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Heat transport in dielectric crystals at low temperature: A variational formulation based on extended irreversible thermodynamics

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We propose a variational principle applicable to the heat-transport equations derived some years ago by Guyer and Krumhansl [Phys. Rev. **133**, 1411 (1964); **148**, 766 (1966); **148**, 778 (1966)] to describe heat-wave propagation in dielectric crystals at low temperature. The variational principle appears as a generalization of Prigogine's minimum-entropy production criterion. The analysis is based on extended irreversible thermodynamics, which is used to derive the Guyer-Krumhansl equations from macroscopic arguments.

I. INTRODUCTION

In this paper, it is shown that the basic equations proposed by Guyer and Krumhansl[1] to describe heat-wave propagation in dielectric crystals at low temperature may be derived from a variational principle and are compatible with irreversible thermodynamics. The Guyer and Krumhansl equations are of primary importance because they exhibit the existence of a second sound. The interest in second sound started with the early works of Tisza[2] and Landau[3] on liquid helium II. In contrast with ordinary sound wave, second sound is a temperature wave; its propagation in He II has been verified experimentally by Peschkov.[4] It must however be stressed that second sound is not only detectable in He II. Ward and Wilks[5] showed that the existence of second sound depends only on the presence of a phonon gas. Since the latter exists in any solid, second sound should be detectable in any material. Theoretical work by Tsai and MacDonald[6] about propagation of heat pulses in metal at high temperature exhibits also heat waves moving at high velocity. Interesting review papers on heat waves have recently been published by Joseph and Preziosi.[7,8]

The problem raised by the existence of a second sound is of fundamental essence and cannot be interpreted from the classical Fourier law of heat conduction

$$\mathbf{q} = -\lambda \nabla T \ , \tag{1.1}$$

which relates the heat flux vector $\mathbf{q}$ to the temperature gradient ∇T, λ represents the heat conductivity which may depend on the temperature. By combining the Fourier law (1.1) with the energy balance equation

$$c\frac{\partial T}{\partial t} = -\nabla\cdot\mathbf{q} \ , \tag{1.2}$$

wherein c denotes the positive heat capacity per unit volume, one obtains for a constant λ

$$c\frac{\partial T}{\partial t} = \lambda \nabla^2 T \ , \tag{1.3}$$

which is a partial differential equation of diffusive parabolic type. It follows that any temperature perturbation will have an instantaneous effect on any point inside the body or, otherwise stated, that perturbations propagate at infinite velocity. To circumvent this disadvantageous property, Cattaneo[9] proposed to replace the Fourier steady constitutive relation by the rate equation

$$\tau\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\lambda \nabla T \ , \tag{1.4}$$

wherein τ designates a relaxation time. Elimination of $\mathbf{q}$ between (1.4) and the energy balance (1.2) yields the hyperbolic telegraphist equation

$$\tau c\frac{\partial^2 T}{\partial t^2} + c\frac{\partial T}{\partial t} = \lambda \nabla^2 T \ , \tag{1.5}$$

to which corresponds a finite phase velocity given by

$$v = \left[\frac{\lambda}{c\tau}\right]^{1/2} . \tag{1.6}$$

For $\tau = 0$, Cattaneo's law (1.4) reduces to Fourier's law and the phase velocity becomes infinite.

Chester[10] was the first to use the Cattaneo equation to study second-sound propagation from a phonon gas model. He was able to give a quantitative estimate of the relaxation time τ in terms of λ, c, and the phonon velocity in the medium.

The dispersion relation for second sound in solids was derived by Guyer and Krumhansl[1] who based their analysis on Boltzmann equation. Neglecting thermal energy transport by free electrons, which is a reasonable assumption for dielectric crystals, Guyer and Krumhansl obtained the following equations for the temperature and heat flux fields:

$$c\frac{\partial T}{\partial t} = -\nabla\cdot\mathbf{q} \ , \tag{1.7}$$

$$\frac{\partial \mathbf{q}}{\partial t} + \tfrac{1}{3}c c_s^2 \nabla T + \frac{1}{\tau_R}\mathbf{q} = \tau_N \frac{c_s^2}{5}(\nabla^2 \mathbf{q} + 2\nabla\,\nabla\cdot\mathbf{q}) \ . \tag{1.8}$$

c_s is the mean speed of the phonons, τ_R a relaxation time for the momentum nonconserving resistive processes, the so-called umklapp processes, and τ_N the relaxation time associated with the momentum preserving processes. Equation (1.7), is the well-known energy balance while (1.8) is the evolution equation for the heat flux; it reduces to Cattaneo's relation (1.4) when τ_N is zero. However, in contrast with Cattaneo's equation, expression (1.8) does not reduce to Fourier's law in steady conditions.

Our aim in this paper is twofold. Our first purpose is to derive the Guyer and Krumhansl results (1.7) and (1.8) from macroscopic bases. This is achieved within the framework of extended irreversible thermodynamics which has proven recently to be a useful tool for describing material systems out of equilibrium.[11-16] The second objective of this work is to show that the Guyer and Krumhansl relations may be recovered from a variational principle stating that steady states are characterized by a minimum of dissipated energy. This criterion was originally proposed by Prigogine[17-20] who derived this result within the context of classical irreversible thermodynamics. Prigogine's formulation concerned the minimum entropy production and was subordinated to some conditions like linear constitutive equations of the Fourier type and was unable to cope with Guyer and Krumhansl equations. We show here that the steady Guyer-Krumhansl equations can be recovered from the principle of minimum dissipated energy at the condition to work within the framework of extended irreversible thermodynamics. This result is important as it asserts the physical background of this formalism.

The structure of the paper is as follows. In Sec. II, it is shown that the Guyer and Krumhansl fundamental relations can be derived from extended irreversible thermodynamics whose basic tenets are briefly recalled. In Sec. III, a variational principle is proposed: it is seen that the stationarity conditions to be satisfied in order for the dissipated energy to be minimum yields, as Euler-Lagrange equations, the steady parts of Guyer and Krumhansl relations. It is also indicated how to modify the criterion to include the unsteady contributions. Conclusions are drawn in Sec. IV.

II. EXTENDED THERMODYNAMIC DERIVATION OF GUYER-KRUMHANSL EQUATIONS FOR SECOND SOUND

Extended irreversible thermodynamics (EIT) is at the present time recognized as an important branch of thermodynamics. To briefly summarize its contents, consider the problem of heat conduction.

In classical irreversible thermodynamics,[17-20] the single independent state variable is the internal energy per unit volume u (or equivalently the absolute temperature T) which satisfies an evolution equation of the form

$$\frac{\partial u}{\partial t}=-\nabla\cdot\mathbf{q}+r\ , \tag{2.1}$$

wherein r designates a source of heat. To solve (2.1), one needs constitutive relations expressing u and $\mathbf{q}$ in terms of T, for instance,

$$u=cT\ , \tag{2.2}$$

$$\mathbf{q}=-\lambda\nabla T\ . \tag{2.3}$$

From now on, unless otherwise stated, the heat capacity c and the heat conductivity λ are assumed to be constant, i.e., temperature independent. Combination of (2.1), (2.2), and (2.3) leads to the classical parabolic equation (1.3) for the temperature.

The philosophy behind EIT is completely different from that of classical irreversible thermodynamics. Firstly, the space of state variables is enlarged to include, besides the classical variable u, the heat flux vector $\mathbf{q}$ and the flux of the heat $\underline{J}^q$. The variables u (or T), $\mathbf{q}$, and $\underline{J}^q$ are on the same footing and are supposed to obey evolution equations, respectively given by (2.1) and

$$\tau\frac{\partial\mathbf{q}}{\partial t}=-\nabla\cdot\underline{J}^q+\sigma^q\ , \tag{2.4}$$

$$\tau_q\frac{\partial\underline{J}^q}{\partial t}=-\nabla\cdot\underline{J}^{qq}+\underline{\sigma}^{qq}\ , \tag{2.5}$$

where τ and τ_q are relaxation times. Observe that (2.4) and (2.5) have strictly the same form as (2.1) in that they contain both a flux term and a source term. The fluxes $\underline{J}^q$ and $\underline{J}^{qq}$ are second- and third-order tensors, respectively. The source σ^q is a vector and $\underline{\sigma}^{qq}$ a second-order tensor. To solve the problem, it is necessary to formulate constitutive equations for the unknown quantities appearing in (2.1), (2.4), and (2.5), namely u (or T), $\underline{J}^{qq}$, σ^q, and $\underline{\sigma}^{qq}$. In the forthcoming, we use T as independent variable and μ as dependent quantity. Moreover, we restrict the analysis to linear developments and assume that the quantities u, $\underline{J}^{qq}$, σ^q and $\underline{\sigma}^{qq}$ depend linearly on the variables T, $\mathbf{q}$, and $\underline{J}^q$. We do not alter the generality of the formalism by supposing that $\underline{J}^q$ is symmetric. It is also convenient to introduce the notation

$$\mathcal{V}=\{T,\mathbf{q},\underline{J}^q\}\ ,$$

where $\mathcal{V}$ is the whole set of variables in the curly brackets.

General (linear) constitutive equations are given by

$$u=cT\ , \tag{2.6}$$

$$\underline{J}^{qq}=3A(\mathbf{q}\underline{I})^s\ , \tag{2.7}$$

$$\sigma^q=-\mathbf{q}\ , \tag{2.8}$$

$$\underline{\sigma}^{qq}=-\underline{J}^q+(B_1(T)+B_2\mathrm{tr}\underline{J}^q)\underline{I}\ . \tag{2.9}$$

All the coefficients are constants except B_1 which may depend on the temperature. The notation is classical: $\underline{I}$ designates the identity tensor, tr the trace, $\mathbf{q}\underline{I}$ the dyadic product, and the superscript s symmetrization; later on we shall also utilize the double scalar product $\mathbf{L}:\mathbf{M}=L_{ij}M_{ji}$. After substitution of (2.6)–(2.9) in the evolution equations (2.1), (2.4), and (2.5), one finds

$$c\frac{\partial T}{\partial t}=-\nabla\cdot\mathbf{q}+r\ , \tag{2.10}$$

$$\tau\frac{\partial\mathbf{q}}{\partial t}=-\nabla\cdot\underline{J}^q-\mathbf{q} \tag{2.11}$$

$$\tau_q\frac{\partial\underline{J}^q}{\partial t}=-A[(\nabla\cdot\mathbf{q})\underline{I}+2(\nabla\mathbf{q})^s]-\underline{J}^q+(B_1+B_2\mathrm{tr}\underline{J}^q)\underline{I}\ . \tag{2.12}$$

From now on, it is assumed that τ_q is negligibly small. Deriving $\underline{J}^q$ from (2.12) and substituting in (2.11), one obtains the following equation:

$$\tau\frac{\partial \mathbf{q}}{\partial t}=-\lambda\nabla T-\mathbf{q}+A\nabla^2\mathbf{q}+B\nabla(\nabla\cdot\mathbf{q}) , \tag{2.13}$$

wherein the coefficients λ and B are defined, respectively, by

$$\lambda=\frac{1}{1-3B_2}\frac{\partial B_1}{\partial T}, \quad B=2A\frac{1-\frac{1}{2}B_2}{1-3B_2} .$$

Clearly, expressions (2.10) and (2.13) are the same as Guyer and Krumhansl's equations (1.7) and (1.8) at the condition to omit the source term in (2.10) and to perform the following identifications:

$$\tau=\tau_R, \quad \lambda=\tfrac{1}{3}cc_s^2\tau_R, \quad A=\tfrac{1}{5}\tau_N\tau_R c_s^2, \quad B=2A . \tag{2.14}$$

A first derivation of Guyer-Krumhansl equations from EIT was proposed in a previous work.[16] However, the procedure followed in this work was completely different from the present approach as it postulated the existence of mechanical stresses induced by heat flux gradients as well as a generalized Gibbs equation. Here, there is no need to appeal for such extra relations but in contrast, we introduce the flux of the heat flux as additional variable.

Moreover, since we have in mind the formulation of a variational principle, the expression of the entropy production generated by the propagation of second sound in crystals is now calculated. Therefore, we postulate the existence of an entropy function s (Refs. 11–16) per unit volume, obeying the balance equation

$$\frac{\partial s}{\partial t}+\nabla\cdot\mathbf{J}^s-\frac{r}{T}=\sigma^s \tag{2.15}$$

with a non-negative production term

$$\sigma^s\geq 0 . \tag{2.16}$$

Assuming that s and the entropy flux $\mathbf{J}^s$ are given by the constitutive equations

$$s=s(\mathcal{V}), \quad \mathbf{J}^s=\mathbf{J}^s(\mathcal{V}) ,$$

it is an easy matter to calculate the entropy production σ^s by performing the operations indicated in the left-hand side of (2.15). Since T has been selected as variable, it is natural to work with the Helmholtz free energy $f=u-Ts$ as a potential function. Up to second-order terms in the fluxes, f and $\mathbf{J}^s$ are given by

$$f=f_1+\tfrac{1}{2}f_2\mathbf{q}\cdot\mathbf{q}+\tfrac{1}{2}f_3\underline{J}^q:\underline{J}^q , \tag{2.17}$$

$$\mathbf{J}^s=\gamma_0^*\mathbf{q}+\gamma_1^*\underline{J}^q\cdot\mathbf{q}+\gamma_2^*(\mathrm{tr}\underline{J}^q)\mathbf{q} , \tag{2.18}$$

wherein all the coefficients may depend on T. On the other side, EIT (Refs. 11–16) predicts that f_2 and f_3 are proportional to the relaxation times τ and τ_q as confirmed later on by equation (2.24a) for the coefficient f_2. After elimination of $\underline{J}^q$ between (2.12) written in the approximation $\tau_q=0$ and (2.18), the entropy flux can be expressed in terms of the heat flux alone, namely,

$$\mathbf{J}^s=\gamma_0\mathbf{q}+\gamma_1(\nabla\mathbf{q})^s\mathbf{q}+\gamma_2(\nabla\cdot\mathbf{q})\mathbf{q} , \tag{2.19}$$

wherein γ_0, γ_1, and γ_2 stand, respectively, for

$$\gamma_0=\gamma_0^*+\frac{B_1}{1-3B_2}\gamma_1^*+\frac{3B_1}{1-B_2}\gamma_2^* ,$$

$$\gamma_1=-2A\gamma_1^* ,$$

$$\gamma_2=-\frac{A(1+B_2)}{1-3B_2}\gamma_1^*-\frac{5A}{1-B_2}\gamma_2^* .$$

Eliminating the source term r between the energy balance (2.1) and the entropy balance (2.15), one obtains the following expression for the dissipated energy, which is equal to the entropy production times the temperature:

$$T\sigma^s=-s\frac{\partial T}{\partial t}-\frac{\partial f}{\partial t}-\nabla\cdot\mathbf{q}+T\nabla\cdot\mathbf{J}^s\geq 0 . \tag{2.20}$$

Using the chain derivation rule to calculate $\partial f/\partial t$ and the relations (2.13), (2.17), and (2.19), expression (2.20) becomes up to second-order terms in the heat flux

$$\begin{aligned}T\sigma^s=&-\left[\frac{\partial f}{\partial T}+s\right]\frac{\partial T}{\partial t}-\frac{1}{\tau}f_2\mathbf{q}\cdot(-\lambda\nabla T-\mathbf{q}+A\nabla^2\mathbf{q}+B\nabla\,\nabla\cdot\mathbf{q})-\nabla\cdot\mathbf{q}+T\left[\frac{\partial\gamma_0}{\partial T}\nabla T\cdot\mathbf{q}+\gamma_0\nabla\cdot\mathbf{q}\right]\\&+T\gamma_1(\nabla\mathbf{q}:\nabla\mathbf{q}^T+\mathbf{q}\cdot\nabla^2\mathbf{q})+T\gamma_2[(\nabla\cdot\mathbf{q})(\nabla\cdot\mathbf{q})+\mathbf{q}\cdot\nabla(\nabla\cdot\mathbf{q})]\geq 0 .\end{aligned} \tag{2.21}$$

It is important to stress that (2.27) is linear in the derivative $\partial T/\partial t$. Following a procedure classical in rational thermodynamics,[21] this quantity can be assigned arbitrary values because there is an energy supply r that ensures that the energy balance is satisfied. Therefore the positiveness of σ^s could be violated except the coefficient of $\partial T/\partial t$ is zero, from which follows the classical result

$$\frac{\partial f}{\partial T}+s=0 . \tag{2.22}$$

In view of this result and after grouping the various second-order terms, inequality (2.21) reduces to

$$\begin{aligned}T\sigma^s=&\frac{1}{\tau}f_2\mathbf{q}\cdot\mathbf{q}+\left[\frac{\lambda}{\tau}f_2+T\frac{\partial\gamma_0}{\partial T}\right]\mathbf{q}\cdot\nabla T-\left[\frac{A}{\tau}f_2-T\gamma_1\right]\mathbf{q}\cdot\nabla^2\mathbf{q}-\left[\frac{B}{\tau}f_2-T\gamma_2\right]\mathbf{q}\cdot\nabla(\nabla\cdot\mathbf{q})\\&+T\gamma_1\nabla\mathbf{q}:\nabla\mathbf{q}^T+T\gamma_2(\nabla\cdot\mathbf{q})(\nabla\cdot\mathbf{q})-\nabla\cdot\mathbf{q}(1-T\gamma_0)\geq 0 .\end{aligned} \tag{2.23}$$

The positiveness of σ^s allows us also to identify the coefficients f_2, γ_0, γ_1, and γ_2 as

$$f_2=\frac{\tau}{\lambda T},\quad \gamma_0=\frac{1}{T},\quad \gamma_1=\frac{A}{\lambda T^2},\quad \gamma_2=\frac{B}{\lambda T^2}\ ; \tag{2.24}$$

moreover, they have to comply the inequalities

$$\frac{f_2}{\tau}>0,\quad \gamma_1>0,\quad \gamma_2>0\ . \tag{2.25}$$

In terms of the notation introduced by Guyer and Krumhansl [see relation (2.14)], the results (2.24) are written as

$$f_2=\frac{3}{cc_s^2},\quad \gamma_1=\frac{3}{5}\frac{\tau_N}{cT^2},\quad \gamma_2=\frac{6}{5}\frac{\tau_N}{cT^2}\ , \tag{2.26}$$

In view of the inequalities (2.25) and the general result $c>0$, it is inferred that

$$\tau_R>0,\quad \tau_N>0\ . \tag{2.27}$$

To our knowledge, it is the first time that it is proved on macroscopic grounds that the relaxation times τ_R and τ_N are positive. The results (2.26) are also worth to be mentioned as they indicate that the coefficients f_2, γ_1, and γ_2 are no longer undefined quantities but are related to the basic physical parameters c, c_s, τ_R, and τ_N. The final expressions of $T\sigma^s$ and $\mathbf{J}^s$ are thus given by

$$T\sigma^s=\frac{3}{\tau_R cc_s^2}\mathbf{q}\cdot\mathbf{q}+\frac{3}{5}\frac{\tau_N}{c}[\nabla\mathbf{q}:\nabla\mathbf{q}^T+2(\nabla\cdot\mathbf{q})(\nabla\cdot\mathbf{q})]\ , \tag{2.28}$$

$$\mathbf{J}^s=\frac{\mathbf{q}}{T}+\frac{3}{5}\frac{\tau_N}{cT^2}[(\nabla\mathbf{q})^s\cdot\mathbf{q}+2(\nabla\cdot\mathbf{q})\mathbf{q}]\ . \tag{2.29}$$

Compared to classical irreversible thermodynamics, $T\sigma^s$ and $\mathbf{J}^s$ contain supplementary terms arising from the presence of the "normal" relaxation time τ_N.

III. A VARIATIONAL FORMULATION

Variational principles have always played a privileged role in physics.[22–24] This is justified because a variational criterion presents the advantage of concision: a single equation stands for a set of differential equations, initial and boundary conditions. From a practical point of view, variational principles provide specific methods, like the Rayleigh-Ritz method, for solving the differential equations governing the physical process. Besides their power of synthesis and their utility in numerical analysis, variational formulations are also frequently interesting from a physical point of view because in many problems, the functional submitted to variation possesses a physical meaning.

A. Steady state

A variational principle meeting the above properties is Prigogine's minimum-entropy production principle expressing that purely dissipative processes (without convection) evolve in such a way that in the steady state, the total entropy production is stationary, truly a minimum. Mathematically, this is expressed by

$$\delta\int\sigma_P^s dV=0\ , \tag{3.1}$$

wherein δ is the usual variational symbol, dV an elementary volume element, and σ_P^s the entropy production per unit volume derived from classical irreversible thermodynamics; for heat conduction, one has[18,19]

$$\sigma_P^s=\mathbf{q}\cdot\nabla T^{-1}\ . \tag{3.2}$$

Prigogine's principle is only applicable to processes governed by Fourier's law with a heat conductivity λ varying like T^{-2}. In the case of λ proportional to T^{-1}, Prigogine's principle must be changed into[20]

$$\delta\phi_P(T)=0 \tag{3.3a}$$

with

$$\phi_P(T)=\int T\sigma_P^s dV\ . \tag{3.3b}$$

It is a simple exercise to check that the corresponding Euler-Lagrange equation is

$$\nabla\cdot\mathbf{q}=0\ . \tag{3.4}$$

It is now shown that the minimum dissipated energy principle still holds for describing the Guyer-Krumhansl steady equations. The difference with Prigogine's criterion is that the expression of the dissipated energy to be used is not that derived from classical irreversible thermodynamics but rather expression (2.28) of the dissipated energy obtained in the context of EIT. Moreover, a look at the Guyer-Krumhansl steady equations

$$\nabla\cdot\mathbf{q}=0\ , \tag{3.5}$$

$$\frac{1}{\tau_R}\mathbf{q}+\tfrac{1}{3}cc_s^2\nabla T=\tau_N\frac{c_s^2}{5}(\nabla^2\mathbf{q}+2\nabla\,\nabla\cdot\mathbf{q}) \tag{3.6}$$

indicates that the energy balance equation (3.5) may be considered as a constraint to be satisfied by the heat flux vector whose behavior is governed by Eq. (3.6). Equation (3.5) plays a role similar to the incompressibility condition in fluid mechanics.

The heat flux that satisfies the Guyer-Krumhansl equation is the one corresponding to the minimum of dissipated energy:

$$\delta\int T\sigma^s dV=0\ . \tag{3.7}$$

To prove the stationary character, we shall introduce a Lagrange multiplier γ. Consider the variational integral

$$\phi(\mathbf{q})=\int\left[\frac{3}{2\tau_R cc_s^2}\mathbf{q}\cdot\mathbf{q}+\frac{3}{10}\frac{\tau_N}{c}[\nabla\mathbf{q}:\nabla\mathbf{q}^T+2(\nabla\cdot\mathbf{q})(\nabla\cdot\mathbf{q})]\right]dV-\int\gamma\nabla\cdot\mathbf{q}\,dV \tag{3.8}$$

and determine the necessary condition for ϕ to be stationary under arbitrary variations of $\mathbf{q}$, admitting that $\mathbf{q}$ takes prescribed values at the boundaries:

$$\delta\mathbf{q}=\mathbf{0} \text{ (at the boundaries)} . \tag{3.9}$$

Calculation of the first variation of (3.8) and use of the divergence theorem leads to

$$\delta\phi=\int\left[\left[\frac{3}{\tau_R c c_s^2}\mathbf{q}-\frac{3}{5}\frac{\tau_N}{c}[\nabla^2\mathbf{q}+2\nabla(\nabla\cdot\mathbf{q})]+\nabla\gamma\right]\delta\mathbf{q}-(\nabla\cdot\mathbf{q})\delta\gamma\right]dV . \tag{3.10}$$

Clearly, $\delta\phi=0$ under the conditions that

$$\nabla\cdot\mathbf{q}=0 \tag{3.11}$$

and

$$\frac{3}{\tau_R c c_s^2}\mathbf{q}+\nabla\gamma-\frac{3}{5}\frac{\tau_N}{c}(\nabla^2\mathbf{q}+2\nabla(\nabla\cdot\mathbf{q}))=0 , \tag{3.12}$$

which are nothing other than the Guyer-Krumhansl equations after that the Lagrange multiplier has been identified with the absolute temperature T. The minimum property of the variational principle (3.7) is evident as the dissipated energy (2.28) is a quadratic form.

The physical meaning of the function ϕ is thus clear as it represents the dissipated energy calculated in extended irreversible thermodynamics.

B. The nonsteady evolution

It should also be added that there is no problem to obtain from a variational equation the complete Guyer and Krumhansl equations with the unsteady contributions $\partial T/\partial t$ and $\partial\mathbf{q}/\partial t$. The variational principle whose Euler-Lagrange relations are the Guyer and Krumhansl unsteady expressions is

$$\begin{aligned}\delta\int\Bigg[&\tfrac{1}{2}cT(\mathbf{x},t)\circ\frac{\partial T(\mathbf{x},t)}{\partial t}+\mathbf{q}(\mathbf{x},t)\circ\nabla T(\mathbf{x},t)+\tfrac{1}{2}cT(\mathbf{x},0)T(\mathbf{x},t)-cT_0(\mathbf{x})T(\mathbf{x},t)+\frac{3}{cc_s^2}\mathbf{q}(\mathbf{x},t)\circ\frac{\partial\mathbf{q}(\mathbf{x},t)}{\partial t}\\&+\frac{3}{2cc_s^2\tau_R}\mathbf{q}(\mathbf{x},t)\circ\mathbf{q}(\mathbf{x},t)+\frac{3}{cc_s^2}[\tfrac{1}{2}\mathbf{q}(\mathbf{x},0)\cdot\mathbf{q}(\mathbf{x},t)-\mathbf{q}_0(\mathbf{x})\cdot\mathbf{q}(\mathbf{x},t)]\\&+\frac{3}{10}\frac{\tau_N}{c}[\nabla\mathbf{q}(\mathbf{x},t):\circ\nabla\mathbf{q}(\mathbf{x},t)^T+2\nabla\cdot\mathbf{q}(\mathbf{x},t)\circ\nabla\cdot\mathbf{q}(\mathbf{x},t)]\Bigg]dV\,dt=0 ,\end{aligned} \tag{3.13}$$

wherein the $\circ$ denotes the convolution product

$$u(\mathbf{x},t)\circ v(\mathbf{x},t)=\int_0^t u(\mathbf{x},t-t')v(\mathbf{x},t')dt' . \tag{3.14}$$

$T_0(\mathbf{x})$ and $\mathbf{q}_0(\mathbf{x})$ are the initial values of $T(\mathbf{x},t)$ and $\mathbf{q}(\mathbf{x},t)$, respectively. Of course, it would be illusory to try to assign a physical signification to the principle (3.13). Nevertheless, such a result may be useful for solving practical unsteady problems.

IV. SUMMARY AND FINAL REMARKS

In the first part of this work, the basic equations governing heat waves propagation in dielectric crystals at low temperature are established within the framework of a macroscopic theory: extended irreversible thermodynamics. In this formalism, the heat flux $\mathbf{q}$ and its flux $\underline{J}^q$ are considered as independent variables just like the absolute temperature T. It is shown that the evolution equations derived for T and $\mathbf{q}$ are the same as those obtained by Guyer and Krumhansl, at the condition to let the relaxation time associated to $\underline{J}^q$ tend to zero. The corresponding entropy flux and entropy production are also calculated. With respect to their classical expressions derived in classical irreversible thermodynamics, the entropy flux and the entropy production contain additional contributions due to the nonzero relaxation time of the phonon momentum preserving processes. It should be observed that all these results are obtained in a straightforward way following strictly the line of thought of extended irreversible thermodynamics. In particular, the basic coefficients appearing in the formalism are well defined and expressed in terms of the heat capacity c, the sound velocity c_s, and the relaxation times τ_N and τ_R associated to the normal and nonconserving momentum processes: these four quantities represent the four parameters of the theory.

The second original contribution of this paper concerns a variational formulation of the Guyer and Krumhansl equations. A first variational principle is proposed whose Euler-Lagrange equations are the steady Guyer and Krumhansl equations. The functional submitted to variation possesses the interesting property to be interpretable in physical terms as it represents the dissipated energy resulting from the heat-wave process in the crystal. This variational criterion provides a generalization of Prigogine's celebrated minimum entropy production principle which was formulated some years ago in the context of classical irreversible thermodynamics. We have also constructed another variational principle applicable to the unsteady heat propagation. Its mathematical structure is more complicated as it involves convolution products; moreover, we were unable to give it a physical interpretation. Nevertheless such a formulation remains useful for solving numerical problems via variational methods, like the Rayleigh-Ritz method, which are appreciated for their simplicity and their powerfulness. Finally, it should be recalled that most of the variational principles found in irreversible thermodynamics, like the Prigogine principle of minimum entropy production at fixed fluxes,[18] Glansdorff-Prigogine local potential,[20] Biot,[25] Gyarmati,[26] Rosen,[27] Lebon-Lambermont,[28] and Ziegler[29] criterions have been classified as quasivariational principles.[23,30] This means that during the variational procedure some quantities are frozen, i.e., not submitted to variation; of course such a procedure is completely *ad hoc.* It is worth to mention that the variational principles derived in the present work do not pertain to the class of so-called quasivariational principles: in contrast they are "true" variational principles because the variation is taken with respect to all the variables.

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