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Steady-state ensemble for the complex Ginzburg-Landau equation with weak noise

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The complex Ginzburg-Landau equation with weak noise, the normal form of the amplitude equation for the order parameter in a spatially distributed system undergoing a continuous Hopf bifurcation, is solved in certain limits for its time-independent probability distribution, which governs the steady state in one spatial dimension. The method used consists of solving the Hamilton-Jacobi equation of the nonequilibrium potential associated with the steady-state distribution. The solution is obtained in the limit of weak spatial diffusion of the order parameter. The nonequilibrium potential serves as a Lyapunov functional for the order-parameter field. We use our result to discuss the Newell-Kuramoto instability and the Eckhaus-Benjamin-Feir instability in one spatial dimension, and to calculate potential barriers of the saddles separating plane-wave attractors. The latter ones provide us with a global measure of stability for these attractors.

I. INTRODUCTION

Macroscopic systems, e.g., fluids, driven sufficiently strongly, e.g., by temperature gradients or shear forces, exhibit a plethora of instabilities, in which the asymptotic state of the system changes qualitatively, e.g., from time independent to time dependent, or from symmetric to asymmetric, or from regular (in space and/or time) to chaotic.

If the asymptotic state for weak external and time-independent driving is simple, e.g., time independent (a fixed point in an appropriate state space), the first instability which occurs when the time-independent driving is increased is usually also simple and, in many cases, it takes the form of a supercritical Hopf bifurcation,[1] in which the fixed point bifurcates into a limit cycle. If $\psi(t)$ is the complex amplitude of the limit cycle, its time dependence sufficiently close to the bifurcation point is governed by the normal form

$$\dot{\psi}=a\psi-b|\psi|^2\psi \ , \tag{1.1}$$

where a is a parameter that can be taken real without restriction of generality by splitting off the frequency of the limit cycle. The parameter a changes sign at the bifurcation point and is positive or negative in the supercritical or subcritical domain, respectively. The parameter $b=b_r+ib_i$ is complex, in general. Its real part b_r, assumed to be positive from now on, describes a nonlinear saturation of the amplitude of the limit cycle at a finite value $|\psi|=(a/b_r)^{1/2}$ in the supercritical domain. In spatially extended systems the amplitude $\psi(\mathbf{x},t)$ depends on space and time and the normal form (1.1) has to be amended by a diffusion term $D\nabla^2\psi(\mathbf{x},t)$, where $D=D_r+iD_i$ is complex, in general. This way one obtains the complex time-dependent Ginzburg-Landau (TDGL) equation.[2-14] We shall assume that $D_r>0$. (In some cases a convective spatial derivative term $-\mathbf{v}\cdot\nabla\psi$ has also to be included, where $\mathbf{v}$ is a real constant vector,[11,12] but for simplicity we shall not consider such a term in the following.) If, furthermore, the system is subject to noise, which is short ranged and homogeneous in its statistical properties in space and time, the equation of motion takes the form

$$\dot{\psi}=a\psi-b|\psi|^2\psi+D\nabla^2\psi+\xi(\mathbf{x},t) \ . \tag{1.2}$$

$\xi(\mathbf{x},t)$ is the complex noise source, which as usual we shall take as Gaussian and δ correlated in space and time

$$\langle \xi(\mathbf{x},t)\xi^*(\mathbf{x}',t')\rangle=\eta Q\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \ . \tag{1.3}$$

$\langle\ \rangle$ indicates the average over the noise. ηQ is a real, positive constant, where η is a formal "smallness parameter," i.e., a bookkeeping device in an asymptotic analysis of Eqs. (1.2) and (1.3) in the weak-noise limit $\eta\to 0$. In the following, the vector character of x is assumed to be understood, if necessary, and will no longer be made explicit by the notation.

A special case of the normal form (1.2) applies also to instabilities of systems in thermodynamic equilibrium (phase transitions for systems without conservation laws and a two-component order parameter). In that case detailed balance under the time reversal transformation $t\to -t$, $\psi\to\psi^*$ holds, which restricts Eq. (1.2) to $b=b^*$, $D=D^*$. As an important consequence of this restriction an explicit form for the thermodynamic potential $\Phi_{\rm th}$ as a functional of the order parameter can be written down,

$$\Phi_{\rm th}=\Phi_{\rm GL}(\{\psi,\psi^*\}) \equiv \frac{2}{Q}\int_V dx(-a|\psi|^2+\tfrac{1}{2}b_r|\psi|^4+D_r|\nabla\psi|^2) \ , \tag{1.4}$$

which generates the drift in Eq. (1.2) via

$$\dot{\psi}(x)=-\tfrac{1}{2}Q\frac{\delta\Phi}{\delta\psi^*(x)}+\xi(x,t) \ , \tag{1.5}$$

with $\Phi=\Phi_{\rm th}$. The simultaneous appearance of Q in Eqs.

(1.5) and (1.3) is an expression of the fluctuation-dissipation theorem for the present case. Furthermore, the form (1.5) guarantees that the equilibrium probability distribution functional is exactly given by the Boltzmann-Einstein formula

$$W(\{\psi,\psi^*\})=\text{const}\times\exp\left[-\frac{\Phi(\{\psi,\psi^*\})}{\eta}\right]. \tag{1.6}$$

The special case (1.4) is known as the Ginzburg-Landau functional. In the general case of Eq. (1.2) with complex b and D the results (1.4) and (1.5) no longer apply. On general grounds[15] a time-independent probability density is approached in the steady state even for complex b and D. Therefore, one may define a "nonequilibrium potential" via Eq. (1.6), i.e., via

$$\Phi(\{\psi,\psi^*\})=-\lim_{\eta\to 0}\eta\ln W(\{\psi,\psi^*\}) \tag{1.7}$$

and study its properties. It is easy to show that Eq. (1.5) is then generalized to

$$\dot\psi(x)=-\tfrac{1}{2}Q\frac{\delta\Phi}{\delta\psi^*(x)}+R(x)+\xi(x,t)\ , \tag{1.8}$$

where $R=R(\{\psi,\psi^*\})$ must satisfy

$$\int dx\left[R(x)\frac{\delta\Phi}{\delta\psi(x)}+R^*(x)\frac{\delta\Phi}{\delta\psi^*(x)}\right]=0\ . \tag{1.9}$$

The general Eq. (1.8) resembles the thermodynamic form (1.5) quite closely, including the fluctuation dissipation theorem. It does even more so, if the additional term $R(x)$ in (1.8) is compared with the reversible drift term $R(x)=ia_i\psi(x)$, which is also allowed in Eq. (1.5), i.e., in thermal equilibrium, but has been transformed away already in Eq. (1.2) by our choice of a as real. A thermodynamic R term in Eq. (1.5) must transform as $R\to -R^*$ under time reversal; it must satisfy Eq. (1.9), but, in addition, also

$$\int dx\left[\frac{\delta R}{\delta\psi}+\frac{\delta R^*}{\delta\psi^*}\right]=0\ , \tag{1.10}$$

which expresses Liouville's theorem, i.e., the "incompressibility" of the reversible flow R in the state space. In the nonequilibrium case, R in Eq. (1.8) does not transform as $R\to -R^*$ under the microscopically defined transformation of time reversal and Eq. (1.10) will *not*, in general, be satisfied.

We may mention at this point the special case $D_-\equiv D_r b_i - D_i b_r = 0$, where Eq. (1.2), in infinitely extended systems, is equivalent to Eqs. (1.8) and (1.9) with the potential Φ given by the Ginzburg-Landau expression (1.4) (Ref. 16) and $R(x)$ given by $R(x)=ib_i[-|\psi|^2\psi+(D_r/b_r)\nabla^2\psi]$. In the general case, the property (1.7) determines the nonequilibrium potential uniquely, as long as the steady-state probability density is unique, which is true for Eq. (1.2). Equations (1.2), (1.8), and (1.9) determine Φ as a solution of the "Hamilton-Jacobi" equation

$$\int dx\left[\tfrac{1}{2}Q\left|\frac{\delta\Phi}{\delta\psi}\right|^2+(a-b|\psi|^2+D\nabla^2)\psi\frac{\delta\Phi}{\delta\psi}+\text{c.c.}\right]=0\ , \tag{1.11}$$

together with the boundary condition, following from Eq. (1.7), that Φ is minimal in the attractors of the deterministic system, Eq. (1.2) with $\xi=0$. As is well known from the Hamilton-Jacobi theory of classical mechanics, the solution of Eq. (1.11) can be expressed by the minimum of an action integral. In the present context this takes the form

$$\Phi(\{\psi,\psi^*\})=\min\left[\int_{\{\psi(-\infty),\psi^*(-\infty)\}\in A}^{\{\psi,\psi^*\}}L(\{\psi(\tau),\psi^*(\tau),\dot\psi(\tau),\dot\psi^*(\tau)\})d\tau+C(A)\right]\ , \tag{1.12}$$

where the Lagrangian L is simply related to the form of (1.11) and reads

$$L=\frac{1}{Q}\int dx\,|\dot\psi-a\psi+b|\psi|^2\psi-D\nabla^2\psi|^2\ . \tag{1.13}$$

The boundary conditions on the integral in (1.12) are again dictated by Eq. (1.7). The lower boundary of the integral is taken in the infinite past, where ψ,ψ^* take values on the attractor A. The upper boundary is taken at $\tau=0$, where $\{\psi,\psi^*\}$ denote an arbitrary point in the state space. The parameter $C(A)$ gives the value of the potential in the attractor A and has to be determined independently.[17] The minimum in Eq. (1.12) is taken over all paths connecting the initial state on A and the final state, and over all simultaneously existing attractors A.

In the present work it is our aim to construct the nonequilibrium potential for the complex TDGL equation in the physically important limiting case of weak spatial diffusion, where Eqs. (1.11)–(1.13) can be evaluated asymptotically in some small parameter. Results of our work have already been briefly reported in Ref. 18.

For a number of discrete (lumped) dynamical systems nonequilibrium potentials have been determined in earlier work[19] either from the Hamilton-Jacobi equation or from the actual integral. The essential new point of the present work is the fact, that the dynamical system is spatially distributed, i.e., the dynamical system is a nonlinear partial differential equation and Eq. (1.11) is a functional partial differential equation.

A number of earlier papers have also dealt with the problem of nonequilibrium potentials in spatially distributed systems. In Ref. 20 a modal expansion of ψ in Eq. (1.2) was employed and Φ was constructed as a power series in the mode amplitudes up to fourth order [cf. Eqs. (4.51)–(4.54) of Ref. 20]. In the limit of dominating diffusion it reduces to the result derived in Appendix A. Another power series expansion of Φ up to the fourth order in the Fourier amplitudes of ψ equivalent to Ref. 20 was also given in the work by Walgraef, Dewel, and Borckmans.[21] Szépfalusy and Tél[16] have determined the nonequilibrium potential of Eq. (1.2) in a local quadratic expansion around the attractors $\psi=0$ for $a<0$ and

$\psi=(a/b_r)^{1/2}$ for $a>0$. As will be seen below their result for $a>0$ can be used to determine $C(A)$ in Eq. (1.12). This is not so with the other polynomial approximants mentioned above since their validity is restricted to a close vicinity of the state $\psi\equiv 0$. Fluctuations in spatially distributed systems undergoing bifurcations have also been studied on the basis of the master equation.[22-24] Again, power-series expansions of the nonequilibrium potential in the mode amplitudes up to fourth order have been worked out also for the Hopf bifurcation case.[23]

In contrast to this previous work on nonequilibrium potentials in spatially distributed systems, our method of construction will not be based on, and is more general than, power-series expansions in the field amplitude. For a general discussion of the limitations of the power-series method cf. Ref. 25. Instead we shall construct the functional Φ for ψ varying on length scales large compared to the coherence length $(D_r/a)^{1/2}$ of Eq. (1.2). Formally, this is realized by expanding Φ, with respect to D, which we call the "small D" or "weak-diffusion" expansion. In Appendix A we also apply our methods to the case of rapidly varying ψ, where the diffusion term in Eq. (1.2) dominates compared to the nonlinear term.

In the supercritical domain $a>0$, Eq. (1.2) (for the moment considered for $\xi=0$) has a number of further instabilities. As shown by Newell[4] and Kuramoto,[6] the spatially constant arbitrary phase of ψ on the attractor $|\psi|=(a/b_r)^{1/2}$ may become unstable in a neighborhood of $k=0$ under appropriate conditions, leading to "phase turbulence."[6,9,10,14] Furthermore, in addition to the spatially homogeneous attractor, further attractors exist where ψ is a traveling wave which may become unstable under appropriate conditions, in particular, if for a given a the wave number k is sufficiently large, or, if for given k the control parameter a decreases (Eckhaus-Benjamin-Feir instability[26,8]). The influence of noise on these further instabilities is also of great interest. In the second part of this work we therefore apply our results to their study.

The paper is organized as follows. In Sec. II we connect the concept of nonequilibrium potential with the steady-state solution of the corresponding Fokker-Planck functional equation. Section III is devoted to the calculation of the potential in the limit of weak spatial diffusion up to second order, at least in the phase fluctuations. Details of the evaluation of the action integral and of the value $C(A)$ on the attractor for $a>0$ are delegated to Appendixes B and C, respectively. The plane-wave attractors and their stability are investigated in Sec. IV. The potential barriers among neighboring plane-wave attractors are determined in the framework of the weak diffusion expansion. These barriers characterize the global stability of the attractors. The paper is concluded, in Sec. V, by a few remarks on general properties of the nonequilibrium potential in spatially extended systems.

II. STEADY-STATE ENSEMBLE AND NONEQUILIBRIUM POTENTIAL

The probability density $W(\{\psi,\psi^*\},t)$ of the stochastic field ψ satisfies the Fokker-Planck functional equation [where $\delta/\delta\psi(x)$ denotes the functional derivative with respect to $\psi(x)$, formally for constant $\psi^*(x)$]

$$\frac{\partial W}{\partial t}=\int dx\left\{-\frac{\delta}{\delta\psi(x)}[(a-b|\psi|^2)\psi+D\nabla^2\psi]W+\frac{\eta}{2}Q\frac{\delta^2 W}{\delta\psi(x)\delta\psi^*(x)}+\text{c.c.}\right\}. \tag{2.1}$$

The second-order functional derivative term in Eq. (2.1) is singular and needs regularization at short distances where we introduce a cutoff λ. We are interested in the time-independent solution of Eq. (2.1), which gives the probability density of the steady-state ensemble. Making the ansatz

$$W=\exp[\chi-\Phi/\eta+O(\eta)]\ , \tag{2.2}$$

we find the equations

$$\int dx\left\{\frac{Q}{2}\frac{\delta\Phi}{\delta\psi(x)}\frac{\delta\Phi}{\delta\psi^*(x)}+[(a-b|\psi|)^2\psi+D\nabla^2\psi]\frac{\delta\Phi}{\delta\psi}+\text{c.c.}\right\}=0 \tag{2.3}$$

and

$$\int dx\left[\left\{(a-b|\psi|^2)\psi+D\nabla^2\psi+Q\frac{\delta\Phi}{\delta\psi^*}\right\}\frac{\delta\chi}{\delta\psi}+\frac{Q}{2}\frac{\delta^2\Phi}{\delta\psi\delta\psi^*}+(a-2b|\psi|^2+D\nabla^2)\lambda^{-d}+\text{c.c.}\right]=0\ . \tag{2.4}$$

In Eq. (2.4) we made use of the short-distance cutoff λ to put formally

$$\frac{\delta\psi(x)}{\delta\psi(x)}=\lambda^{-d}\ , \tag{2.5}$$

where d is the spatial dimensionality of the system. The potential Φ in Eq. (2.2) satisfies Eq. (1.7). It is the nonequilibrium potential, the quantity determining the leading part of W asymptotically for $\eta\to 0$. Therefore, (2.3) is identical to (1.11). The function χ in Eq. (2.2) determines the "prefactor" of the steady-state ensemble.

III. THE LIMIT OF WEAK SPATIAL DIFFUSION

In the bulk of this paper we consider the case of weak spatial diffusion. Formally, we assume D (both real and imaginary parts) to be small, of order ϵ, and other parameters, as, e.g., $|b|$, to be of order unity. Then it is possible to expand both the nonequilibrium potential Φ and the prefactor χ in powers of ϵ. We write

$$\begin{aligned}\Phi&=\Phi_0+\Phi_1+\cdots\ ,\\ \chi&=\chi_0+\chi_1+\cdots\ ,\end{aligned} \tag{3.1}$$

where Φ_1 and χ_1 are expected to be of order ϵ (or $|D|$).

A. Solution of the Hamilton-Jacobi equation

The Hamilton-Jacobi equation (2.3) and the prefactor-equation (2.4) in leading order ($D=0$) then read

$$\int dx \left[\frac{Q}{2}\left|\frac{\delta\Phi_0}{\delta\psi}\right|^2+\frac{\delta\Phi_0}{\delta\psi}(a-b|\psi|^2)\psi+\text{c.c.}\right]=0\ ,$$
$$\int dx \left[\left[(a-b|\psi|^2)\psi+Q\frac{\delta\Phi_0}{\delta\psi^*}\right]\frac{\delta\chi_0}{\delta\psi}+\text{c.c.}\right]$$
$$=-\int dx\left[\frac{Q}{2}\frac{\delta^2\Phi_0}{\delta\psi\delta\psi^*}+(a-2b|\psi|^2)\lambda^{-d}+\text{c.c.}\right]\ . \tag{3.2}$$

It is easy to check that the solution for Φ_0 is

$$\Phi_0=\frac{2}{Q}\int dx\left[-a+\frac{b_r}{2}|\psi|^2\right]|\psi|^2\ , \tag{3.3}$$

which is, in fact, minimal in the deterministic attractor to zeroth order $\psi\equiv0$ for $a<0$ and $|\psi|\equiv(a/b_r)^{1/2}$ for $a>0$. The zeroth-order solution does not contain any spatial derivative; therefore it is advantageous to work in the real-space representation also when carrying the calculation to the next order.

Inserting Eq. (3.3) for Φ_0 the inhomogeneity of the equation for χ_0 vanishes, and we have

$$\chi_0=\text{const.}$$

We shall content ourselves with this result of zeroth order for the prefactor, but we proceed to compute the potential to first order.

Keeping in (1.11) all terms of order ϵ we obtain for Φ_1

$$\int dx\left[Q\frac{\delta\Phi_0}{\delta\psi^*}\frac{\delta\Phi_1}{\delta\psi}+(a-b|\psi|^2)\psi\frac{\delta\Phi_1}{\delta\psi}+\text{c.c.}\right]$$
$$=-\int dx\left[D\frac{\delta\Phi_0}{\delta\psi}\nabla^2\psi+\text{c.c.}\right]\ . \tag{3.4}$$

By inserting the functional derivative of Φ_0 one finds

$$\int dx\left[(-a+b^*|\psi|^2)\psi\frac{\delta\Phi_1}{\delta\psi}+\text{c.c.}\right]$$
$$=\frac{2}{Q}\int dx[(a-b_r|\psi|^2)(D\psi^*\nabla^2\psi+\text{c.c.})]\ . \tag{3.5}$$

This *linear* functional-differential equation can be solved by the method of characteristics. Since in the homogeneous part there is no coupling between neighboring coordinates x, the characteristic equations do not contain x explicitly. By introducing a fictitious time τ they can be written as

$$\frac{d\psi}{d\tau}=(-a+b^*|\psi|^2)\psi \tag{3.6}$$

for all x. This has to be solved in an interval $\tau_0<\tau<\tau_e$ by fixing the value of ψ at the end τ_e. Without restricting generality we can choose $\tau_e=0$ since the problem is autonomous. The solution is easily found to be

$$\psi(\tau)=\psi\left[\frac{b_r|\psi|^2}{a}+\left[1-\frac{b_r|\psi|^2}{a}\right]e^{2a\tau}\right]^{-1/2+ib_i/2b_r}e^{-ia(b_i/b_r)\tau}\ ,\quad \tau\le0\ , \tag{3.7}$$

where ψ denotes the value at the end point $\psi\equiv\psi(\tau=0)$ and ψ is the field which will appear as the variable of the potential. Therefore, it is ψ where the x dependence appears, $\psi\equiv\psi(x)$. The left-hand side of Eq. (3.5) is just $d\Phi_1/d\tau$ taken along the characteristics. Consequently, a particular solution of (3.5) takes the form of a time integral of the right-hand side, i.e.,

$$\Phi_{1\,\text{part}}(\tau_0)=\frac{2}{Q}\int dx\int_{\tau_0}^{0}d\tau[a-b_r|\psi(\tau)|^2][D\psi^*(\tau)\nabla^2\psi(\tau)+\text{c.c.}]\ . \tag{3.8}$$

Here the coupling by diffusion between neighboring sites in x is explicit and must be taken into account through the implicit dependence of $\psi(\tau)$ on its prescribed end value $\psi(x)$. By evaluating $\nabla^2\psi(\tau)$ and using that

$$a-b_r|\psi(\tau)|^2=(a-b_r|\psi|^2)e^{2a\tau}\left[\frac{b_r|\psi|^2}{a}+\left[1-\frac{b_r|\psi|^2}{a}\right]e^{2a\tau}\right]^{-1}\ , \tag{3.9}$$

and that the time integral in (3.8) can be converted into an integral over a new variable $u=\exp(2a\tau)$, we find

$$\begin{aligned}\Phi_{1\,\text{part}}(\tau_0)=\frac{1}{Q}\int dx\left[1-\frac{b_r|\psi|^2}{a}\right](&I_1(\tau_0)[D_r(\psi^*\nabla^2\psi+\psi\nabla^2\psi^*)+iD_i(\psi^*\nabla^2\psi-\psi\nabla^2\psi^*)]\\&+I_2(\tau_0)\{(D_rb_i-D_ib_r)i(\psi^*\nabla\psi-\psi\nabla\psi^*)\nabla|\psi|^2\\&\qquad-(D_rb_r+D_ib_i)[(\nabla|\psi|^2)^2+|\psi|^2\nabla^2|\psi|^2]\}/a\\&+I_3(\tau_0)(3D_rb_r^2-D_rb_i^2+4D_ib_ib_r)|\psi|^2(\nabla|\psi|^2)^2/(2a^2))\ ,\end{aligned} \tag{3.10}$$

where the "time" integrals $I_n(\tau_0)$ are

$$I_n(\tau_0)\equiv\int_{e^{2a\tau_0}}^{1}du\frac{(1-u)^{n-1}}{\{(b_r/a)|\psi|^2+[1-(b_r/a)|\psi|^2]u\}^{n+1}}\ ,\quad n=1,2,3\ . \tag{3.11}$$

In order to have the general solution of Eq. (3.5), the general solution of the homogeneous equation has to be added

$$\Phi_1 = \Phi_{1\,\text{part}}(\tau_0) + \Phi_{1\,\text{hom}}(\tau_0) \ , \tag{3.12}$$

where $\Phi_{1\,\text{hom}}(\tau_0)$ can be an arbitrary real valued functional of $\psi(\tau_0)$ and, through (3.7), of $\psi(x)$. $\Phi_{1\,\text{hom}}(\tau_0)$ must be chosen in such a way that the complete potential $\Phi = \Phi_0 + \Phi_1$ is minimal on the attractor specified up to first order in ϵ. In the framework of the Hamilton-Jacobi formalism it is difficult to devise a systematic procedure for fulfilling this condition. However, a solution of this problem can be given by determining Φ from the action integral, which leads back to Eq. (3.5) and the solution just presented, but in addition automatically incorporates the boundary condition.

B. Evaluation of the action integral

Our aim here is to evaluate the action

$$\Phi_a \equiv \int_{\tau_0 \to -\infty}^{\tau=0} d\tau\, L(\{\psi(\tau), \psi^*(\tau), \dot{\psi}(\tau), \dot{\psi}^*(\tau)\}) \ , \tag{3.13}$$

where $\psi(\tau)$ is a solution of the Euler-Lagrange (or Hamiltonian) equations defined by the Lagrangian (1.13). The trajectories $\psi(\tau)$ have to be chosen in such a way that in the infinite past $\tau_0 \to -\infty$ they start on the attractor A, and at $\tau=0$ they end at $\psi(\tau=0)=\psi(x)$. In view of Eq. (1.12) this property automatically ensures the minimum of the potential on the attractor since $\Phi_a = 0$ for $\psi, \psi^* \in A$, and $\phi_a \geq 0$ elsewhere due to (1.13).

The Hamiltonian associated with the Lagrangian (1.13) of the complex TDGL model is easily obtained as

$$H(\{\psi, \psi^*, \pi, \pi^*\}) = \int dx \left[\frac{Q}{2}|\pi|^2 + \pi(a\psi - b|\psi|^2\psi + D\nabla^2\psi) + \text{c.c.} \right] , \tag{3.14}$$

where $\pi = \delta L/\delta\dot{\psi}$ is the generalized momentum field. The canonical equations are then

$$\dot{\psi} = \frac{\delta H}{\delta \pi}, \quad \dot{\pi} = -\frac{\delta H}{\delta \psi} \ , \tag{3.15}$$

where derivation with respect to the time τ is denoted by a dot.

Since Φ_a is an action, the Hamilton-Jacobi theory tells us that

$$\pi = \frac{\delta \Phi_a}{\delta \psi} \tag{3.16}$$

is the momentum on the trajectory along which the action is to be evaluated. Consequently, $\pi = 0$ holds on the attractor A, where Φ_a is minimal and Eq. (3.16) specifies the unstable manifold in the infinite-dimensional phase space along which the deterministic attractor A can be reached for $\tau \to -\infty$.

In the limit of weak diffusion the Lagrangian splits into two parts.

$$L = L_0 + L_1 \ , \tag{3.17}$$

where

$$L_0 = \frac{1}{Q}\int dx\, |\dot{\psi} - (a - b|\psi|^2)\psi|^2 \tag{3.18}$$

and L_1 is of order ϵ:

$$L_1 = \frac{1}{Q}\int dx \{ D[\dot{\psi}^* - (a - b^*|\psi|^2)\psi^*]\nabla^2\psi + \text{c.c.} \} \ . \tag{3.19}$$

From (3.13) then follows, correspondingly, $\Phi_a = \Phi_{a_0} + \Phi_{a_1}$, where Φ_{a_0} is identical to Φ_0, given by Eq. (3.3). It is easy to specify the required Hamiltonian trajectory in leading (ϵ^0) order. From the canonical equations we have

$$\dot{\psi} = Q\pi^* + (a - b|\psi|^2)\psi \ . \tag{3.20}$$

Furthermore, we know that

$$\pi^* = \frac{\delta \Phi_0}{\delta \psi^*} = -\frac{2}{Q}(a - b_r|\psi|^2)\psi \ , \tag{3.21}$$

and hence

$$\dot{\psi} = (-a + b^*|\psi|^2)\psi \ . \tag{3.22}$$

This is nothing else than the differential equation of the characteristics of the Hamilton-Jacobi equation (3.6) as expected. By prescribing the endpoint to be $\psi \equiv \psi(x)$ the solution is given by (3.7). It is obvious that for $\tau \to -\infty$ this trajectory really approaches the deterministic attractor for $D=0$.

When evaluating the action to first order in ϵ the Lagrangian $L_0 + L_1$ must be integrated along a trajectory specified up to first order in ϵ. However, since the action is extremal, there is no contribution to the time integral from the first-order correction of the trajectory, i.e., the time integral of L_0 continues to yield Φ_0 up to corrections of second order in ϵ, which are neglected in our present first-order calculation. A contribution of first order in ϵ might still arise from the fact that the lower boundary of the time integral (the attractor) changes to first order in ϵ. This leads to a term proportional to $\delta L_0/\delta\dot{\psi} = \pi$. Since the momentum is zero on the attractor this contribution also vanishes. Thus, we find that the first-order correction Φ_{a_1} is simply the time integral of L_1 taken along the unperturbed trajectory (3.7). By inserting (3.22) into (3.19) one obtains

$$\Phi_{a_1} = \frac{2}{Q}\int_{\tau_0 \to -\infty}^{0} d\tau \int dx [a - b_r|\psi(\tau)|^2] \times [D\psi^*(\tau)\nabla^2\psi(\tau) + \text{c.c.}] \ , \tag{3.23}$$

where $\psi(\tau)$ is given by (3.7).

A comparison of (3.23) and (3.8) shows that one should choose $\tau_0 \to -\infty$ in the Hamilton-Jacobi formalism since $\Phi_{1\,\text{part}}$ then automatically fulfills the boundary condition as

$$\Phi_{1\,\text{part}}(\tau_0 \to -\infty) = \Phi_{a_1} \ . \tag{3.24}$$

Consequently, by recalling (1.12), the homogeneous solution $\Phi_{1\,\mathrm{hom}}$ must be nothing else than the potential on the attractor

$$\Phi_{1\,\mathrm{hom}}(\tau_0 \to -\infty) = C(A) \ . \tag{3.25}$$

Our next task is to specify $C(A)$ which will then lead to the complete determination of the potential to first order. In order to do this we must distinguish between the two regions $a<0$ and $a>0$. The deterministic attractor is drastically different in these regions, already in zeroth order. We also note that the lower limit $\exp(2a\tau_0)$ of the integrals $I_n(\tau_0 \to -\infty)$, Eq. (3.11), is infinity and zero for $a<0$ and $a>0$, respectively.

C. The potential on the attractor

Since our aim is to specify the potential up to first order in D we have to know how the attractors look like in the same order. Below the bifurcation threshold, i.e., for $a<0$, the situation is quite simple. The time-independent solution of the deterministic version of Eq. (1.2) must be $\psi=0$ both in zeroth and in first order in D. This means that the attractor is unique, there cannot be any fluctuations on it, consequently, the potential on the attractor is constant, which can always be chosen to be zero, i.e.,

$$C(\{\psi,\psi^*\})=0 \ . \tag{3.26}$$

Above the bifurcation threshold, i.e., for $a>0$, the case is completely different. The zeroth-order result for the attractor specifies only the modulus r_0 of $\psi(-\infty)$ as $r_0=(a/b_r)^{1/2}$. The phase $\varphi(-\infty)$ on the attractor is undetermined and a continuum of degenerate attractors exists. Thus $C(A)$ in Eq. (3.25) becomes a functional of the phase $\varphi(-\infty)$. In order to determine this functional it is necessary to calculate the potential Φ in a small neighborhood of the attractor and then to put

$$C(A)=\Phi(\{r_0\exp[i\varphi(-\infty)], r_0\exp[-i\varphi(-\infty)]\}) \ . \tag{3.27}$$

As the potential is a smooth functional in a sufficiently small neighborhood of the attractor it can be expanded as a power series in $\psi-r_0$, ψ^*-r_0. Fortunately, the required power series expansion has already been performed in the work of Szépfalusy and Tél,[16] and we shall make use of their result. In Ref. 16 the expansion is performed with respect to the amplitude $\delta(x)$ defined by

$$\psi(x)=r_0+\delta(x) \ . \tag{3.28}$$

Defining Fourier transforms (in a finite system of volume V with cyclic boundary conditions) by

$$\begin{aligned}\delta(x)&=\frac{1}{\sqrt{V}}\sum_k e^{-ikx}\delta_k \ ,\\ \psi(x)&=\frac{1}{\sqrt{V}}\sum_k e^{-ikx}\psi_k \ ,\end{aligned} \tag{3.29}$$

the result of Ref. 16 takes the form [cf. also Appendix C, where we derive a more general result containing Eq. (3.30) as a special case]

$$\Phi(\{\delta_k,\delta_k^*\})=\Phi_{\mathrm{GL}}^{(G)}(\{\delta_k,\delta_k^*\}) - \frac{D_r b_i - D_i b_r}{Q}\sum_k \left[\frac{ik^2\delta_k\delta_{-k}}{b^*+\dfrac{b_r D^*}{a}k^2}+\mathrm{c.c.}\right] . \tag{3.30}$$

The first part represents the Gaussian approximation of the Ginzburg-Landau potential (1.4). For our present purposes the result (3.30) should be expanded in the diffusion coefficient, and we shall retain only the terms of first and second order of this expansion. Furthermore, we wish to use Eq. (3.30) in Eq. (3.27), and we therefore specialize it for the attractor A on which only phase fluctuations $\varphi(-\infty)$ are possible. Therefore we put $\delta(x)=r_0\{\exp[i\varphi(x,-\infty)]-1\}\approx ir_0\varphi(x,-\infty)$, i.e.,

$$k\delta_k=\frac{-r_0}{\sqrt{V}}\int dx\, e^{ikx}\nabla\varphi(x,-\infty) \ , \tag{3.31}$$

and obtain in this way

$$C(A)=\frac{1}{Q}\int dx\left[2\left[\frac{aD_r}{b_r}-\frac{ab_iD_-}{b_r|b|^2}\right](\nabla\varphi)^2 - \frac{2D_-}{|b|^4}[D_i(b_r^2-b_i^2)-2D_rb_rb_i](\nabla^2\varphi)^2 + \alpha(\nabla\varphi)^4\right]+\mathrm{const} \ , \tag{3.32}$$

where $\varphi\equiv\varphi(x,-\infty)$ and the constant is independent of $\nabla\varphi(x,-\infty)$. Here and in what follows we use the abbreviations

$$\begin{aligned}D_-&\equiv D_rb_i-D_ib_r \ ,\\ D_+&\equiv D_rb_r+D_ib_i \ .\end{aligned} \tag{3.33}$$

The term with the constant α has been added in Eq. (3.32), because a term of this form is expected to appear in second order. The constant α cannot be determined from Eq. (3.30) as the latter result is restricted to second order in the perturbation δ, while $(\nabla\varphi)^4$ is of fourth order in δ. However, α will be fixed by an expansion generalizing Eq. (3.30) in Sec. III F.

By means of the characteristics (3.7) the result (3.32) still has to be expressed in terms of $\psi(x)$. We note that

$$\nabla\varphi(x,-\infty)=\nabla\varphi(x)+\frac{b_i}{b_r}\frac{\nabla r(x)}{r(x)} \ . \tag{3.34}$$

From Eq. (3.7) for $\tau\to-\infty$ we obtain $\psi(-\infty)$ in terms of $\psi(x)$. Thus to first order in the D expansion we obtain

$$C(\{\psi,\psi^*\})=\frac{2aD_+}{|b|^2Q}\times\int dx\left[\frac{|\nabla\psi|^2}{|\psi|^2}+\frac{b_i^2-b_r^2}{4b_r^2}\frac{(\nabla|\psi|^2)^2}{|\psi|^4}-\frac{b_i}{2b_r}\frac{i(\psi^*\nabla\psi-\psi\nabla\psi^*)\cdot\nabla|\psi|^2}{|\psi|^4}\right]+\text{const}\ . \tag{3.35}$$

As this expression for fixed ψ no longer depends on $\varphi(x,-\infty)$ labeling the continuum of attractors for $a>0$, the minimum over all attractors A in Eq. (1.12) can be dropped in this case. The contributions to C of second order in D are also contained in Eq. (3.32), but not in (3.35). They will be given explicitly only when they are needed (cf. Sec. III F). Now we are in a position to add up Φ_a and C to find the final results for the potential to first order.

D. The potential below the bifurcation threshold ($a<0$)

Using the fact that in this region the integrals I_n (3.11) take the form

$$I_n(\tau_0\to-\infty)=\frac{(-1)^n}{n[1-(b_r/a)|\psi|^2]^n}\ , \tag{3.36}$$

we obtain from (3.10)

$$\Phi_{a_1}=\frac{1}{Q}\int dx\left[-D_r(\psi^*\nabla^2\psi+\psi\nabla^2\psi^*)-iD_i(\psi^*\nabla^2\psi-\psi\nabla^2\psi^*)+\frac{1}{2[1-(b_r/a)|\psi|^2]}\left[\frac{D_-}{a}i(\psi^*\nabla\psi-\psi\nabla\psi^*)\cdot\nabla|\psi|^2-\frac{D_+}{a}[(\nabla|\psi|^2)^2+|\psi|^2\nabla^2|\psi|^2]\right]-\frac{1}{3[1-(b_r/a)|\psi|^2]^2}\left[\frac{3}{2a^2}D_+b_r-\frac{1}{2a^2}D_-b_i\right]|\psi|^2(\nabla|\psi|^2)^2\right]\ . \tag{3.37}$$

By means of (3.1), (3.3), (3.26), and (3.37) one finds the complete potential up to first order in $|D|$. There is no need to take the minimum in Eq. (1.12) since the action is single valued. It is worth eliminating double spatial derivatives by means of Gauss's theorem, which leads to the appearance of a contribution from a surface integral. We can write

$$\Phi(\{\psi,\psi^*\})=\Phi_B(\{\psi,\psi^*\})+\Phi_S(\{\psi,\psi^*\})\ . \tag{3.38}$$

The surface contribution is of order $|D|$ and has the form

$$\Phi_S(\psi,\psi^*)=\frac{1}{Q}\int \mathbf{df}\cdot\left[-D_r\nabla|\psi|^2-D_i i(\psi^*\nabla\psi-\psi\nabla\psi^*)-D_+\frac{|\psi|^2\nabla|\psi|^2}{2(a-b_r|\psi|^2)}\right]\ . \tag{3.39}$$

The bulk contribution splits into the Ginzburg-Landau part (1.4) and a remainder

$$\Phi_B(\{\psi,\psi^*\})=\Phi_{\rm GL}(\{\psi,\psi^*\})+\Phi_{n\rm GL}(\{\psi,\psi^*\})\ , \tag{3.40}$$

where $\Phi_{n\rm GL}$ is of order $|D|$ and reads

$$\Phi_{n\rm GL}(\psi,\psi^*)=\frac{D_-}{Q}\int dx\left[\frac{i(\psi^*\nabla\psi-\psi\nabla\psi^*)\cdot\nabla|\psi|^2}{2(a-b_r|\psi|^2)}+\frac{b_i|\psi|^2(\nabla|\psi|^2)^2}{6(a-b_r|\psi|^2)^2}\right]\ . \tag{3.41}$$

It is worth emphasizing that the potential we have found is of nonpolynomial type. Note that the denominators cannot vanish as long as $a<0$ since b_r is assumed to be positive.

E. The potential above the bifurcation threshold ($a>0$)

In the region $a>0$ the integrals I_n read

$$I_n(\tau_0\to-\infty)=\frac{1}{n[(b_r/a)|\psi|^2]^n}\ . \tag{3.42}$$

Therefore we have

$$\Phi_{a_1}=\frac{1}{Q}\int dx\left[\frac{a}{b_r}-|\psi|^2\right]\left[[D_r(\psi^*\nabla^2\psi+\psi\nabla^2\psi^*)+iD_i(\psi^*\nabla^2\psi-\psi\nabla^2\psi^*)]\frac{1}{|\psi|^2}+\left[\frac{D_-}{2b_r}i(\psi^*\nabla\psi-\psi\nabla\psi^*)\cdot\nabla|\psi|^2-\frac{D_+}{2b_r}[(\nabla|\psi|^2)^2+|\psi|^2\nabla^2|\psi|^2)]\right]\frac{1}{|\psi|^4}+\frac{3D_+b_r-D_-b_i}{6b_r^2}\frac{(\nabla|\psi|^2)^2}{|\psi|^4}\right]\ . \tag{3.43}$$

The result becomes again clearer if double spatial derivatives are eliminated by means of Gauss's theorem. Adding Φ_0 from (3.3), Φ_{a_1} from (3.43), and C from (3.35) the complete contribution is single valued and the potential is again the sum of a bulk and of a surface contribution (3.38), where now

$$\Phi_S(\{\psi,\psi^*\})=\frac{1}{Q}\int d\mathbf{f}\cdot\left[\frac{a}{b_r|\psi|^2}-1\right]\left[\left[D_r-\frac{D_+}{2b_r}\right]\nabla|\psi|^2+D_i i(\psi^*\nabla\psi-\psi\nabla\psi^*)\right] . \tag{3.44}$$

The bulk contribution is of the form (3.40), where the remainder to the Ginzburg-Landau term is found to be, for $a>0$,

$$\Phi_{n\mathrm{GL}}(\{\psi,\psi^*\})=\frac{D_-}{Q}\frac{a}{b_r}\int dx\left\{-\frac{2b_i}{|b|^2}|\nabla\ln\psi|^2+i(\nabla\ln\psi/\psi^*)\cdot\nabla\ln|\psi|^2\left[\frac{1}{2b_r}\left[1-\frac{b_r}{a}|\psi|^2\right]-\frac{b_r}{|b|^2}\right]\right.$$
$$\left.+(\nabla\ln|\psi|^2)^2\left[-\frac{b_i}{6b_r^2}\left[1-\frac{b_r}{a}|\psi|^2\right]+\frac{b_i}{|b|^2}\right]\right\} . \tag{3.45}$$

The nonequilibrium potential is again of nonpolynomial type. Note that in this region a formal divergence shows up for $|\psi|\to 0$. This divergence is formal because it occurs outside the domain of validity of Eq. (3.45) as ψ is not slowly varying. Instead the diffusion term dominates in (1.2) for $|\psi|\to 0$ and the potential of Appendix A can be used. Finally, it is worth noting that the results (3.41) and (3.45) fit together at the bifurcation threshold $a=0$.

F. Dominant second-order contributions to the potential

Due to the phase symmetry of Eq. (1.2) in the case $a>0$ phase fluctuations of long wavelengths occur with high probability, while fluctuations in $|\psi|$ are suppressed in comparison due to the nonlinear stabilization described by Eq. (1.2). It is therefore desirable to proceed with our expansion in D to the next (second) order at least in as much as phase fluctuations are concerned. The calculation proceeds as in Sec. III A and is given in Appendix B. The calculation is simplified by our decision to treat only the phase gradients in second order in D. The result is

$$\Phi_{a_2}=\frac{D_-}{2b_r^2Q}\int dx(\nabla^2\varphi)^2\left[\frac{D_-}{b_r}\left[1-\frac{a}{b_r|\psi|^2}\right]\right.$$
$$\left.+\left[2D_i+\frac{D_-}{b_r}\right]\ln\frac{a}{b_r|\psi|^2}\right] . \tag{3.46}$$

It should be noted that ϕ_{a_2} vanishes for $|\psi|^2=a/b_r$. We still have to add the second-order contribution C_2 from $C(\{\psi,\psi^*\})$ in order to obtain the complete potential to second order. To this end we have to use Eq. (3.32) with Eq. (3.34) and Eq. (3.7) in order to express $[\nabla\varphi(-\infty)]^4$, $[\nabla^2\varphi(-\infty)]^2$ in terms of $[\nabla\varphi(x)]^4$, $[\nabla^2\varphi(x)]^2$ at the end point of the characteristics. However, the latter task is trivial in our present approximation, where we neglect gradients of $|\psi|$ in the second-order calculation, because then simply

$$[\nabla\varphi(-\infty)]^4=[\nabla\varphi(x)]^4,\quad [\nabla^2\varphi(-\infty)]^2=[\nabla^2\varphi(x)]^2 . \tag{3.47}$$

Thus from Eq. (3.32)

$$C_2(A)=\frac{1}{Q}\int dx\{\alpha[\nabla\varphi(x)]^4+\beta[\nabla^2\varphi(x)]^2\} , \tag{3.48}$$

where α is not yet known and

$$\beta=-\frac{2D_-}{|b|^4}[D_i(b_r^2-b_i^2)-2D_rb_rb_i]$$
$$=-\frac{2D_-\mathrm{Re}(iD^*b^2)}{|b|^4} . \tag{3.49}$$

We now proceed to determine α. In principle, a could be obtained by a local expansion of the potential on the degenerate attractor $|\psi_0|=(a/b_r)^{1/2}$ up to fourth order in $\nabla\varphi$, but this calculation becomes very tedious in practice. A much easier computation is instead a local second-order expansion of Φ in $\nabla\varphi$ and ∇r in the neighborhood of the plane-wave attractors appearing as local minima of the potential in first order for $a>0$,

$$\psi_{k_0}=[(a-D_rk_0^2)/b_r)]^{1/2}e^{ik_0x+i\varphi_0} , \tag{3.50}$$

where φ_0 is an arbitrary constant phase. The local expansion around (3.50) is given in Appendix (C) [cf. Eq. (C7)]. A corresponding local expansion can also be performed for the potential given by (3.40) and (3.45), for $a\geq 0$, extended by second-order terms, for which we make the general ansatz, polynomial in $(\nabla\varphi)^2$, $(\nabla r)^2$, and in the scalar product $(\nabla\varphi\cdot\nabla r)$ (a polynomial form is always legitimate in the vicinity of attractors):

$$\Phi_2=\Phi_{a_2}+\frac{1}{Q}\int dx\left[\alpha[\nabla\varphi(x)]^4+\frac{\alpha_{11}}{r^2(x)}[\nabla\varphi(x)]^2[\nabla r(x)]^2+\frac{\alpha_{12}}{r^2(x)}\{(\nabla\varphi(x)\cdot\nabla r(x)]^2-(\nabla\varphi)^2(\nabla r)^2\}\right.$$
$$\left.+\frac{\alpha_2}{r(x)}[\nabla\varphi(x)]^2[\nabla\varphi(x)\cdot\nabla r(x)]+\beta[\nabla^2\varphi(x)]^2\right] . \tag{3.51}$$

Here Φ_{a_2} is given by Eq. (3.46) and β by Eq. (3.49). We included all terms linear and quadratic, but not those cubic and quartic in ∇r. Therefore, the following calculation determines $C_2(A)$ to an approximation better than required to fix α in Eq. (3.48). The ansatz (3.51) goes even beyond the leading phase-gradient terms of second order, because it retains also the $(\nabla\varphi)^3\nabla r$ and $(\nabla\varphi)^2(\nabla r)^2$ terms (cf. below). The local expansion around (3.50) contains, among other terms, a quadratic form in $\nabla\delta\varphi$, $\nabla\delta r$, the gradients of the phase and amplitude fluctuations [cf. (C4) and (C8)]. Its coefficients depend on the undetermined coefficients in Eq. (3.51). The two local expansions can be compared. As shown in Appendix C the comparison is successful [i.e., the polynomial ansatz (3.51) is possible] only in the one-dimensional case, where the term with the coefficient α_{12} vanishes. For the other coefficients we obtain in Appendix C

$$\begin{aligned}\alpha&=\frac{1}{3}\frac{D_-D_rb_i}{b_r|b|^2}\ ,\\ \alpha_{11}&=\frac{2}{3}\frac{D_-D_rb_i}{b_r^3|b|^2}(b_i^2-2b_r^2)\ ,\\ \alpha_2&=\frac{2}{3}\frac{D_-D_r}{b_r^2|b|^2}(b_i^2-b_r^2)\ .\end{aligned} \tag{3.52}$$

While the terms with α_{11}, α_2 are not relevant to the order in which ϕ_{a_2} was calculated in (3.46) [where terms linear and quadratic in ∇r multiplied by $(a/b_r|\psi|^2)-1$ or $\ln(a/b_r|\psi|^2)$ have been neglected], the inclusion of these terms will lead to higher symmetry later [cf. Eqs. (4.14) and (4.16)] and we therefore decide to keep them. For global stability to this order we have to require that $\alpha>0$, i.e.,

$$D_-b_i>0\ , \tag{3.53}$$

which we assume in the following. We note that (3.51) for fixed $r(x)$, $\varphi(x)$ is independent of k_0. Therefore taking the minimum over k_0 in Eq. (1.12) is not necessary. In the case $d\geq 2$, however, a polynomial ansatz of the form (3.51) independent of k_0, even if including terms of still higher arbitrary order, cannot be matched with the direct local expansion around (3.50) without introducing unacceptable divergencies for $|\psi|^2=a/b_r$. We have to conclude that for $d\geq 2$ the potential must contain terms which are not analytic in D arising from taking the minimum over k_0 (cf. Appendix C). Henceforth we shall restrict ourselves to the case $d=1$, where Eqs. (3.51) and (3.52) apply.

IV. ATTRACTORS AND THEIR STABILITY IN ONE DIMENSION

In this section we study, for one-dimensional systems, extrema of the potential, in particular its local minima which describe attractors of Eq. (1.2). The *local* stability properties of a given attractor can be determined from the form of the potential in its neighborhood. A *global* measure of stability is provided by the height $\delta\Phi$ of the potential barrier which must be surmounted, with the help of the fluctuations included in Eq. (1.2), in order to reach the basin of attraction of another attractor. In leading order in the noise intensity η the mean lifetime in the vicinity of a given attractor is then proportional to $\exp(\delta\Phi/\eta)$. We shall consider here a special class of attractors, the plane-wave attractors

$$\psi(x)=r_0(k)e^{ikx}\ , \tag{4.1}$$

characterized by the wave number k, and use the potential to determine both their local and global stability properties. We suppose periodic boundary conditions in which case no surface contributions Φ_S are present. The local stability properties of the attractors (4.1) can also be derived from Eq. (1.2) via a linear stability analysis. This part of our calculation therefore merely demonstrates the method and serves mainly as a check for the explicit expressions we obtained for the potential. On the other hand, the global stability properties are entirely outside the realm of linear stability analysis. The results we obtain here cannot be obtained by any other method.

A. The attractor $\psi\equiv 0$—Hopf bifurcation

As mentioned in Sec. III C, in the region below the bifurcation threshold one single attractor $\psi\equiv 0$ exists only. One immediately sees from (3.41) that for $a<0$ the non-Ginzburg-Landau part of the bulk contribution is of fourth and higher order in ψ. Therefore, the quadratic approximation of the potential around the attractor $\psi\equiv 0$ follows completely from $\Phi_{\rm GL}(\{\psi,\psi^*\})$ and has the form

$$\Phi_B^{(G)}(\{\psi,\psi^*\})=\frac{2}{Q}\int dx(-a|\psi|^2+D_r|\nabla\psi|^2)\ , \tag{4.2}$$

where the superscript (G) stands for Gaussian approximation. Thus the presence of any nonvanishing $\psi(x)$ makes the potential higher than its value taken at the attractor $\psi_0=0$; the larger $|a|$, the more stable the attractor. It also follows from (4.2) that the $k=0$ component ψ_0 becomes marginally stable for $a\to 0$ in accord with the fact that the deterministic solution $\psi\equiv 0$ loses its stability at the bifurcation point $a=0$. Note that at this point the potential is not of polynomial type due to the presence of a singular $\Phi_{n\rm GL}$ (3.41).

B. Extremum conditions for $a>0$

The potential Φ defines a local Lagrangian via

$$\Phi=\frac{1}{Q}\int dx\, L(\nabla^2\varphi,\nabla\varphi,\nabla r,r)\ . \tag{4.3}$$

Its extrema $\psi(x)=r(x)\exp[i\varphi(x)]$ satisfy the variational principle $\delta\Phi=0$ which leads to the Euler-Lagrange equations

$$\begin{aligned}&\nabla\left[\frac{\partial L}{\partial\nabla\varphi}-\nabla\frac{\partial L}{\partial\nabla^2\varphi}\right]=0\ ,\\ &\nabla\left[\frac{\partial L}{\partial\nabla r}\right]-\frac{\partial L}{\partial r}=0\ .\end{aligned} \tag{4.4}$$

Equation (4.4) has the form of a conservation law

$\nabla J=0$, $J=\text{const}$, in agreement with the phase symmetry of Eq. (1.2) and of Φ, where

$$J=\frac{\partial L}{\partial \nabla \varphi}-\nabla \frac{\partial L}{\partial \nabla^2 \varphi} . \tag{4.5}$$

Another conservation law is implied by the fact that L does not explicitly depend on x. We find in the manner familiar from analytical mechanics the conserved "energy" E:

$$E=[(\nabla^2\varphi)-(\nabla\varphi)\nabla]\frac{\partial L}{\partial \nabla^2\varphi}+\nabla\varphi\frac{\partial L}{\partial \nabla\varphi}+\nabla r\frac{\partial L}{\partial \nabla r}-L , \tag{4.6}$$

where the somewhat unusual first term on the right-hand side arises because L depends also on the second spatial derivative of φ. The remaining differential equations (4.4) are, in general, nonintegrable, because Eqs. (4.4) constitute a dynamical system of fourth order without any further conserved quantities. Therefore, it must be expected that chaotic extrema of Φ exist. It is tempting to speculate that chaotic minima of Φ do exist and, in the long wavelength limit, to which the form (4.3) is restricted, correspond to numerically determined solutions of Eq. (1.2),[6,9,10,14] which by all appearances are spatially chaotic attractors. Further evidence in favor of or against this hypothesis could only be gained by a numerical investigation of Eqs. (4.4) which has not yet been carried out.

In the following we shall restrict our attention to simpler, nonchaotic, extrema of Φ for which the term with $\nabla^4\varphi$ in Eq. (4.4) is negligible compared to the remaining $\nabla^2\varphi$ term. Then Eqs. (4.5) and (4.6) simplify, because we may neglect $\partial L/\partial \nabla^2\varphi$. Explicitly the Lagrangian in this approximation may be written [see Eqs. (3.3), (3.45), and (3.51)]

$$\begin{aligned} L=&\alpha(\nabla\varphi)^4+\frac{\alpha_{11}}{r^2}(\nabla\varphi)^2(\nabla r)^2+\frac{\alpha_2}{r}(\nabla\varphi)^3(\nabla r)\\ &+2f_1(a,r)(\nabla\varphi)^2+4f_2(a,r)\nabla r\nabla\varphi\\ &+2f_3(a,r)(\nabla r)^2-U_0(a,r) \end{aligned} \tag{4.7}$$

with

$$\begin{aligned} &U_0(a,r)=+2ar^2-b_r r^4 ,\\ &f_1(a,r)=D_r r^2-\frac{D_- b_i}{|b|^2}\frac{a}{b_r} ,\\ &f_2(a,r)=\frac{aD_-}{|b|^2 r}\left[1-\frac{|b|^2}{2b_r^2}\left[1-\frac{b_r r^2}{a}\right]\right] ,\\ &f_3(a,r)=D_r+\frac{D_- b_i}{b_r r^2}\left[\frac{a}{|b|^2}-\frac{1}{3}\frac{a-b_r r^2}{b_r^2}\right] , \end{aligned} \tag{4.8}$$

where $\alpha,\alpha_{11},\alpha_2,D_-$ are defined in Eqs. (3.52) and (3.33), respectively. The "angular momentum" J and the "energy" E are then given by

$$\begin{aligned} J=&4\alpha(\nabla\varphi)^3+\frac{2\alpha_{11}}{r^2}(\nabla\varphi)(\nabla r)^2+\frac{3\alpha_2}{r}(\nabla\varphi)^2\nabla r\\ &+4f_1(a,r)\nabla\varphi+4f_2(a,r)\nabla r , \end{aligned} \tag{4.9}$$

$$\begin{aligned} E=&\tfrac{3}{4}J\nabla\varphi+\frac{3\alpha_{11}}{2r^2}(\nabla\varphi)^2(\nabla r)^2+\frac{3}{4}\frac{\alpha_2}{r}(\nabla\varphi)^3(\nabla r)\\ &-f_1(a,r)(\nabla\varphi)^2+f_2(a,r)\nabla r\nabla\varphi\\ &+2f_3(a,r)(\nabla r)^2+U_0(a,r) . \end{aligned} \tag{4.10}$$

Equations (4.9) and (4.10) are integrable, because in principle they can be solved for ∇r, $\nabla\varphi$ by algebraic methods and then integrated by quadratures. Unfortunately, the resulting algebra cannot be carried through in practice, and we have to resort to further simplifications. Consistent with our derivation of L, Eq. (4.7), we shall neglect in E powers of ∇r of higher than second order. In order to eliminate $\nabla\varphi$ from Eq. (4.10) we first have to solve (4.9) for $\nabla\varphi$, which now needs to be done only up to $(\nabla r)^2$ terms. We make the general ansatz

$$\nabla\varphi=f(r,J)+g(r,J)\nabla r+h(r,J)(\nabla r)^2 . \tag{4.11}$$

For f we find the equation

$$J=4[\alpha f^2+f_1(a,r)]f , \tag{4.12}$$

while g and h satisfy

$$\begin{aligned} &g(r,J)=-\frac{f_2(a-D_r f^2,r)}{f_1(a-D_r f^2,r)} ,\\ &h(r,J)=-\frac{\alpha_{11}/2r^2+(3\alpha_2/2r)g+3\alpha g^2}{f_1(a-D_r f^2,r)}f . \end{aligned} \tag{4.13}$$

Here we made use of the first two of the relations

$$\begin{aligned} &f_1(a,r)+3\alpha f^2=f_1(a-D_r f^2,r) ,\\ &f_2(a,r)+\frac{3\alpha_2}{4r}f^2=f_2(a-D_r f^2,r) ,\\ &f_3(a,r)+\frac{\alpha_{11}}{2r^2}f^2=f_3(a-D_r f^2,r) . \end{aligned} \tag{4.14}$$

Inserting Eq. (4.11) into Eq. (4.10) and making use of Eqs. (4.12)–(4.14) a straightforward but lengthy algebra yields E in the compact form

$$E=\tfrac{1}{2}K(r)(\nabla r)^2+U_{\text{eff}}(r) \tag{4.15}$$

with

$$K(r)=4f_3(a-D_r f^2,r)-4\frac{f_2^2(a-D_r f^2,r)}{f_1(a-D_r f^2,r)} , \tag{4.16}$$

$$U_{\text{eff}}(r)=U_0(a,r)+3\alpha f^4(r,J)+2f_1(a,r)f^2(r,J) . \tag{4.17}$$

Equation (4.15) must be satisfied by all extrema of the potential with $|\beta\nabla^4\varphi| \ll 2|f_1(a,r)\nabla^2\varphi|$. Once Eq. (4.15) is solved the corresponding (extremal) value can easily be calculated by noting that the potential Φ here plays the role of the action in classical mechanics. With energy and angular momentum conservation the extremal action takes the form

$$\Phi_{\text{extr}}=\frac{1}{Q}\int dx(J\nabla\varphi-E)+\Phi_r(\{r\}) , \tag{4.18}$$

where the radial part $\Phi_r(\{r\})$ is given by

$$\Phi_r(\{r\})=\frac{1}{Q}\int dx\left[K(r)(\nabla r)^2+\left[\frac{\alpha_2}{r}f^3(r,J)+4f_2(a,r)f(r,J)\right]\nabla r\right] . \tag{4.19}$$

In the following we shall study plane-wave states $r(x)\equiv\text{const}$, $\varphi(x)=kx$, and states differing only locally in some small region from such plane states.

C. Plane-wave attractors

Assuming

$$r(x)\equiv r_0 \tag{4.20}$$

we reduce the angular momentum J, Eq. (4.11), and the "energy" E, Eq. (4.15), to

$$\begin{aligned} J&=4\alpha f^3(r_0,J)+4f_1(a,r_0)f(r_0,J) ,\\ E&=U_{\text{eff}}(r_0)=U_0(a,r_0)+3\alpha f^4(r_0,J)\\ &\quad+2f_1(a,r_0)f^2(r_0,J) . \end{aligned} \tag{4.21}$$

The state (4.20) is a local minimum of Φ,

$$\Phi=\frac{1}{Q}\int dx[J\nabla\varphi+E-2U_{\text{eff}}(r)] , \tag{4.22}$$

if

$$\left.\frac{\partial U_{\text{eff}}}{\partial r}\right|_{r=r_0}=0 , \tag{4.23}$$

$$\left.\frac{\partial^2 U_{\text{eff}}}{\partial r^2}\right|_{r=r_0}<0 , \tag{4.24}$$

$$K(r_0)>0 . \tag{4.25}$$

The conditions (4.24) and (4.25) ensure stability of the attractor (the minimum of Φ) agains fluctuations of r and ∇r, respectively. Equation (4.23) leads to

$$a-b_r r_0^2+D_r f^2+f_1(a-D_r f^2,r_0)\frac{\partial f^2}{\partial r_0^2}=0 , \tag{4.26}$$

while from Eq. (4.21) we obtain by differentiation of the expression for J

$$\frac{\partial f^2}{\partial r_0^2}=-\frac{2f^2 D_r}{f_1(a-D_r f^2,r_0)} . \tag{4.27}$$

Inserting this result in (4.26) we find

$$r_0^2=[a-D_r f^2(r_0,J)]/b_r . \tag{4.28}$$

The function $f(r,J)$, on the other hand, is determined by Eqs. (4.11) and (4.12). From (4.11) it follows with $r=r_0$ that $\nabla\varphi=f(r_0,J)\equiv k$, i.e., the attractor is a plane wave, where the wave number k is determined via (4.12)

$$J=4[\alpha k^2+f_1(a,r_0)]k . \tag{4.29}$$

As J is a constant of integration, which can be chosen freely, k becomes an arbitrary constant which labels the different plane-wave attractors. Equation (4.24) can be evaluated by differentiation of (4.26) and (4.27) and eliminating the second derivative of f^2 with respect to r_0^2. The resulting conditions reads, with $f=k$,

$$[-b_r f_1(a-D_r k^2,r_0)+2D_r^2 k^2]f_1(a-D_r k^2,r_0)<0 . \tag{4.30}$$

Equation (4.25) yields the condition

$$D_r+\frac{D_- D_i}{D_+}>0 , \tag{4.31}$$

where D_+ is defined by (3.33). The conditions (4.24) and (4.25) ensure stability with respect to fluctuations of r and ∇r, respectively. In order to ensure also stability against phase fluctuations the "moment of inertia" of the attractor must be positive

$$\left.\frac{\partial J}{\partial(\nabla\varphi)}\right|_{\substack{r=r_0\\ \nabla\varphi=k}}>0 , \tag{4.32}$$

which, by (4.12) and (4.14), implies

$$f_1(a-D_r k^2,r_0)>0 , \tag{4.33}$$

which reads, more explicitly

$$D_r b_r+D_i b_i\equiv D_+>0 . \tag{4.34}$$

If (4.34) is violated, the system is unstable against fluctuations of $\nabla\varphi$ (Newell-Kuramoto instability[4,6]). In this case the term $\beta(\nabla^2\varphi)^2$ in the Lagrangian is no longer negligible, and new chaotic attractors may appear with long wavelengths as long as $|D_+|$ remains small. If (4.34) is satisfied the stability condition (4.30) reduces to

$$k^2<\frac{D_+}{3D_+b_r+2D_-b_i}\frac{ab_r}{D_r}\equiv k_c^2 . \tag{4.35}$$

The right-hand side is positive, as D_+ is positive by (4.34) and a, b_r, D_r, and D_-b_i [see (3.53)] are also positive. The condition (4.35) is only violated if k^2 becomes too large. This is the Eckhaus-Benjamin-Feir instability for the complex TDGL equation.[8] Thus Eqs. (4.31), (4.34), and (4.35) are the general local stability conditions for plane-wave attractors. These attractors with $0\le k^2\le k_c^2$ coexist for $a>0$. They have the Φ value

$$\Phi_k=\frac{2k^2D_+la}{|b|^2Q}\left[1-\frac{k^2}{6k_c^2}\right]+\text{const} , \tag{4.36}$$

where l is the length of the system. The absolute minimum of Φ_k with respect to k is at $k=0$. There is a maximum at $k^2=3k_c^2$ outside the stable regime, however.

For $k^2 \to k_c^2$ the second derivative $\partial^2\Phi_k/\partial k^2 \to 0$. Thus, like in thermodynamics, instability is associated to a loss of convexity of the potential Φ. In the stable regime the plane-wave attractors are separated by barriers of the potential Φ. The height of these barriers furnishes an objective measure of the nonlinear stability of a given attractor. It is therefore of interest to calculate them, which is the purpose of the next section, where we apply the method worked out by Langer and Ambegaokar[27] for the real case.

D. Potential barriers

Let us now study states which differ only little from plane waves, i.e., $\varphi(x)=kx+\delta\varphi$, hence $f(r)=k+\delta f(r)$. Equation (4.12) may be expanded in δf to quadratic order and solved explicitly for δf to yield

$$\delta f(r)=\frac{J-J_k(r)}{4f_1(a-D_rk^2,r)}-\frac{3\alpha k[J-J_k(r)]^2}{16f_1^3(a-D_rk^2,r)} \tag{4.37}$$

with

$$J_k(r)=4[-2\alpha k^2+f_1(a-D_rk^2,r)]k \;. \tag{4.38}$$

Inserted into Eq. (4.17) this yields, after some algebra,

$$U_{\text{eff}}(r)=U_0(a,r)-3\alpha k^4+\frac{(J+8\alpha k^3)^2}{8f_1(a-D_rk^2,r)} \;. \tag{4.39}$$

The "radial kinetic energy" (4.16) is simplified by neglecting δf, i.e., putting $f=k$. Thus the energy conservation (4.15) in the present case reduces to

$$E=2\left[f_3(a-D_rk^2,r)-\frac{f_2^2(a-D_rk^2,r)}{f_1(a-D_rk^2,r)}\right](\nabla r)^2 + U_0(a,r)-3\alpha k^4+(J+8\alpha k^3)^2/[8f_1(a-D_rk^2,r)] \;. \tag{4.40}$$

We are interested in states coinciding with a plane-wave attractor $r=r_0$, $\varphi=kx+\text{const}$ for $x\to\pm\infty$, hence $J_k(r(\pm\infty))=J$ and

$$J=-8\alpha k^3+4f_1(a-D_rk^2,r_0)k =4\frac{D_+ak}{|b|^2}\left[1-\frac{k^2}{3k_c^2}\right]\equiv J(k)\;, \tag{4.41}$$

$$E=U_0(a,r_0)-3\alpha k^4+2f_1(a-D_rk^2,r_0)k^2 =\frac{a^2}{b_r}+2k^2\frac{D_+a}{|b|^2}\left[1-\frac{k^2}{2k_c^2}\right]\equiv E(k)\;, \tag{4.42}$$

as follows from (4.38) and (4.40) with (4.8). Solving Eq. (4.40) for ∇r we obtain

$$\nabla r=\pm\frac{1}{\sqrt{2}}\left[\frac{[E+3\alpha k^4-U_0(a,r)]f_1(a-D_rk^2,r)-(J+8\alpha k^3)^2/8}{f_3(a-D_rk^2,r)f_1(a-D_rk^2,r)-f_2^2(a-D_rk^2,r)}\right]^{1/2} \;. \tag{4.43}$$

The numerator in the large parentheses is a polynomial of third order in r^2. It must have a double root at $r^2=r_0^2=(a-D_rk^2)/b_r$ because $r^2=r_0^2$, $\varphi=kx$ is a minimum of Φ. Denoting its second root by r_1^2 the numerator takes the form $b_rD_r(r^2-r_0^2)^2(r^2-r_1^2)$. The value of r_1^2 can be evaluated with the help of (4.41) and (4.42). After some algebra we find

$$r_1^2=r_0^2-\frac{aD_+}{D_r|b|^2}\left[1-\frac{k^2}{k_c^2}\right]\;, \tag{4.44}$$

where k_c is defined in Eq. (4.35). The denominator in the parentheses of (4.43) may be evaluated by using (4.8). It is of the form $(Ar^{-2}+B+Cr^2)$ with

$$A=\frac{D_-^2(a-D_rk^2)^2}{12|b|^2b_r^4}(b_i^2-3b_r^2)\;,$$

$$B=\frac{-(a-D_rk^2)D_-}{3|b|^2b_r^3}[|b|^2b_iD_r+(3b_r^2-b_i^2)D_-/2]\;,$$

$$C=D_r^2+b_iD_rD_-/(3b_r^2)-(D_-^2/4b_r^2)\;. \tag{4.45}$$

Equation (4.43) can be integrated to yield

$$x=\left[\frac{1}{2b_rD_r}\right]^{1/2}\int_{r_1^2}^{r^2}dy\frac{1}{y(r_0^2-y)}\left[\frac{A+By+Cy^2}{y-r_1^2}\right]^{1/2}\;. \tag{4.46}$$

Equation (4.46) gives, in implicit form, an extremum $r(x)$ of the potential Φ, which satisfies

$$r(0)=r_1,\quad r(\pm\infty)=r_0\;. \tag{4.47}$$

We shall evaluate the quadrature (4.46) only approximately by replacing the slowly varying part of the integrand by its value taken at $y=r_0^2$, i.e., we write

$$A/y^2+B/y+C\simeq A/r_0^4+B/r_0^2+C\;. \tag{4.48}$$

Equation (4.46) then yields

$$r_2(x)=r_0^2-(r_0^2-r_1^2)\text{sech}^2\{x[(r_0^2-r_1^2)\kappa/2]^{1/2}\} \tag{4.49}$$

with

$$\kappa=\frac{D_r|b|^2}{D_rD_++D_iD_-}\;. \tag{4.50}$$

Note that κ is positive by (4.31) and (4.34). Equation (4.49) gives the saddle-point configuration between two plane-wave attractors with slightly different wave number. To determine the relation between k_i, the wave number of an initial plane-wave state, and k, the wave number of the asymptotic ($x\to\pm\infty$) part of the saddle-point configuration (4.49) separating this plane-wave state from another, it must be required[27] that the total phase change over the length l ($l\to\infty$) is fixed,

$$\Delta\varphi \equiv \varphi(l/2)-\varphi(-l/2)\equiv k_i l=\int_{-l/2}^{l/2}\nabla\varphi(x)dx\ , \qquad (4.51)$$

where $\varphi(x)$ is the phase belonging to the saddle-point configuration. This condition ensures that the phase perturbation is merely local. The potential barrier $\delta\Phi$ is then expressed as

$$\delta\Phi=\frac{1}{Q}\int_{-l/2}^{l/2}[L(\{r(x),\nabla\varphi(x)\})-L(\{r_i,k_i\})]dx \qquad (4.52)$$

with

$$r_i^2=\frac{a-D_r k_i^2}{b_r}\ . \qquad (4.53)$$

We use Eq. (4.18) and evaluate the difference in the integrand of Eq. (4.52) to first order in $\delta k=k_i-k$. From Eqs. (4.42) and (4.41) we evaluate $\delta E=E(k_i)-E(k)$ and $\delta J=J(k_i)-J(k)$,

$$\delta E=\frac{\partial E(k)}{\partial k}\delta k=4k\frac{D_+a}{|b|^2}\left[1-\frac{k^2}{k_c^2}\right]\delta k\ ,$$
$$\delta J=\frac{\partial J(k)}{\partial k}\delta k=4\frac{D_+a}{|b|^2}\left[1-\frac{k^2}{k_c^2}\right]\delta k\ . \qquad (4.54)$$

With Eqs. (4.18) and (4.19), we may write

$$L=J\nabla\varphi+K(r)(\nabla r)^2+\left[\frac{\alpha_2}{r}f^3(r,J)+4f_2(a,J)f(r,J)\right]\nabla r-E\ , \qquad (4.55)$$

and Eq. (4.52), for trajectories with $r(l/2)=r(-l/2)$, reduces to

$$\delta\Phi=\frac{1}{Q}\left[\int_{-l/2}^{l/2}K(r)(\nabla r)^2dx-\Delta\varphi\delta J+\delta E l\right]\ . \qquad (4.56)$$

As Eq. (4.56) needs to be accurate only to first order in δk we may write $\Delta\varphi=k_i l\simeq kl$ in the first-order term $\Delta\varphi\delta J$. From Eq. (4.54) it follows that $(\delta E-k\delta J)=0$, hence the last two terms in Eq. (4.56) cancel. It is remarkable that the difference δk therefore cancels out completely in Eq. (4.56), which is fortunate, as its computation is tedious in our general case. Introducing $y=r^2$ as new variable of integration via Eq. (4.46) we find the compact result

$$\delta\Phi=\frac{2\sqrt{2}\sqrt{b_r}}{Q\sqrt{D_r}}\int_{r_1^2}^{r_0^2}\frac{(r_0^2-y)(y-r_1^2)^{1/2}}{(y-\bar{r}_1^2)}\times\left[\frac{A}{y^2}+\frac{B}{y}+C\right]^{1/2}dy \qquad (4.57)$$

with

$$\bar{r}_1^2=\frac{D_-b_i}{D_r|b|^2}r_0^2\ . \qquad (4.58)$$

Equation (4.57) is exact (within our D expansion) and gives the potential barrier by a quadrature. We evaluate the integral (4.57) close to the Eckhaus-Benjamin-Feir instability, i.e., for $k^2\to k_c^2$. Asymptotically we obtain

$$\delta\Phi\simeq\frac{8\sqrt{2}}{15}\frac{\sqrt{b_r}}{Q\sqrt{D_r}}\frac{(A/r_0^4+B/r_0^2+C)^{1/2}}{r_0^2-\bar{r}_1^2}\times\left[\frac{aD_+}{D_r|b|^2}\right]^{5/2}\left[1-\frac{k^2}{k_c^2}\right]^{5/2}\ . \qquad (4.59)$$

Inserting the expressions for $A,B,C,r_0^2,\bar{r}_1^2,k_c^2$ we find

$$\delta\Phi\simeq\frac{4\sqrt{2}}{15Q}\frac{b_r(3D_+b_r+2D_-b_i)}{D_r^3|b|^6}\times\sqrt{D_rD_++D_iD_-}(aD_+)^{3/2}\left[1-\frac{k^2}{k_c^2}\right]^{5/2}\ , \qquad (4.60)$$

i.e., $\delta\Phi\sim(1-k^2/k_c^2)^{5/2}$. Equation (4.60) generalizes a well-known result for the real Ginzburg-Landau equation (thermodynamic equilibrium)[27] to which it reduces for $b_i=0=D_i$.

In the opposite extreme case $k=0$, the integral (4.57) changes because then $\bar{r}_1^2=r_1^2$. We find, using the approximation (4.48) again,

$$\delta\Phi\simeq\frac{8\sqrt{2}}{3Q}\frac{b_r}{|b|^4D_r^2}\sqrt{D_rD_++D_iD_-}(aD_+)^{3/2}\ . \qquad (4.61)$$

The potential barrier vanishes as $D_+^{3/2}$ for $D_+\to 0$, i.e., if the Newell-Kuramoto instability is approached and it vanishes like $(D_rD_++D_iD_-)^{1/2}$ near the newly found stability border (4.31). Similarly, the potential barrier disappears also for $a\to 0$. For the discussion of lifetimes in the real TDGL model see Ref. 28.

V. CONCLUSIONS

In this paper we gave the first example of a nonequilibrium potential with field variables which correctly reflected all known instabilities of the deterministic system and which enabled us to also derive global properties not obtainable via a deterministic analysis. We summarize the features of the potential which are not present in systems with a finite number of degrees of freedom and which may be typical for spatially extended cases.

(i) In systems possessing degenerated attractors (in at least one variable) the determination of the value of the potential $C(A)$ on these attractors is unavoidable. This is a much less trivial task than for discrete systems (where it required to fix a few constants of integration only[17]) and can be solved by working out suitable local expansions around such attractors.

(ii) Boundary conditions may play an important role and can properly be taken into account by means of surface contributions to the potential.

(iii) Our approximation breaks down at the repeller state $\psi\equiv 0$ for $a>0$, where our result becomes singular; however, a singularity of the exact potential at this state cannot be excluded and may occur also in other cases.

(iv) At least in the one-dimensional case extremizing states of the potential (attractors, repellers, and saddles of the deterministic system) can always be obtained as solutions of certain Euler-Lagrange equations with a finite number of degrees of freedom, where the "potential density" plays the role of the Lagrangian and the spatial coordinate is the analog of time. This opens the possibility for finding systematically all such states including, perhaps, also ones which have not yet been found by other methods.

(v) The vanishing of potential barriers among certain attractor states signals the fact that these states lose their stability and are no longer attractors.

(vi) The potential depends on the spatial dimension involving the appearance of nonanalytic terms in higher than one dimension.

Besides these new features, of course, all properties known for the nonequilibrium potential in systems with a finite number of degrees of freedom like, e.g., nondifferentiality, can, in principle, be observed also in spatially extended cases. While this feature did not show up in the one-dimensional case studied in the present work we obtained indications that it is present for the case $d \geq 2$.

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APPENDIX A: THE LIMIT OF DOMINATING DIFFUSION

In order to give another limiting solution of Eqs. (2.3) and (2.4) and to compare with earlier work let us consider the case where the terms with a,b in Eq. (1.2) or (2.1) are both small ($\sim\epsilon$) of the same order and the diffusion term D is of order ϵ^0. Then we may expand both Φ and χ in powers of ϵ. In order to solve the resulting functional differential equations for the coefficients we introduce the Fourier-transformed field, in a finite volume V with cyclic boundary conditions

$$\psi_k = V^{-1/2}\int dx\, e^{ikx}\psi(x) \tag{A1}$$

and its complex conjugate. With the expansion

$$\begin{aligned}\Phi &= \Phi_0+\epsilon\Phi_1+\cdots\ ,\\ \chi &= \chi_0+\epsilon\chi_1+\cdots\ ,\end{aligned} \tag{A2}$$

we have the zero-order solutions

$$\begin{aligned}\Phi_0 &= \frac{2}{Q}\sum_k D_r k^2|\psi_k|^2\ ,\\ \chi_0 &= 0=\text{const}\ ,\end{aligned} \tag{A3}$$

and the first-order equations:

$$\sum_q\left[D^*q^2\psi_q\frac{\partial}{\partial\psi_q}+Dq^2\psi_q^*\frac{\partial}{\partial\psi_q^*}\right]\Phi_1=-\frac{2}{Q}D_r\sum_q q^2\left[a|\psi_q|^2-\frac{1}{2V}\sum_{k,k'}b\psi_k^*\psi_q^*\psi_{k'}\psi_{k+q-k'}+\text{c.c.}\right]\ , \tag{A4}$$

$$\sum_q\left[D^*q^2\psi_q\frac{\partial}{\delta\psi_q}+Dq^2\psi_q^*\frac{\partial}{\delta\psi_q^*}\right]\chi_1=\frac{-2aV}{\lambda^d}+\frac{4b_r}{V\lambda^d}\sum_q|\psi_q|^2-Q\sum_q\frac{\partial^2\Phi_1}{\partial\psi_q\partial\psi_q^*}\ . \tag{A5}$$

These inhomogeneous linear partial differential equations of first order are easily solved by the methods of characteristics. The characteristics satisfy

$$\dot{\psi}_q(\tau)=D^*q^2\psi_q(\tau)\ . \tag{A6}$$

Physically, they describe the most probable paths, to zeroth order in ϵ, by which the system leaves the deterministic attractor at $\tau=-\infty$ and moves to the end point ψ_q at time $\tau=0$. To zeroth order in ϵ the attractor is given by the minimum of Φ_0, i.e., by

$$A:\ \ \psi_q=\tilde{\psi}_0\delta_{q,0}\ , \tag{A7}$$

with arbitrary complex $\tilde{\psi}_0$. We note that the attractor (or union of all attractors) to zeroth order in ϵ is a two-dimensional manifold, parametrized by $\tilde{\psi}_0$. The characteristics satisfying the required initial and final conditions at $\tau=-\infty$ and $\tau=0$, respectively, are

$$\psi_q(\tau)=\psi_q e^{D^*q^2\tau}(1-\delta_{q,0})+\psi_0\delta_{q,0}\ . \tag{A8}$$

It satisfies $\psi_q(-\infty)=\psi_0\delta_{q,0}\in A$, because we can always choose $\tilde{\psi}_0=\psi_0$ in Eq. (A7). It is now easy to determine ψ_1 from Eq. (A4) by integrating the inhomogeneity in this equation along the characteristics (A8). We obtain

$$\Phi_1=\frac{1}{Q}\sum_q{}'\left[-a|\psi_q|^2+\frac{2D_r}{V}\sum_{k,k'}\frac{bq^2\psi_k^*\psi_q^*\psi_{k'}\psi_{k+q-k'}}{D(k^2+q^2)+D^*[k'^2+(k+q-k')^2]}+\text{c.c.}\right]+C(\{\psi_0,\psi_0^*\})\ , \tag{A9}$$

where $\sum'_q$ excludes the value $q=0$. The function $C(\{\psi_0,\psi_0^*\})$ is not determined by Eq. (A4). It gives the value of the nonequilibrium potential, to first order in ϵ (i.e., in a and b), in the point $\tilde{\psi}_0=\psi_0$ of A, Eq. (A7). Fortunately, it is easy to determine $C(\{\psi_0,\psi_0^*\})$ by projecting Eq. (1.2) on A, i.e., on the spatially homogeneous field $\psi(x,t)=(1/V^{1/2})\psi_0(t)$, yielding

$$\dot{\psi}_0=a\psi_0-\frac{b}{V}|\psi_0|^2\psi_0+\xi_0(t) \tag{A10}$$

with

$$\langle \xi_0(t)\xi_0^*(t')\rangle=\eta Q V\delta(t-t') . \tag{A11}$$

The exact nonequilibrium potential of Eq. (A10) then is of first order in a and b and can be identified with $C(\psi_0,\psi_0^*)$

$$C(\psi_0,\psi_0^*)=-\frac{2a}{Q}|\psi_0|^2+\frac{b_r}{QV}|\psi_0|^4+\text{const} . \tag{A12}$$

It should be noted that the addition of $C(\{\psi_0,\psi_0^*\})$ is not equivalent to including the $q=0$ terms in $\sum_q$, because it removes the indeterminacy of the quartic term for the case where all wave numbers vanish. We also note that the potential $\Phi=\Phi_0+\Phi_1$ has local minima for $\psi_q=0$ for all q if $a<0$ and for $\psi_q=V^{1/2}(a/b_r)^{1/2}\delta_{q,0}$ if $a>0$, as it should.

In order to determine χ, we have to evaluate the inhomogeneity of Eq. (A5) by using the result (A9), (A12). We find that remarkably, the inhomogeneity vanishes (after each individual term has been regularized by using the short-distance cutoff), i.e., we obtain the solution

$$\chi_1=\text{const}+\tilde{\chi}_1(\{\psi_0,\psi_0^*\}) , \tag{A13}$$

where, again $\tilde{\chi}_1$ remains undetermined, and has to be found from Eq. (A10). As the steady-state distribution of Eq. (A10) is given exactly by

$$W(\{\psi_0,\psi_0^*\})=\text{const}\times\exp\left[-\frac{C(\{\psi_0,\psi_0^*\})}{\eta}\right] , \tag{A14}$$

we conclude that $\tilde{\chi}_1(\{\psi_0,\psi_0^*\})=0$. In conclusion, we have therefore succeeded to determine the steady-state probability density in the form

$$W(\{\psi,\psi^*\})=\text{const}\times\exp\left[-\frac{\Phi_0}{\eta}-\epsilon\frac{\Phi_1}{\eta}+O(\epsilon^2/\eta,\eta)\right] , \tag{A15}$$

where the parameters a and b are assumed to be small (of order ϵ) compared to Dq^2 for $q\neq 0$. The expansion can be continued, if desired, generating in Φ the higher even powers of the Fourier amplitudes ψ_q. The coefficients of these powers contain increasing powers of the wave numbers in the denominators, but the wave-number sums remain constrained in such a way, that vanishing denominators do not appear.

Finally, let us note that a power series expansion of Φ_1 can also be used directly to find a solution of Eqs. (2.3) and (2.4). This method has been employed in previous work.[20] However, for reasons discussed and exemplified in detail in Ref. 25 an expansion in terms of a small parameter is more systematic and therefore preferable. In particular, as the calculation reported in this section has demonstrated, the crucial boundary conditions (minima in attractors), which a solution of Eq. (2.3) has to satisfy in order to determine the true nonequilibrium potential, can be systematically satisfied when expanding in a small parameter, while the boundary conditions may or may not be satisfied if a direct power-series solution is constructed.

APPENDIX B: SECOND ORDER OF THE D EXPANSION

Here we provide the second-order calculation whose result is given in Eq. (3.46). From the Hamilton-Jacobi equation we obtain in second order

$$\begin{aligned}\int dx\left[(-a+b^*|\psi|^2)\psi\frac{\delta\Phi_2}{\delta\psi}+(-a+b|\psi|^2)\psi^*\frac{\delta\Phi_2}{\delta\psi^*}\right] &= -Q\int dx\left|\frac{\delta\Phi_1}{\delta\psi}\right|^2 \\ &\quad -\left[D\int dx\frac{\delta\Phi_1}{\delta\psi}\psi[i\nabla^2\varphi-(\nabla\varphi)^2]+\text{c.c.}\right] ,\end{aligned} \tag{B1}$$

where, on the right-hand side we have already used the fact that only the gradients of the phase φ are retained in our second-order analysis. For the same reason only the phase-gradient terms of $\delta\Phi_1/\delta\psi$ need to be retained, i.e., we may use

$$\frac{\delta\Phi_1}{\delta\psi}=\frac{2}{Q}\left[D_r\psi^*(\nabla\varphi)^2+i\frac{\nabla^2\varphi}{\psi}\left\{D_r|\psi|^2-\frac{ab_i}{b_r|b|^2}D_-+iD_-\frac{a}{|b|^2}\left[1-\frac{|b|^2}{2b_r^2}\left[1-\frac{b_r}{a}|\psi|^2\right]\right]\right\}\right] . \tag{B2}$$

Using this expression and Eq. (3.6), Eq. (B1) can be reduced to

$$\frac{d\Phi_2(\tau)}{d\tau}=-\frac{1}{Q}\int dx[\nabla^2\varphi(\tau)]^2\left[\frac{D_-^2}{b_r^2|\psi(\tau)|^2}\left[\frac{a}{b_r}-|\psi(\tau)|^2\right]^2+\frac{2D_iD_-}{b_r}\left[|\psi(\tau)|^2-\frac{a}{b_r}\right]\right] . \tag{B3}$$

Remarkably, the $(\nabla\varphi)^4$ terms have dropped out. Integrating along the characteristics (3.7) using the integrals (3.11) and (3.42) we obtain Eq. (3.46) for $a>0$.

APPENDIX C: EXPANSION AROUND PLANE-WAVE ATTRACTORS

Here we wish to provide the potential locally expanded near plane-wave attractors

$$-\psi_{k_0}(x)=r_0 e^{i(k_0x+\varphi_0)} \tag{C1}$$

with

$$r_0=\left[\frac{a-D_r k_0^2}{b_r}\right]^{1/2} . \tag{C2}$$

The method we use is a direct generalization of the calculation for $k_0=0$ performed in Ref. 16. Let us write

$$\psi(x)=[r_0+\delta(x)]e^{ik_0x+i\varphi_0} , \tag{C3}$$

where $\delta(x)$ is a small fluctuation $|\delta(x)|\ll r_0$. Alternatively we may write

$$\psi(x)=[r_0+\delta r(x)]e^{ik_0x+i\varphi_0+i\delta\varphi(x)} \tag{C4}$$

with amplitude and phase perturbations δr and $\delta\varphi$, respectively. Equations (C4) and (C3) are equivalent if $\delta\varphi(x)$ remains small everywhere, in which case we have

$$\delta(x)=\delta r(x)+ir_0\delta\varphi(x) . \tag{C5}$$

It is convenient to introduce the Fourier transforms of $\delta,\delta r,\delta\varphi$ according to Eq. (3.29). A straightforward calculation along the lines of Ref. 16 then leads to the Gaussian approximant

$$\Phi^{(G)}(\{\delta_k,\delta_k^*\})=\Phi_{\mathrm{GL}}^{(G)}(\{\delta_k,\delta_k^*\})-\frac{iD_-}{Q}\sum_k\left[\frac{r_0^2k^2}{b^*r_0^2+D^*k^2}\delta_k\delta_{-k}-\mathrm{c.c.}\right] \tag{C6}$$

where r_0 is given by (C2), which generalizes the result for $k_0=0$ reported in Eq. (3.30). Here $\Phi_{\mathrm{GL}}^{(G)}$ is the local Gaussian expansion of the Ginzburg-Landau potential. For further use it is convenient to reexpress (A6) in δr_k, $\delta\varphi_k$

$$\Phi^{(G)}-\Phi_{\mathrm{GL}}^{(G)}=-\frac{iD_-}{Q}\sum_k\left[\frac{r_0^2k^2}{b^*r_0^2+D^*k^2}[|\delta r_k|^2-r_0^2|\delta\varphi_k|^2+ir_0(\delta\varphi_k\delta r_k^*+\delta\varphi_k^*\delta r_k)]-\mathrm{c.c.}\right] . \tag{C7}$$

Equation (C7) is valid for small δr, $\nabla\delta\varphi$, and arbitrary dimensionality. We now wish to compare the result (C7) with the corresponding local expansion of our potential $\Phi_{n\mathrm{GL}}+\Phi_2$ given by Eqs. (3.45), (3.46), and (3.51). Here the ansatz for $C_2(A)$ [i.e., the second term of (3.51)] is assumed to be independent of $\mathbf{k}_0$ and the coefficients $\alpha,\alpha_{11},\alpha_{12},\alpha_2$ of Eq. (3.51) are not yet determined. As the only goal is to determine these unknown coefficients it is sufficient to record merely the quadratic form in $\nabla\delta\varphi$, $\nabla\delta r$ contained in the local expansion:

$$\begin{aligned}\Phi_{n\mathrm{GL}}^{(G)}+\Phi_2^{(G)}=\frac{1}{Q}\int dx\Bigg[&-\frac{2D_-b_ia}{b_r|b|^2}(\nabla\delta\varphi)^2+\frac{2D_-}{b_rr_0}\left[r_0^2+\frac{a(b_r^2-b_i^2)}{b_r|b|^2}\right](\nabla\delta r\cdot\nabla\delta\varphi)\\&+\frac{2D_-b_i}{3b_r^3r_0^2|b|^2}[a(2b_r^2-b_i^2)+b_r|b|^2r_0^2](\nabla\delta r)^2+4\alpha(\mathbf{k}_0\cdot\nabla\delta\varphi)^2+2\alpha k_0^2(\nabla\delta\varphi)^2\\&+\frac{\alpha_{11}k_0^2}{r_0^2}(\nabla\delta r)^2+\frac{\alpha_{12}}{r_0^2}[(\mathbf{k}_0\cdot\nabla\delta r)^2-k_0^2(\nabla\delta r)^2]\\&+\frac{\alpha_2k_0^2}{r_0}(\nabla\delta r\cdot\nabla\delta\varphi)+\frac{2\alpha_2}{r_0}(\mathbf{k}_0\cdot\nabla\delta\varphi)(\mathbf{k}_0\cdot\nabla\delta r)\Bigg] .\end{aligned} \tag{C8}$$

We note that Φ_{a_2} of Eq. (3.46) does not contribute to Eq. (C8) as it contains $(\nabla^2\varphi)^2$ terms only.

We now wish to compare the result (C8) with Eq. (C7) and therefore need to consider Eq. (C7) only in the limit $k^2\to0$, where $r_0^2k^2/(b^*r_0^2+D^*k^2)$ reduces to k^2/b^*. We take this limit and transform back to real space. Then let us split $\nabla=\nabla_\parallel+\nabla_\perp$ with $\nabla_\parallel=\mathbf{k}_0(\mathbf{k}_0\cdot\nabla)k_0^{-2}$, $\nabla_\perp=\nabla-\nabla_\parallel$ and let us first compare the coefficients of the $\nabla_\parallel$ terms. These are the only possible terms in one dimension. This comparison yields the results given in Eq. (3.52). In the general case $d\geq2$ additional terms with $\nabla_\perp$ appear in Eq. (C8). Of these the $(\nabla_\perp\delta r)^2$ term is matched with (C7) by choosing $\alpha_{12}=0$. However, it is impossible to match the remaining $(\nabla_\perp\delta\varphi)^2$, $(\nabla_\perp\delta\varphi)\cdot(\nabla_\perp\delta r)$ terms without spoiling the matching for the $\nabla_\parallel$ terms. The matching would be possible if still higher-order terms are included in our ansatz, but the coefficients then contain divergencies for $r^2\to a/b_r$, which is not acceptable. We are forced to conclude from this failure that the ansatz (3.51) is not possible for $d\geq2$ with coefficients quadratic in D. Nonanalytic terms appear because it is not possible for $d\geq2$ to reconcile, within an analytical D expansion, the appearance of plane-wave attractors with rotational invariance. More specifically, for $d\geq2$ the contribution of Eq. (3.51), for fixed $\nabla\psi$, must depend on the direction of the wave vector $\mathbf{k}_0$ of each plane-wave attractor A. The minimum over A in Eq. (1.12) will then yield a nonanalytic dependence on $\nabla\psi$ and D. This problem does not occur for $d=1$, hence nonanalytic terms are not to be expected there. In this connection we note that the result for the potential for $d=1$ is *not* contained in the result for $d \geq 2$ as the special case where r and φ depend on one spatial variable only. Rather the cases $d=1$ and $d \geq 2$ differ fundamentally. The reason is that the fluctuating force ξ in Eq. (1.2) can depend only on a single spatial coordinate for $d=1$ (and therefore becomes highly anisotropic if this case is embedded in $d \geq 2$), while for a rotationally invariant system with $d \geq 2$ the force ξ must depend on all spatial coordinates. This rotational invariance enforced by the fluctuations, is the reason for the appearance of nonanalytic terms in D for $d \geq 2$. Our applications in Sec. IV are restricted to the one-dimensional case where the nonanalytic terms are not present.

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