signal field. We examine this assumption by obtaining analytic expressions for the probability statistics for the output of such a detector for a single mode signal with both signal and local oscillator treated quantum mechanically. We investigate the conditions under which the statistics of the quadrature phase are reproduced in the actual output of the detector. We show that the most obvious condition—that the number of quanta in the local oscillator be much larger than the number in the signal—is not sufficient to ensure that a balanced homodyne detector acts like an ideal detector of quadrature phases. Furthermore, we obtain the explicit conditions that are necessary and sufficient to reproduce the interference features in the homodyne statistics of a superposition of coherent states.

I. INTRODUCTION

The homodyne detector[1] is a fundamental device for measuring phase-sensitive properties of optical fields. Its phase sensitivity comes from beating a signal field against a local oscillator, which acts as a phase reference. Recently, the quantum limits of measurement have been probed using these devices in optical experiments[2-4] involving squeezed states[5] of light. It is usually assumed that the observable measured by the homodyne detector is one of the quadrature phase amplitudes;[6] this is strictly true in the limit that the local oscillator's amplitude is taken to infinity. In this paper we investigate the effects of a finite-amplitude fully-quantum-mechanical local oscillator.

The electric field of a near-monochromatic plane wave can be written[6]

$$\begin{aligned} \hat{E} &\propto \frac{1}{2}\left[\hat{a}e^{i(kx-\Omega t)} + \hat{a}^\dagger e^{-i(kx-\Omega t)}\right] \\ &= \hat{a}_1\cos(\Omega t - kx) + \hat{a}_2 \sin(\Omega t - kx)\ , \end{aligned} \tag{1.1}$$

where $\hat{a} = \hat{a}_1 + i\hat{a}_2$ is the annihilation operator for this plane-wave mode, and $\hat{a}_1$ and $\hat{a}_2$ are the Hermitian quadrature phase amplitudes. If we were to visualize the electric field $\hat{E}$ of Eq. (1.1) plotted on a complex plane then we would see that $\hat{a}_1$ and $\hat{a}_2$ are analogs of the position and momentum of a mechanical oscillator. For a classical local oscillator of the form

$$E_{\rm LO} = 2|E_{\rm LO}|\cos(\Omega t - kx)\ , \tag{1.2}$$

the beating between $\hat{E}$ in Eq. (1.1) and $E_{\rm LO}$ (averaged over several optical periods) has its only significant contribution from the term

$$|E_{\rm LO}|\hat{a}_1\ . \tag{1.3}$$

This term is, apart from a scale factor involving the local oscillator amplitude, the quadrature phase $\hat{a}_1$. In this simplified treatment the homodyne detector's output is a continuous variable; it is continuous in the sense that the observable can take on a continuum of values. Such a device with a truly classical local oscillator could be thought of as an ideal detector of quadrature phase amplitudes.

A quantum treatment of the local oscillator in the balanced homodyne detector is shown schematically in Fig. 1. The signal $\hat{a}$ and local oscillator $\hat{b}$ fields combine at the 50-50 beamsplitter to give the sum and difference fields $\hat{c}$ and $\hat{d}$ in the two arms of the detector. These combined fields are incident on a pair of ideal photodetectors that in each counting interval will yield a photocurrent proportional to the number of quanta counted,[7] i.e.,

$$\begin{aligned} \hat{I}_1 &\propto \hat{c}^\dagger\hat{c}\ , \\ \hat{I}_2 &\propto \hat{d}^\dagger\hat{d} \end{aligned} \tag{1.4}$$

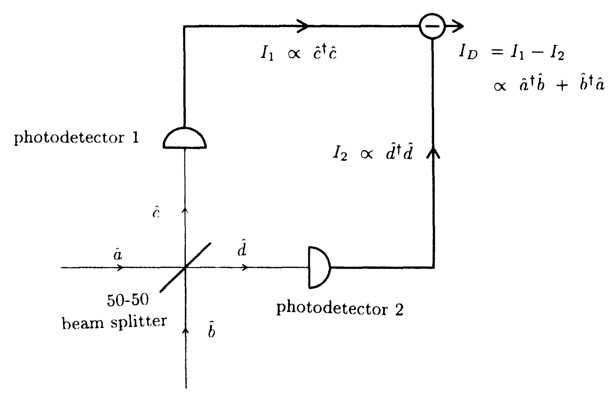

FIG. 1. Schematic of a balanced homodyne detector. The signal and local oscillator combine at a 50-50 beam splitter to give the sum and difference amplitudes $\hat{c} = (\hat{a} + \hat{b})/\sqrt{2}$ and $\hat{d} = (\hat{a} - \hat{b})/\sqrt{2}$, respectively. The detector's output is given by the difference photocurrent between the two detectors—which are assumed to have equal gains.

where

$$\hat{c} = \frac{\hat{a}+\hat{b}}{\sqrt{2}} ,$$
$$\hat{d} = \frac{\hat{a}-\hat{b}}{\sqrt{2}} . \tag{1.5}$$

The normalized current from each will be a non-negative integer. Now, with the gains of the photodetectors equal (for the balanced configuration) the difference photocurrent has the form

$$\hat{I}_D = \hat{I}_1 - \hat{I}_2 \propto \hat{b}^\dagger \hat{a} + \hat{a}^\dagger \hat{b} , \tag{1.6}$$

which unlike Eq. (1.3) is not a continuous observable. It should be noted that throughout this paper the terms "homodyne statistics" and "probability statistics of the difference photocurrent" are used interchangeably.

There is a difficulty with Eq. (1.4) that is worth mentioning.[8,9] If the signal and local oscillator are coming in a continuous stream, then it will be difficult to separate the counting intervals so that quanta are counted from distinguishable spatial modes. One way around this problem is to use a pulsed scheme with signal and local oscillator fields strongly overlapping spatially within the detectors. A study of this, however, would require a broadband analysis which is outside the scope of this paper.

The quantum nature of the local oscillator has been investigated by several authors.[10-12] In each case, however, only corrections to the first and second moments of the difference photocurrent have been calculated, and no attempt has been made to see how closely the discrete output from the quantum device approximates the "continuous" output expected of an ideal detector of quadrature phase amplitude.

In Sec. II of this paper we develop a general formalism for calculating the probability statistics of the difference photocurrent, Eq. (1.6), and we obtain closed-form expressions for the cases of a coherent local oscillator and signal either in a coherent state or in a superposition of coherent states.

We analyze the asymptotic limits of these expressions in Sec. III (not just a few moments) as the amplitude of the local oscillator tends to infinity, to see how the whole distribution of the difference photocurrent varies.

There is an obvious condition to ensure that the statistics of the balanced homodyne detector can reproduce the details of an ideal detector of quadrature phase: that the number of quanta in the local oscillator is much larger than the number in the signal field. We show, in Sec. IV, that this condition is not sufficient. We go on to derive explicit conditions for a homodyne detector to approximate an ideal detector of quadrature phase statistics when the signal is either in a coherent state or a superposition of coherent states, and the local oscillator is in a coherent state.

II. STATISTICS OF HOMODYNE DETECTION

In this section we will calculate the probability statistics for a balanced homodyne detector. We shall determine the probability statistics for the difference photocurrent $\hat{a}^\dagger\hat{b}+\hat{b}^\dagger\hat{a}$ to take the value N by first calculating its characteristic function, $\chi(k)$. This probability distribution will then be given by

$$P(N) \equiv \int dk \frac{1}{2\pi} \chi(k) \exp(-ikN) . \tag{2.1}$$

In quantum theory the characteristic function is defined in a manner analogous to the way it is defined classically, so we have

$$\chi(k) \equiv \mathrm{tr}\{\hat{\rho}_a \hat{\rho}_b \exp[ik(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})]\} , \tag{2.2}$$

where $\hat{\rho}_a$ and $\hat{\rho}_b$ are the states of the signal and local oscillator, respectively. In order to obtain a general expression for this characteristic function we shall represent the density matrices in terms of a c-number distribution called the positive-P representation.[13] Thus,

$$\hat{\rho}_a = \int d^2\alpha_1 d^2\alpha_2 \frac{|\alpha_1\rangle\langle\alpha_2|}{\langle\alpha_2|\alpha_1\rangle} P_a(\alpha_1, \alpha_2^*) , \tag{2.3}$$

for the signal, where $|\alpha_1\rangle$ and $|\alpha_2\rangle$ are coherent states for the signal with respective amplitudes α_1 and α_2, and $P_a(\alpha_1, \alpha_2^*)$ is a positive-P representation of the signal state $\hat{\rho}_a$. Similarly, a positive-P representation for the local oscillator yields

$$\hat{\rho}_b = \int d^2\beta_1 d^2\beta_2 \frac{|\beta_1\rangle\langle\beta_2|}{\langle\beta_2|\beta_1\rangle} P_b(\beta_1, \beta_2^*) . \tag{2.4}$$

In Eqs. (2.3) and (2.4) we have used a convention of writing α_2 for α_2^* and β_2 for β_2^* simply for the convenience of notation. We note that this representation is completely general, assuming only that the signal and local oscillator are initially uncorrelated. Thus, the expressions we obtain can be applied to nonclassical states such as sub-poissonian, squeezed, and superposed coherent states. Indeed, the discussion of Sec. IV is mainly concerned with the homodyne detector's ability to reproduce the interference features seen in the quadrature phase statistics for coherent superpositions.

In terms of these distributions the characteristic function of Eq. (2.2) may be written solely in terms of coherent state matrix elements,

$$\chi(k) = \int d^2\alpha_1 d^2\alpha_2 \frac{1}{\langle\alpha_2|\alpha_1\rangle} d^2\beta_1 d^2\beta_2 \frac{1}{\langle\beta_2|\beta_1\rangle} \times P_a(\alpha_1, \alpha_2^*) P_b(\beta_1, \beta_2^*) \chi_{21} , \tag{2.5}$$

where

$$\chi_{21} \equiv \langle\alpha_2, \beta_2| \exp[ik(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})] |\alpha_1, \beta_1\rangle . \tag{2.6}$$

In order to calculate χ_{21} we need only place the exponential operator in Eq. (2.6) in normal order; this step becomes easy if we rely on the theorem

$$(1-\lambda)^{\hat{a}^\dagger\hat{a}} =: \exp(-\lambda\hat{a}^\dagger\hat{a}) : , \tag{2.7}$$

where the colon bracketing : : means to place the enclosed operator in normal-order form without regard to operator algebra; e.g.,

$$: \hat{a}^\dagger \hat{a} :=: \hat{a}\hat{a}^\dagger := \hat{a}^\dagger \hat{a} \ . \tag{2.8}$$

To use Eq. (2.7) we need only recall that the difference current may be written

$$\hat{I}_D \propto \hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a} = \hat{c}^\dagger \hat{c} - \hat{d}^\dagger \hat{d} \ , \tag{2.9}$$

so that

$$\chi_{21} = \langle \alpha_2, \beta_2 | : \exp[-(1-e^{ik})\hat{c}^\dagger \hat{c} - (1-e^{-ik})\hat{d}^\dagger \hat{d}] : |\alpha_1, \beta_1\rangle \ . \tag{2.10}$$

This matrix element can now be evaluated immediately to give

$$\chi_{21} = \exp[-(1-e^{ik})c_2^* c_1 - (1-e^{-ik})d_2^* d_1]\langle \alpha_2 | \alpha_1\rangle \langle \beta_2 | \beta_1 \rangle \ , \tag{2.11}$$

where

$$c_i = \frac{\alpha_i + \beta_i}{\sqrt{2}} \ , \quad i = 1,2 \ ; \quad d_i = \frac{(\alpha_i - \beta_i)}{\sqrt{2}} \ , \quad i = 1,2 \ . \tag{2.12}$$

Because χ_{21} alone in Eq. (2.5) depends on the Fourier variable k, this part of the integral may be performed separately:

$$\begin{aligned}\int dk \frac{1}{2\pi} \chi_{21} \exp(-ikN) &= \langle \alpha_2 | \alpha_1 \rangle \langle \beta_2 | \beta_1 \rangle \exp(-c_2^* c_l - d_2^* d_l) \\ &\quad \times \sum_{n,m} \frac{(c_2^* c_1)^n (d_2^* d_1)^m}{n! m!} \int dk \frac{1}{2\pi} \exp[ik(n-m-N)] \ . \end{aligned} \tag{2.13}$$

The integral over k is seen to lead to a sum of Dirac δ functions, which spike at integral values of N. This stems from using a continuous form of the Fourier transform. We may reduce this expression to a more standard discrete form by integrating about small intervals at integral values of N. In this way Eq. (2.13) becomes

$$\langle \alpha_2 | \alpha_1 \rangle \langle \beta_2 | \beta_1 \rangle \exp(-c_2^* c_1 - d_2^* d_1) \left(\frac{c_2^* c_1}{d_2^* d_1} \right)^{N/2} I_{|N|}(2\sqrt{c_2^* c_1 d_2^* d_1}) \ , \tag{2.14}$$

where $I_N(x)$ is the modified Bessel function of integral order.[14] Care needs to be taken in choosing the square-root branches when Eq. (2.14) is evaluated; for instance, one could take the square roots of each term $c_2^* c_1$ and $d_2^* d_1$ separately, thus ensuring a consistent choice of branch cut throughout.

Combining these results gives the discrete form for the distribution in Eq. (2.1) as

$$P_N = \int d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2 P_a(\alpha_1, \alpha_2^*) P_b(\beta_1, \beta_2^*) \exp(-c_2^* c_1 - d_2^* d_1) \left(\frac{c_2^* c_1}{d_2^* d_1} \right)^{N/2} I_{|N|}(2\sqrt{c_2^* c_1 d_2^* d_1}) \ . \tag{2.15}$$

For the remainder of this section we shall concentrate on some specific choices for the states of the signal and local oscillator. In particular, we shall for the rest of the paper consider the case of the local oscillator in a coherent state $\hat{\rho}_b = |\beta\rangle\langle\beta|$, in which case Eq. (2.15) can be reduced to

$$P_N = \int d^2\alpha_1 d^2\alpha_2 P_a(\alpha_1, \alpha_2^*) \exp(-\alpha_2^* \alpha_1 - |\beta|^2) \left(\frac{c_2^* c_1}{d_2^* d_1} \right)^{N/2} I_{|N|}(2\sqrt{c_2^* c_1 d_2^* d_1}) \ , \tag{2.16}$$

with $\beta_1 = \beta_2 = \beta$.

For the signal also in a coherent state $|\alpha\rangle\langle\alpha|$ the homodyne statistics of Eq. (2.16) reduces to

$$P_N = \exp(-|\alpha|^2 - |\beta|^2) \left| \frac{\alpha+\beta}{\alpha-\beta} \right|^N I_{|N|}(|\alpha+\beta|\,|\alpha-\beta|) \ , \tag{2.17}$$

with the special case of the signal in vacuum $|0\rangle\langle 0|$ being given by

$$P_N = \exp(-|\beta|^2) I_{|N|}(|\beta|^2) \ . \tag{2.18}$$

There is one last case for which we write down the explicit form of the exact homodyne statistics: when the signal is in the superposition state

$$\mathcal{N}[\cos\theta |\alpha_1\rangle + (\sin\theta) e^{i\phi} |\alpha_2\rangle] \ , \tag{2.19}$$

where

$$\mathcal{N}^2 = \frac{1}{1+\sin(2\theta)\mathrm{Re}(e^{i\phi}\langle\alpha_1|\alpha_2\rangle)} . \tag{2.20}$$

In this case the homodyne statistics are

$$\begin{aligned} P_N = \mathcal{N}^2 \Bigg[& (\cos^2\theta)\exp(-|\alpha_1|^2-|\beta|^2)\left|\frac{\alpha_1+\beta}{\alpha_1-\beta}\right|^N I_{|N|}(|\alpha_1+\beta||\alpha_1-\beta|) \\ & +(\sin^2\theta)\exp(-|\alpha_2|^2-|\beta|^2)\left|\frac{\alpha_2+\beta}{\alpha_2-\beta}\right|^N I_{|N|}(|\alpha_2+\beta||\alpha_2-\beta|) \\ & +\sin(2\theta)\mathrm{Re}\left(e^{-i\phi}\langle\alpha_2|\alpha_1\rangle\exp(-c_2^*c_1-d_2^*d_1)\left(\frac{c_2^*c_1}{d_2^*d_1}\right)^{N/2} I_{|N|}(2\sqrt{c_2^*c_1d_2^*d_1})\right)\Bigg] \end{aligned} \tag{2.21}$$

where c_i and d_i are defined in Eq. (2.12).

In the next section we shall consider the asymptotic forms of the above expressions as the local oscillator amplitude grows large.

III. ASYMPTOTIC FORMS

In the preceding section, we obtained exact expressions for the probability statistics in direct-counting experiments from a balanced homodyne detector.

The coherent state is the quantum state that most closely resembles the classical description of a harmonic oscillator. This resemblance becomes exact as the excitation of the coherent state tends to infinity. Thus, one could think of the process of taking a coherent state's excitation to infinity as the classical limit for that state.

This line of reasoning will also hold for coherent states of light representing individual modes of an electromagnetic field; whose equations of free evolution are just those of a harmonic oscillator. Thus, the classical limit of the local oscillator's coherent state $|\beta\rangle$ will be approached as $\beta \to \infty$.

We shall now repeat the calculations of Sec. II to obtain asymptotic expansions of the homodyne statistics as $|\beta|$ is taken to be large. We could perform asymptotic expansions of our results in Sec. II by applying the known asymptotic expansions of the modified Bessel functions; we shall instead take a more general approach and perform the asymptotic expansion of the homodyne statistics directly.

We wish to expand the operator

$$\hat{\Theta} \equiv \langle\beta|\exp[ik(\hat{a}^\dagger\hat{b}+\hat{b}^\dagger\hat{a})]|\beta\rangle , \tag{3.1}$$

to $O(|\beta|^{-2})$ about the classical result. To do this we first expand

$$\hat{X}_n \equiv \langle\beta|(\hat{a}^\dagger\hat{b}+\hat{b}^\dagger\hat{a})^n|\beta\rangle , \tag{3.2}$$

to $O(|\beta|^{n-2})$. Using a shift operator we may write Eq. (3.2) as the vacuum expectation:

$$\begin{aligned} \hat{X}_n &= \langle 0|[(\hat{a}^\dagger\beta+\hat{a}\beta^*)+(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger)]^n|0\rangle \\ &= (\hat{a}^\dagger\beta+\hat{a}\beta^*)^n+\hat{S}_n+O(|\beta|^{n-4}) , \end{aligned} \tag{3.3}$$

where

$$\hat{S}_n \equiv \langle 0|\mathcal{S}[(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger)^2(\hat{a}^\dagger\beta+\hat{a}\beta^*)^{n-2}]|0\rangle \tag{3.4}$$

is the second term in the binomial expansion of $\hat{X}_n$; the symbol $\mathcal{S}$ reminds us to symmetrize among the bracketed factors following it. $\hat{S}_n$ may be evaluated by writing it explicitly as

$$\begin{aligned} \hat{S}_n &= \sum_{l=0}^{n-2}\sum_{m=0}^{n-2-l}(\hat{a}^\dagger\beta+\hat{a}\beta^*)^l\hat{a}^\dagger(\hat{a}^\dagger\beta+\hat{a}\beta^*)^{n-2-l-m}\hat{a}(\hat{a}^\dagger\beta+\hat{a}\beta^*)^m \\ &= \sum_{l=0}^{n-2}\sum_{m=0}^{n-2-l}[\hat{a}^\dagger(\hat{a}^\dagger\beta+\hat{a}\beta^*)^{n-2}\hat{a}+m\beta\hat{a}^\dagger(\hat{a}^\dagger\beta+\hat{a}\beta^*)^{n-3}+l\beta^*(\hat{a}^\dagger\beta+\hat{a}\beta^*)^{n-3}\hat{a}+lm|\beta|^2(\hat{a}^\dagger\beta+\hat{a}\beta^*)^{n-4}] . \end{aligned} \tag{3.5}$$

After performing these elementary sums we find

$$\begin{aligned} \hat{X}_n = & (\hat{a}^\dagger\beta+\hat{a}\beta^*)^n+\frac{n(n-1)}{2}\hat{a}^\dagger(\hat{a}^\dagger\beta+\hat{a}\beta^*)^{n-2}\hat{a} \\ & +\frac{n(n-1)(n-2)}{6}[\beta\hat{a}^\dagger(\hat{a}^\dagger\beta+\hat{a}\beta^*)^{n-3}+\beta^*\hat{a}(\hat{a}^\dagger\beta+\hat{a}\beta^*)^{n-3}] \\ & +\frac{n(n-1)(n-2)(n-3)}{24}|\beta|^2(\hat{a}^\dagger\beta+\hat{a}\beta^*)^{n-4}+O(|\beta|^{n-4}) . \end{aligned} \tag{3.6}$$

This expression allows us to obtain the asymptotic form for Eq. (3.1) directly from a power series expansion of the exponential

$$\hat{\Theta} = \sum_{n=0}^{\infty} \frac{(ik)^n \hat{X}_n}{n!}$$
$$= \hat{E} + \frac{(ik)^2}{2} \hat{a}^\dagger \hat{E} \hat{a} + \frac{(ik)^3}{6} [\beta \hat{a}^\dagger \hat{E} + \beta^* \hat{E} \hat{a}] + \frac{(ik)^4}{24} |\beta|^2 \hat{E} + O(|\beta|^{-4}) , \tag{3.7}$$

where $\hat{E}$ is just the version with a classical local oscillator; that is, the classical limit:

$$\hat{E} \equiv \exp[ik(\hat{a}^\dagger \beta + \hat{a}\beta^*)] . \tag{3.8}$$

The probability statistics of Eq. (2.1) may now be obtained by substituting Eq. (3.7) into Eq. (2.6), (recalling that in this section we treat the local oscillator as a coherent state) to get

$$P(N) = \int d^2\alpha_1 d^2\alpha_2 \frac{1}{\langle \alpha_2 | \alpha_1 \rangle} P_a(\alpha_1, \alpha_2^*) \int dk \frac{1}{2\pi} e^{-ikN} \langle \alpha_2 | \hat{\Theta} | \alpha_1 \rangle . \tag{3.9}$$

The integral over k in Eq. (3.9) may be performed yielding

$$P(\tilde{N}) = \int d^2\alpha_1 d^2\alpha_2 \frac{1}{\sqrt{2\pi}} P_a(\alpha_1, \alpha_2^*) \exp[-(\tilde{N} - z)^2/2]$$
$$\times \left(1 + \frac{\alpha_2^* \alpha_1}{2|\beta|^2} [(\tilde{N} - z)^2 - 1] + \frac{z}{6|\beta|^2} (\tilde{N} - z)[(\tilde{N} - z)^2 - 3] \right.$$
$$\left. + \frac{1}{24|\beta|^2} [(\tilde{N} - z)^4 - 6(\tilde{N} - z)^2 + 3] + O(|\beta|^{-4}) \right) , \tag{3.10}$$

where

$$\tilde{N} \equiv \frac{N}{|\beta|} , \tag{3.11}$$

and

$$z \equiv \frac{(\alpha_2^* \beta + \alpha_1 \beta^*)}{|\beta|} , \tag{3.12}$$

are just scaled variables appropriate for the output of a homodyne detector with a local oscillator of amplitude $|\beta|$.

It is instructive to look at some special cases of Eq. (3.10). For the signal in a vacuum the homodyne statistics asymptotically approach

$$P(\tilde{N}) \simeq \frac{1}{\sqrt{2\pi}} \exp(-\tilde{N}^2/2) \left(1 + \frac{1}{24|\beta|^2} (\tilde{N}^4 - 6\tilde{N}^2 + 3) + O(|\beta|^{-4}) \right) . \tag{3.13}$$

For the signal in a coherent state $|\alpha\rangle\langle\alpha|$ they become

$$P(\tilde{N}) \simeq \frac{1}{\sqrt{2\pi}} \exp[-(\tilde{N} - x)^2/2] \left(1 + \frac{|\alpha|^2}{2|\beta|^2} [(\tilde{N} - x)^2 - 1] + \frac{x}{6|\beta|^2} (\tilde{N} - x)[(\tilde{N} - x)^2 - 3] \right.$$
$$\left. + \frac{1}{24|\beta|^2} [(\tilde{N} - x)^4 - 6(\tilde{N} - x)^2 + 3] + O(|\beta|^{-4}) \right) , \tag{3.14}$$

where

$$x = \frac{\alpha\beta^* + \alpha^*\beta}{|\beta|} \tag{3.15}$$

is just the scaled quadrature phase variable of the homodyne detector.

We could similarly write down the asymptotic form for the superposition of coherent states in Eq. (2.19); this is easy enough given Eq. (3.10), however, so we shall not write out the result explicitly.

The general result given by Eq. (3.10) allows us to check previous calculations of corrections to the moments of the quadrature phase amplitudes as seen by the homodyne detector. We find

$$\int P(\tilde{N}) d\tilde{N} = 1 + O(|\beta|^{-4}) , \tag{3.16}$$

$$\int P(\tilde{N})\tilde{N}\,d\tilde{N} = \int d^2\alpha_1 d^2\alpha_2 P(\alpha_1,\alpha_2^*)z + O(|\beta|^{-4})$$
$$= \langle\hat{x}\rangle + O(|\beta|^{-4}) \,, \tag{3.17}$$

and

$$\int P(\tilde{N})\tilde{N}^2 d\tilde{N} = \int d^2\alpha_1 d^2\alpha_2 P(\alpha_1,\alpha_2^*)\left(z^2 + \frac{\alpha_2^*\alpha_1}{|\beta|^2}\right) + O(|\beta|^{-4}) \,, \tag{3.18}$$

therefore

$$(\Delta\tilde{N})^2 = (\Delta\hat{x})^2 + \frac{\langle\hat{n}_a\rangle}{|\beta|^2} + O(|\beta|^{-4}) \,, \tag{3.19}$$

where $\Delta\hat{x}$ is the uncertainty of the scaled quadrature phase amplitude

$$\hat{x} \equiv \frac{\hat{a}\beta^* + \hat{a}^\dagger\beta}{|\beta|} \,, \tag{3.20}$$

and

$$\langle\hat{n}_a\rangle \equiv \langle\hat{a}^\dagger\hat{a}\rangle \tag{3.21}$$

is the number of quanta in the signal. The result of Eq. (3.19) is in agreement with previous calculations[10-12] of the fluctuations of a homodyne detector's output.

IV. CONDITIONS FOR AN IDEAL HOMODYNE DETECTOR

In this section we discuss the conditions needed for a homodyne detector to mimic an ideal detector of quadratic phase amplitudes. We shall restrict our attention to explicit conditions for the cases of the signal in a coherent state or a superposition of coherent states; again we assume in this section that the local oscillator is in a coherent state.

If one is only ever interested in using the homodyne detector to measure uncertainties in the quadrature phase amplitudes then Eq. (3.19) is all that is required, and we require

$$\langle\hat{n}_b\rangle = |\beta|^2 \gg \langle\hat{n}_a\rangle \,, \tag{4.1}$$

i.e., the number of quanta in the local oscillator must be much greater than the number in the signal.

There is, however, another mode of operation: namely, to obtain a histogram of the statistics for the photocurrent. Ideally this would reproduce the statistics of the quadrature phase, however, as we have seen in the previous sections this is only strictly achieved for a classical local oscillator. We shall show by example that the condition of Eq. (4.1) is not sufficient.

Let us first consider an example for which the condition of Eq. (4.1) does work well; the case where the signal is in a coherent state. The exact statistics are given by Eq. (2.17) and the asymptotic form by Eq. (3.14). Since the statistics are only significant near the bell of the Gaussian, the corrections due to a quantum-mechanical local oscillator in Eq. (3.14) are small provided

$$|\beta|^2 \gg |\alpha|^2, x, 1 \,. \tag{4.2}$$

Further, if we assume that the local oscillator has more than a few quanta this relation reduces to

$$|\beta|^2 \gg |\alpha|^2 \,, \tag{4.3}$$

which reproduces Eq. (4.1). There is one thing we must check, and that is that the discrete nature of the actual detector does not miss any details of the quadrature phase statistics. The discreteness limits $\tilde{N}$ to consecutive values differing by $1/|\beta|$, but the scale of significant changes in the homodyne statistics of a coherent state is the width of the Gaussian distribution of Eq. (3.14), and this gives the supplementary condition that

$$|\beta| \gg 1 \,, \tag{4.4}$$

which is contained already in Eq. (4.2). Thus, the homodyne statistics of a coherent state closely follow the quadrature phase statistics so long as

$$\langle\hat{n}_b\rangle \gg \langle\hat{n}_a\rangle \,. \tag{4.5}$$

We now present an example for which the condition of Eq. (4.1) is not sufficient in general. We consider a signal in the coherent superposition

$$\mathcal{N}[\cos\theta|\alpha_1\rangle + (\sin\theta)e^{i\phi}|\alpha_2\rangle] \,. \tag{4.6}$$

The density matrix of this state can be written as the sum of three terms

$$\mathcal{N}^2[\cos^2\theta|\alpha_1\rangle\langle\alpha_1| + \sin^2\theta|\alpha_2\rangle\langle\alpha_2| + \sin\theta\,\cos\theta(e^{i\phi}|\alpha_2\rangle\langle\alpha_1| + e^{-i\phi}|\alpha_1\rangle\langle\alpha_2|)] \,, \tag{4.7}$$

each of which will contribute independently to the homodyne statistics. The first two terms (the diagonal terms) will give a contribution to the homodyne statistics that reduce to the quadrature phase statistics respectively when $\langle\hat{n}_b\rangle \gg |\alpha_1|^2$, and $\langle\hat{n}_b\rangle \gg |\alpha_2|^2$.

However, the most important features, which should be carefully reproduced, are the interference effects. The off-diagonal terms in Eq. (4.7) give a contribution that can be read off directly from Eq. (3.10) as being

$$
\begin{aligned}
\frac{\mathcal{N}^2}{\sqrt{2\pi}} \sin(2\theta) \mathrm{Re}\Bigg[& \langle\alpha_2|\alpha_1\rangle e^{i\phi} \exp[-(\tilde{N}-z)^2/2] \\
& \times \Bigg(1 + \frac{\alpha_2^*\alpha_1}{2|\beta|^2}[(\tilde{N}-z)^2-1] + \frac{z}{6|\beta|^2}(\tilde{N}-z)[(\tilde{N}-z)^2-3] + \frac{1}{24|\beta|^2}[(\tilde{N}-z)^4-6(\tilde{N}-z)^2+3] \\
& \quad + O(|\beta|^{-4}) \Bigg) \Bigg] \, . \qquad (4.8)
\end{aligned}
$$

Now if either α_1 or α_2 is larger than about 2, this expression reduces significantly when the interference terms are biggest, i.e., when z is purely imaginary, we get

$$
\frac{\mathcal{N}^2}{\sqrt{2\pi}} \sin(2\theta) \mathrm{Re}\left[\langle\alpha_2|\alpha_1\rangle e^{i\phi} \exp[-(\tilde{N}-z)^2/2] \left(1 - \frac{(z^4 - 4\alpha_2^*\alpha_1 z^2)}{8|\beta^2|} + O(z^2/4|\beta^2|) \right) \right] \, . \qquad (4.9)
$$

The correction terms in Eq. (4.9) are insignificant provided

$$
\langle \hat{n}_b \rangle \gg \frac{|z^4 - 4\alpha_2^*\alpha_1 z^2|}{8}, \frac{|z|^2}{4} \, . \qquad (4.10)
$$

There is one thing left to check in the above conditions, and that is that the discrete nature of the homodyne statistics is capable of showing the fine interference pattern. Since the distance between fringes is $2\pi/|z|$, we will require

$$
\langle \hat{n}_b \rangle \gg \frac{|z|^2}{4\pi^2} \, , \qquad (4.11)
$$

which is already contained within Eq. (4.10).

There are two special cases of the state in Eq. (4.6) that are worthy of note. Firstly, consider the case

$$
\alpha_1 = \alpha \, , \qquad \alpha_2 = -\alpha + \epsilon \, , \quad |\epsilon| < 1/|\alpha| \, . \qquad (4.12)
$$

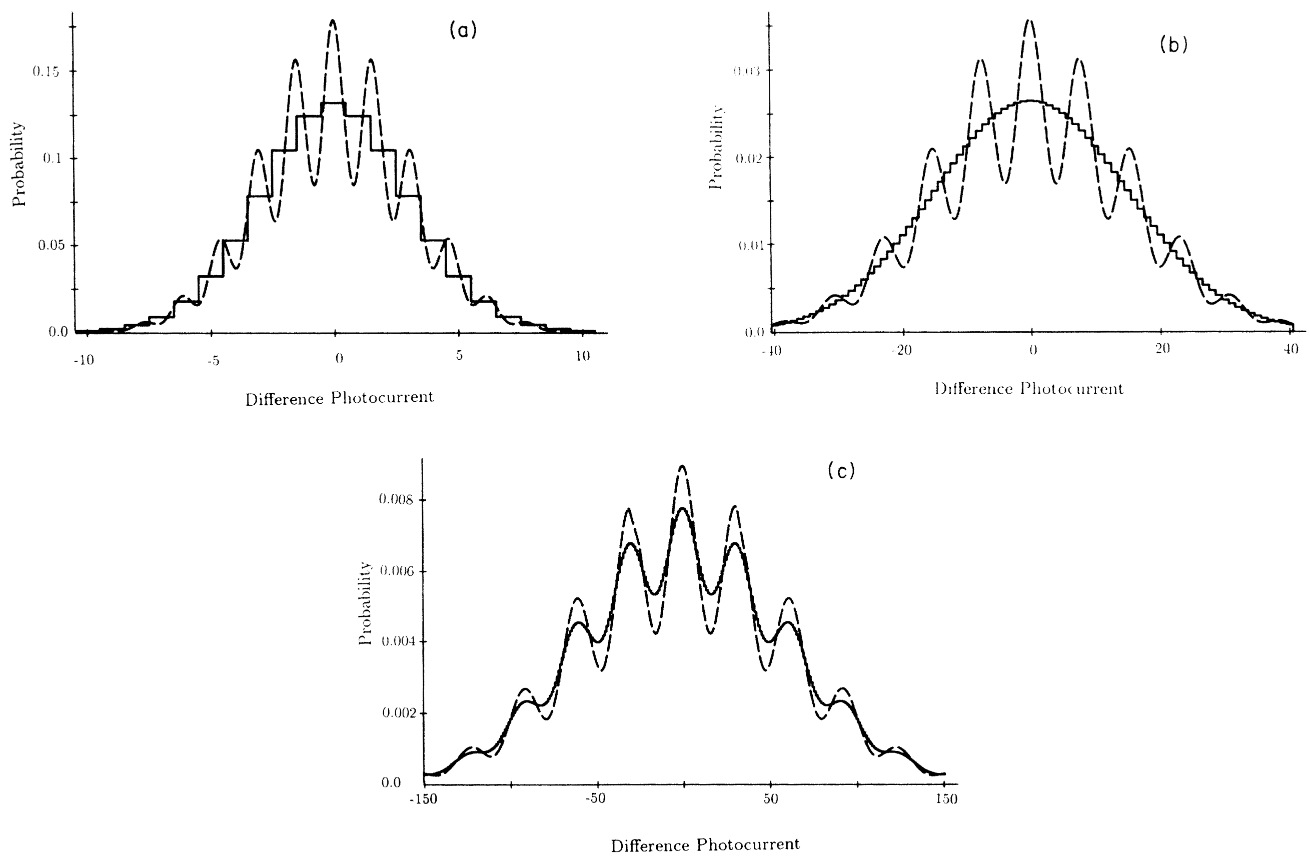

FIG. 2. Plots of the homodyne statistics (solid line) and the quadrature phase statistics (dashed line) for the superposition state $\mathcal{N}(\cos\theta|0\rangle + \sin\theta|\alpha\rangle)$, with $\theta = 10°$, $\langle \hat{n}_a \rangle \approx 4.2$, and (a) $\langle \hat{n}_b \rangle = 9$, (b) $\langle \hat{n}_b \rangle = 225$, and (c) $\langle \hat{n}_b \rangle = 3600$.

In this case, the first condition of Eq. (4.10) may be dropped, and the second condition reduces to

$$\langle \hat{n}_b \rangle \gg \langle \hat{n}_a \rangle \ . \tag{4.13}$$

The other extreme case is when

$$\alpha_1 = 0 \ , \qquad \alpha_2 = \alpha \ . \tag{4.14}$$

In this case, the first condition of Eq. (4.10) dominates, yielding

$$\langle \hat{n}_b \rangle \gg \frac{|\alpha|^4}{8} \approx \frac{\langle \hat{n}_a \rangle^2}{8 \sin^4 \theta} \ . \tag{4.15}$$

To demonstrate this result we show both the homodyne statistics (solid line) and the quadrature phase statistics (dashed line) in Fig. 2 for the state

$$\mathcal{N}(\cos\theta |0\rangle + \sin\theta |\alpha\rangle) \ , \tag{4.16}$$

with $\theta = 10°$ and $\alpha = 12$, i.e., $\langle \hat{n}_a \rangle \approx 4.2$. Figure 2 is shown with the local oscillator strengths of $\langle \hat{n}_b \rangle =$ (a) 9, (b) 225, and (c) 3600. A good match is achieved only when $\langle \hat{n}_b \rangle$ reaches around 40 000, much larger than would have been expected from the simplistic condition of Eq. (4.1).

V. CONCLUSION

The balanced homodyne detector is the paradigm of phase-sensitive detection schemes in quantum optics. We have studied the effect of a quantum local oscillator in a coherent state for this detector when it is coupled to a single signal mode. We have calculated explicit general expressions for the output statistics of the difference current of this detector (applicable even to nonclassical states), and have compared them to the statistics of the variable called the quadrature phase amplitude. We have shown that the simplistic condition that the number of quanta in the local oscillator be much larger than the number in the signal is not sufficient to ensure that the homodyne detector approximates an ideal detector of the quadrature phase amplitude. Furthermore, we have derived explicit conditions for this approximation to be achieved when the signal being detected is a coherent state or a superposition of coherent states.

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