First-passage-time statistics in disordered media

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A method is presented for the study of first-passage-time statistics in one-dimensional disordered media. We introduce a projection-operator formalism that allows the calculation of an averaged backwards master equation. A generalized Dynkin's equation is also presented. We apply this approach to the random-trap model. The exact mean first-passage time to leave a finite domain is calculated for weak disorder. To study strong disorder we introduce a mean-field-like approximation that gives exact results at long times. Long-time tails in the survival probability and in the first-passage-time distribution are found, predicting a divergent mean first-passage time for the case of strong disorder. Our findings confirm numerical results found previously.

I. INTRODUCTION

In the context of studies of transport of classical particles and excitations in disordered media, the model of the nearest-neighbor random walk (RW) in a one-dimensional disordered chain describes a number of physically relevant situations.[1] The model assumes that the position of the walker in a particular chain undergoes a Markovian RW defined by a master equation of the form

$$\begin{aligned}\partial_t P(n,t|n_0,t_0,\{w_i^\pm\}) = {} & w_{n-1}^+ P(n-1,t|n_0,t_0,\{w_i^\pm\}) \\ & + w_{n+1}^- P(n+1,t|n_0,t_0,\{w_i^\pm\}) \\ & -(w_n^+ + w_n^-)P(n,t|n_0,t_0,\{w_i^\pm\}) ,\end{aligned} \tag{1.1}$$

where w_n^+ (w_n^-) is the transition probability per unit time from site n to $n+1$ ($n-1$). A particular configuration of the disordered chain is described by the set $\{w_i^\pm\}$. $P(n,t|n_0,t_0,\{w_i^\pm\})$ is the probability of finding the walker at site n at time t, for a particular configuration $\{w_i^\pm\}$ of the medium and with the condition of having been at site n_0 at time t_0.

Disorder is described as assigning a probability $P\{w_i^\pm\}$ to each possible configuration of the chain. This assignment of probabilities, together with (1.1), defines a RW that, in contrast to that defined for a particular set $\{w_i^\pm\}$, is no longer Markovian. The statistical properties of the new process can, in principle, be obtained by averaging over configurations of the disorder the expressions for the quantities of interest obtained for fixed $\{w_i^\pm\}$.

Non-Markovian behavior manifests itself in the appearance of a frequency dependence in the diffusion coefficient associated with the RW. If this dependence is such that the diffusion coefficient vanishes at zero frequency, the motion is subdiffusive, that is, the second moment of the position, $\langle n(t)^2\rangle$, grows slower than linearly with time for $t\to\infty$. Models leading to this behavior are usually called models of *strong disorder*, whereas models for which $\langle n^2(t)\rangle \sim t$ are of *weak disorder*.

Although the behavior of the moments of the position of the walker has been widely studied for different models of disorder, there is another manifestation of non-Markovian behavior that remains to be explored in detail: A non-Markovian process it not completely described by a two-time probability distribution like that appearing in (1.1), but the whole hierarchy of m-time probability distributions is needed. Then, the effect of disorder in quantities depending not only on two-time probability distributions is expected to be important. One such quantity is the mean first-passage time (MFPT), for which the memory effect must be taken into account in order to write the correct backwards equation. Due to this fact, extreme care must be taken if one attempts to obtain exact results for the MFPT in disordered systems.

The first-passage time (FPT) is a random variable defined as the time the walker takes to leave a certain interval on the chain. If the interval is semi-infinite, it coincides with the time needed to reach a particular site. The first-passage-time distribution (FPTD) can be obtained from the survival probability, which is defined here as the probability of remaining in the interval of interest (which will be called D) for a time interval $t-t_0$, having started from some site in D at $t=t_0$.

A few attempts of describing diffusion in disordered media by FPT statistics have been recently reported:[2–7] Continuous models are studied in Refs. 2 and 3. Discrete models of weak disorder are considered in Ref. 4. References 5 and 6 show that for the Sinai model the MFPT scales with the size of the interval considered in a different way than the *typical* FPT. An in-depth discussion of some models in discrete time is found in Ref. 7. A different but related problem that has received a lot of attention is that of calculating a survival probability in an infinite chain for normal diffusion in the presence of randomly distributed traps.[1(b),1(c)]

A common feature of the models studied in Refs. 2–7 is that, in absence of trapping, the MFPT is finite if the interval considered is finite. In contrast, a class of models was recently found in which the MFPT to leave any finite interval is infinite.[8] This was used in Ref. 8 to propose the association of the concept of strong disorder with a divergent MFPT. In the case of strong disorder, it was also proposed[8] to characterize the degree of disorder by the long-time tail present in the FPTD or, equivalently, by the divergence at small frequencies of the Laplace transform of the survival probability. It is the aim of the present paper to develop a technique to calculate such quantities, as an alternative to the usual way of characterizing disorder and diffusion by moments of the displacement and diffusion coefficients.

The calculation of the MFPT or the FPTD for non-Markovian processes is not an easy task. Formally exact equations for these quantities can be obtained,[9] but they contain operators that must be *adjusted* in order to prevent contributions from trajectories returning to the interval of interest after having left it. This adjustment can be done for Markov processes by imposing adequate boundary conditions. Special non-Markovian processes, such as the continuous-time random walk (CTRW), can be studied,[10] but no recipe for adjusting the operators is known in the general non-Markovian case.

To overcome this difficulty, the method used in Refs. 2–7 begins by calculating the MFPT or an approximation to it for a fixed configuration $\{w_i^{\pm}\}$ of the disorder. This can be done by Markovian techniques because the RW is Markovian for fixed $\{w_i^{\pm}\}$. The result obtained in this way is then averaged with the probability distribution of the configurations $\{w_i^{\pm}\}$.

In this paper we introduce another method for calculating FPT properties in the presence of disorder. We start with the Markovian equations satisfied by the survival probability for a fixed configuration $\{w_i^{\pm}\}$. Then, the equations themselves (not their solutions) are averaged over disorder by a projection-operator method. In this way we obtain exact equations for the MFPT and related quantities. Systematic approximation procedures are here developed to obtain relevant information from these equations. In particular, the MFPT and the long-time behavior of the FPTD can be calculated within our framework even in the case of strong disorder.

We use this general framework to study several models of disorder. For definiteness we restrict ourselves to nearest-neighbor RW's that are symmetric in the sense that w_n^+ and w_n^- in (1.1) are equal:

$$w_n^+ = w_m^- \equiv w_n , \quad \forall n . \tag{1.2}$$

The class of models satisfying (1.2) is generically called the *random-trap model*.[1] Results for another important model of disordered chain, the *random-barrier model*,[1] will be discussed elsewhere. The type of disorder considered is characterized by the statistical properties of $\{w_i^{\pm}\}$. They are taken to be independent but identically distributed random variables. We have considered the following models.

(a) Weak disorder [model A of Ref. 1(a)]: All the inverse moments $\langle 1/w^k \rangle$, $k=1,2,\ldots$ are finite.

(b) Model B of Ref. 1(a): The probability distribution for each variable w_i is

$$\rho(w) = \begin{cases} 1 & \text{if } w \in (0,1) \\ 0 & \text{otherwise} . \end{cases} \tag{1.3}$$

(c) Model C of Ref. 1(a):

$$\rho(w) = \begin{cases} (1-\alpha) w^{-\alpha} & \text{if } w \in (0,1) \\ 0 & \text{otherwise} . \end{cases} \tag{1.4}$$

(d) Chain with randomly distributed absorbing sites:

$$\rho(w) = p\,\delta(w - w_0) + (1-p)\,\delta(w) . \tag{1.5}$$

Models (b), (c), and (d) are cases of strong disorder. The mean-square displacement $\langle n(t)^2 \rangle$ behaves for long times as $t/\ln t$, $t^{2(1-\alpha)/(2-\alpha)}$, and t^0, respectively.

Exact results for the long- and short-time behavior of the survival probability are directly obtained from our general equations in the weak-disorder case. The study of the long-time limit for strong disorder needs the introduction of a mean-field-like approximation. Within this approximation a survival probability is obtained which exhibits long-time tails. The MFPT to leave any finite interval is infinite for our random-trap models with strong disorder. We show that the mean-field-like approximation introduced here reproduces the exact result for the leading term in the long time limit for models (a) and (b) defined before. For models (c) and (d), our mean-field approximation gives the exact exponent for the divergence law.

The outline of the paper is as follows. In Sec. II we review some useful results for the backwards master equation[11] of Markovian RW's and its solution for the survival probability in a nondisordered chain. They will be used in the subsequent sections. In Sec. III we introduce a projection-operator technique to average the backwards master equation over the configurations of disorder. We obtain an exact evolution equation for the averaged survival probability. A generalization of Dynkin's equation for disordered media is also presented. Its exact solution is found in the case of weak disorder. In Sec. IV we study the different cases of strong disorder by means of a mean-field-like approximation. A short discussion of the results found is presented in Sec. V. Appendix A considers the FPTD in a nondisordered chain, and Appendix B is devoted to the calculation of the Green's function for a RW with absorbing boundaries. These quantities are needed in Secs. II and III of this paper. Finally, the validity of the approximation used in Sec. IV is analyzed in Appendix C.

II. FIRST-PASSAGE-TIME STATISTICS FOR MARKOVIAN PROCESSES

A. Backwards master equation

Let us consider a general time-homogeneous Markovian RW in an infinite chain. The forward master equation has the general form[11]

$$\partial_t P(n,t|n_0,t_0) = \sum_{n'} (H)_{n,n'} P(n',t|n_0,t_0) , \tag{2.1}$$

where $P(n,t|n_0,t_0)$ is the probability of being at site n at time t with the condition to start at n_0 at time $t_0<t$. H is an infinite square matrix whose indices run over the sites in the chain.

$P(n,t|n_0,t_0)$ also fulfills the backwards master equation, which, invoking the time homogeneity of the process, can be written as[11]

$$\partial_t P(n,t|n_0,t_0)=\sum_{n'}(H^+)_{n_0,n'}P(n,t|n',t_0)\ , \tag{2.2}$$

where H^+ is the transpose matrix of H. We are interested in the time at which the walker leaves for the first time the interval $D\equiv[-L,L]$, that is, the FPT through the limits of such an interval. The problem of finding the FPTD can be reduced to the study of $P_D(n,t|n_0,t_0)$, the probability of being at n at time t with the condition of never having left the interval D since the initial time t_0 in which the walker was at n_0.

In order to obtain $P_D(n,t|n_0,t_0)$, we note that this is the two-time conditional probability for a new Markovian walk, defined as follows:[11(a)] jumps of the walker inside D and from D to the outside are governed by the same probability rates appearing in the matrix H as the original process, but jumps from outside D into D are forbidden. In other words, $P_D(n,t|n_0,t_0)$ satisfies equations of the form (2.1) or (2.2) with the matrix H replaced by H_D, such that

$$(H_D)_{n,n'}=\begin{cases}(H)_{n,n'} & \text{if } n'\in D\\ 0 & \text{if } n'\notin D\ .\end{cases} \tag{2.3}$$

This structure and the fact that we only need $P_D(n,t|n_0,t_0)$ for $n,n_0\in D$ allow us to consider only a finite number of elements of H_D. For example, the backwards equation for $P_D(n,t|n_0,t_0)$ can be written

$$\partial_t P_D(n,t|n_0,t_0)=\sum_{n'\in D}(H_D^+)_{n_0,n'}P_D(n,t|n',t_0)\ , \tag{2.4}$$

where now H_D^+ (and also H_D) is a finite matrix having as many rows and columns as sites as are contained in the interval D. All the indices in (2.4) take values only in D.

If $F_{n_0}(t/t_0)$ denotes the survival probability, that is, the probability for the walker starting at t_0 at n_0 to be still at time t in the domain D without ever having left this interval,

$$F_{n_0}(t|t_0)\equiv\sum_{n\in D}P_D(n,t|n_0,t_0) \tag{2.5}$$

the probability density $f_{n_0}(t)$ for the FPT is given by[11]

$$f_{n_0}(t)=-\partial_t F_{n_0}(t)\ . \tag{2.6}$$

Here and in the following we take $t_0=0$.

The evolution equation for the survival probability $F_{n_0}(t)$ follows from Eqs. (2.4) and (2.5):

$$\partial_t F(t)=H_D^+F(t)\ . \tag{2.7}$$

Here $F(t)$ is a vector with components $(F(t))_{n_0}\equiv F_{n_0}(t)$. The initial condition for Eq. (2.7) is $F_{n_0}(t=0)=1$ for all $n_0\in D$.

Even when it is difficult to carry out the calculation for $F(t)$ in detail, it may still be possible to calculate the moments $T_{n_0}^k$ of the FPT:

$$T_{n_0}^k\equiv\int_0^\infty t^k f_{n_0}(t)dt\ . \tag{2.8}$$

It is easy to obtain a simple equation for all these moments when they are finite: multiplying Eq. (2.7) by $T_{n_0}^{k-1}$, integrating by parts, and using that

$$T_{n_0}^k=k\int_0^\infty t^{k-1}F_{n_0}(t)dt\ , \tag{2.9}$$

the following equation is obtained:

$$H_D^+T^k=-kT^{k-1}\ , \tag{2.10}$$

which is a recurrence relation for all the moments of the FPT starting from that of zeroth order $T_{n_0}^0=1$ for all n_0. For $k=1$, (2.10) is the Dynkin equation.[11]

As an example of the ideas just explained, we consider the symmetric Markovian walk defined by

$$\begin{aligned}\partial_t P(n,t|n_0,t_0)=&\,w_{n-1}P(n-1,t|n_0,t_0)\\&+w_{n+1}P(n+1,t|n_0,t_0)\\&-2w_nP(n,t|n_0,t_0)\ .\end{aligned} \tag{2.11}$$

From the prescription given in Eq. (2.3), for the finite matrix H_D^+, we can write the following expression:

$$H_D^+=\begin{pmatrix}-2w_{-L} & w_{-L} & 0 & 0 & \cdots & & & & 0\\ w_{-L+1} & -2w_{-L+1} & w_{-L+1} & 0 & & & & & 0\\ 0 & w_{-L+2} & -2w_{L+2} & w_{-L+2} & & & & & 0\\ \cdots & & & \cdots & \cdots & & & & \cdots\\ 0 & & & & w_{L-2} & -2w_{L-2} & w_{L-2} & & 0\\ 0 & & & & 0 & w_{L-1} & -2w_{L-1} & & w_{L-1}\\ 0 & \cdots & & & 0 & 0 & w_L & & -2w_L\end{pmatrix}\ . \tag{2.12}$$

The survival probability in the domain $D=[-L,L]$ is the solution of the backwards master equation (2.7) with (2.12). If the domain D has a small number of sites N $(=2L+1)$ the problem of solving this equation reduces to the inversion of an $N\times N$ matrix.

We stress that there is no need to impose any boundary condition to solve the above problem. Nevertheless, if we are interested in a domain with an arbitrary number N of sites, we can solve Eq. (2.7) by another method: Instead of solving the finite-dimensional problem characterized by the matrix equation (2.7), we can formulate the following problem: (a) Use the infinite matrix H^+ of the original walk appearing in Eq. (2.2) so that the backflow of probability into the interval D is, in principle, allowed. (b) Solve the equation

$$\partial_t F(t)=H^+F(t) , \tag{2.13}$$

with the initial condition $F_{n_0}(t=0)=1$ for all $n_0 \in D$, and prevent backflow by using the *artificial* boundary conditions $F_{n_0}(t)=0$ for all t if $n_0 \notin D$. The solution of this problem (into the domain D) is also the solution of Eq. (2.7).

B. First-passage-time distribution in an homogeneous chain

Basic results about the FPTD in nondisordered chains can be easily obtained from Eq. (2.7) and prescriptions (a) and (b). The ordinary RW in an homogeneous chain is obtained by putting $w_n=\mu$ for all n in Eq. (2.11). In this case $F_n(t)$ satisfies the following equation (here and in the following the initial condition n_0 is denoted simply by n):

$$\partial_t F_n(t)=\mu(E^+ + E^- -2)F_n(t) , \tag{2.14}$$

where $E^\pm$ are shifting operators $E^\pm F_n \equiv F_{n\pm 1}$. Equation (2.14) must be solved with the initial condition

$$F_n(t=0)=1 \quad \text{if } n \in D , \tag{2.15}$$

and the *artificial* boundary conditions

$$F_{-L-1}(t)=F_{L+1}(t)=0 \quad \forall t . \tag{2.16}$$

The additional boundary conditions $F_{\pm(L+2)}(t) =F_{\pm(L+3)}(t)= \cdots =0$ are not needed because Eq. (2.14) only involves transitions to nearest neighbors. From Eq. (2.14) we obtain for the Laplace transform $\hat{F}_n(z)$ of the survival probability (see Appendix A for a detailed calculation)

$$\hat{F}_n(z)=\frac{1}{z}\left[1-\frac{x_1^n+x_2^n}{x_1^{L+1}+x_2^{L+1}}\right] , \tag{2.17a}$$

where

$$x_{1,2}(z)\equiv 1+\frac{z}{2\mu}\pm\left[\frac{z}{\mu}\left[1+\frac{z}{4\mu}\right]\right]^{1/2} . \tag{2.17b}$$

From (2.17) and the Laplace representation of Eq. (2.9) it is easy to calculate the MFPT, $T_n \equiv T_n^1$:

$$T_n=\lim_{z\to 0}\hat{F}_n(z)=\frac{(L+1)^2-n^2}{2\mu} . \tag{2.18}$$

Comparing this with (A13) we see that only the presence of $L+1$, instead of L in (2.18), reminds us of the discrete nature of the lattice.

To gain insight in the description of the problem, we comment that the continuous limit of the survival probability is easily obtained from Eq. (2.17). Introducing a lattice spacing l, an interval $\mathcal{D}\equiv[-X,X]$, $X\equiv lL$, and a variable $x_0 \equiv nl$, we obtain in the continuous limit ($l\to 0, L\to\infty, \mu\to\infty$ with the diffusion coefficient $\Delta\equiv\mu l^2$ and X finite) the following survival probability in $\mathcal{D}$:

$$\hat{F}(x_0,z)=\frac{1}{z}\left[1-\frac{\cosh[x_0(z/\Delta)^{1/2}]}{\cosh[X(z/\Delta)^{1/2}]}\right] . \tag{2.19}$$

The FPTD for leaving the domain $\mathcal{D}$ is obtained from (2.19) as $\hat{f}(x_0,z)=1-z\hat{F}(x_0,z)$. The result confirms the expected scaling behavior $T_0 \sim X^2$ for $X\to\infty$. We will prove in Sec. III that this type of scaling is also obtained in weak disordered media.

Due to the importance of the tails of the survival probability, strong disorder breaks down the scaling that one could expect from the second moment of the displacement of the walker in anomalous diffusion [that is, if $\langle n^2(t)\rangle \sim t^\delta$, $\delta<1$, for $t\to\infty$, one would expect $T_0\sim L^{2/\delta}$ for $L\to\infty$]. We will show in Sec. IV that the MFPT in the presence of strong disorder is a divergent quantity, independent of the size L of the interval considered. We would like to stress that this divergence has no relation with other divergences which appear when considering semi-infinite intervals. For example, for a symmetric Markovian RW, the MFPT to reach a given site $L+1$, say, on the right of the initial one, is infinite because of the contributions from trajectories going arbitrarily far away on the left of the initial site. If a drift in the adequate direction is introduced, for instance, using a master equation (2.1) with $H=(E^+-1)b+(E^--1)a$ and $a>b$ (drift to the right), the divergence disappears. This can be seen by solving Dynkin's equation (2.10) for the MFPT to reach the site $L+1$ (which is the same as the mean time needed to leave the interval $(-\infty,L]$) starting from $n\le L$. We find

$$T_n=\frac{L+1-n}{a-b} \quad (a>b) . \tag{2.20}$$

This is a finite quantity. For large L, it gives a linear relation between T_n and L, typical of biased diffusion.

If the drift points to the left, a divergence is found again:

$$T_n=\lim_{A\to-\infty} A\frac{1-(b/a)^{n-L-1}}{a-b}=\infty \quad (a<b) . \tag{2.21}$$

This is so because the drift moves the walker arbitrarily away from site L.

The reason for these divergences is always the infinite size of the segment $(-\infty,L]$ where the walker evolves before reaching the point $L+1$. The aim in this paper is the study of the exit from a finite interval $D=[-L,L]$ in a disordered chain. By considering a finite interval D we make sure that any divergence that we find is due exclusively to the disorder in the chain.

III. FIRST-PASSAGE-TIME STATISTICS IN DISORDERED MEDIA

A. Projection operator average

Our starting point is the master equation (2.11) describing symmetric one-step RW's. When complemented with a statistical model for the w_n's, such as (1.3)–(1.5), it becomes the random-trap model. Equation (2.11) can be written, splitting the transition probabilities w_n in an average μ and a random part ξ_n ($w_n=\mu+\xi_n$, $\langle \xi_n \rangle=0$), as

$$\partial_t P_n(t)=\mu\mathcal{K}P_n(t)+\Theta P_n(t) . \tag{3.1}$$

We write $P(n,t|n_0,t_0)$ as $P_n(t)$. $\mathcal{K}$ is the operator describing the random walk in the absence of disorder $\mathcal{K}=(E^+ + E^- -2)$, and Θ is the contribution to the transition probabilities due to random parts ξ_n. For the random-trap model considered here we have $\Theta P_n \equiv (E^+ + E^- -2)\xi_n P_n$. Other models can be studied simply by changing the expression for Θ. The evolution equation for the survival probability is given by Eq. (2.7). The operator H_D^+ can be constructed from (3.1):

$$\partial_t F(t)=(\mu\mathcal{K}_D+\Theta_D)^+ F(t) . \tag{3.2}$$

The matrix $H_D^+ \equiv (\mu\mathcal{K}_D+\Theta_D)^+$ is given by the expression (2.12) identifying $w_n=\mu+\xi_n$.

The average of Eq. (3.2) over the realizations of ξ_n leads to an averaged backwards equation with effective transition probabilities which incorporate in an exact way the effects of disorder. This average can be formally carried out introducing a projection operator $\mathcal{P}$ that averages over disorder[12,13]

$$\langle F \rangle=\mathcal{P}F, \quad F=\langle F \rangle+(1-\mathcal{P})F . \tag{3.3}$$

Applying the operator $\mathcal{P}$ to Eq. (3.2), we obtain

$$\partial_t \langle F \rangle=\mu\mathcal{K}_D^+\langle F \rangle+\mathcal{P}\Theta_D^+\langle F \rangle+\mathcal{P}\Theta_D^+(1-\mathcal{P})F . \tag{3.4}$$

Also, applying the operator $1-\mathcal{P}$ to Eq. (3.2) we arrive at

$$\partial_t (1-\mathcal{P})F \equiv \mu\mathcal{K}_D^+(1-\mathcal{P})F+(1-\mathcal{P})\Theta_D^+\langle F \rangle + (1-\mathcal{P})\Theta_D^+(1-\mathcal{P})F . \tag{3.5}$$

A formal solution of (3.5) can be written as

$$(1-\mathcal{P})F=\int_0^t dt' G(t|t')[(1-\mathcal{P})\Theta_D^+\langle F(t')\rangle + (1-\mathcal{P})\Theta_D^+(1-\mathcal{P})F(t')] , \tag{3.6}$$

where we have used the fact that the initial condition does not depend on ξ_n: $(1-\mathcal{P})F(t=0)=0$. The function $G(t|t')$ is the Green's function for the nondisordered system, that is, the solution of the initial-value problem

$$\partial_t G(t|t')=\mu\mathcal{K}_D^+ G(t|t') , \quad G(t|t)=\underline{1} . \tag{3.7}$$

$\underline{1}$ is the $(2L+1)\times(2L+1)$ identity matrix. Equation (3.6) can be iteratively solved for $(1-\mathcal{P})F$. Putting this solution into Eq. (3.4) we find a closed equation for the averaged survival probability $\langle F(t)\rangle$, which gives the complete description of the problem of FPT statistics in disordered media:

$$\partial_t \langle F \rangle=\mu\mathcal{K}_D^+\langle F \rangle+\left\langle \sum_{k=0}^{\infty} [\Theta_D^+\tilde{M}(1-\mathcal{P})]^k \Theta_D^+ \right\rangle \langle F \rangle . \tag{3.8}$$

$\tilde{M}$ is a convolution operator, defined by

$$(\tilde{M}A(t))_n \equiv \sum_m \int_0^t dt' G_{nm}(t|t') A_m(t') . \tag{3.9}$$

Equation (3.8) requires that the statistics of the random variables ξ_n for each particular model are specified.

Equation (3.8) contains the Green's function $G(t|t')$. It is the solution of the finite-dimensional matrix equation (3.7). As before [see prescriptions (a) and (b) after Eq. (2.12)] we can substitute the problem addressed in Eq. (3.7) by the problem in an infinite chain with adequate boundary conditions:

$$\partial_t G(t|t')=\mu\mathcal{K}^+ G(t|t') , \tag{3.10}$$

where $\mathcal{K}^+=\mathcal{K}=E^+ + E^- -2$, the initial condition is $G(t|t)=\underline{1}$, and the boundary conditions are

$$G_{L+1,n}(t|t')=G_{-L-1,n}(t|t')=0 . \tag{3.11}$$

Using the method of the images[14] we can write the solution of (3.10) as

$$G_{n,m}=\sum_{k=-\infty}^{\infty} G^0_{n+4k(L+1),m}-G^0_{-n-(4k+2)(L+1),m} , \tag{3.12}$$

where $G^0_{n,n'}(t|t')$ is the *free* Green's function,[11] i.e., the solution of (3.10) with boundary conditions at infinity,

$$G^0_{nm}(t|t')=\exp[-2\mu(t-t')]I_{|n-m|}(2\mu(t-t')) . \tag{3.13}$$

Here $I_r(\tau)$ is a modified Bessel function. In the Laplace $(t\to z)$ representation, the sum (3.12) can be easily evaluated. We obtain for the Laplace transformed Green's function $\hat{G}(z)$ in the interval $[-L,L]$ the expression (see Appendix B)

$$\hat{G}_{nm}(z)=B(z)(1-\epsilon^4 A^4)^{-1} \times [A^{|n-m|}-\epsilon^2 A^2(A^{-(n+m)}+A^{n+m}) + \epsilon^4 A^4 A^{-|n-m|}] , \tag{3.14}$$

where $\epsilon \equiv [A(z)]^L$. A and B are functions of the Laplace variable z and they are given in Appendix B.

FPT statistics can be studied from our general equation (3.8). For instance, the MFPT in disordered media $\langle T_n \rangle=\langle \hat{F}_n(z\to 0)\rangle$ is the solution of

$$\mu\mathcal{K}_D^+\langle T_n \rangle+\lim_{z\to 0}\left\langle \sum_{k=0}^{\infty} [\Theta_D^+\tilde{M}(1-\mathcal{P})]^k \Theta_D^+ \right\rangle \langle T_n \rangle=-1 . \tag{3.15}$$

This equation is the generalization of Dynkin's equation, which takes into account the non-Markovian effects introduced by the averaging over configurations of the disordered medium.

It is well known that generalized master equations (GME's) for the probability distribution of the position of the walker $P_n(t)$ can be used to study transport in disordered media. The kernel associated with the GME can be constructed in the context of the effective-medium approximation[1(a),1(b)] (EMA), CTRW theories,[14] or by exact methods.[12,13] We want to note that in order to solve the problem of FPT statistics we cannot use the adjoint or backwards equation of this GME because of the non-Markovian nature of the problem.[9] It is necessary to *adjust* the adjoint equation in order to eliminate contributions from trajectories returning to the interval of interest after having left it. The non-Markovian kernel in (3.8) takes into account contributions from the adequate trajectories in the correct way so as to prevent backflow of probability into the domain D. This fact is reflected by the appearance of the Green's function $G(t|t')$ [Eq. (3.14)] in the convolution operator $\tilde{M}$ [Eq. (3.9)]. This is a remarkable difference with the case of studying the probability distribution for the position of the walker $P_n(t)$ in a disordered medium. In that case[13] the average probability distribution satisfies an equation similar to (3.8), but the convolution operator contains the *free* Green's function $G^0(t|t')$ [Eq. (3.13)] instead of $G(t|t')$ [Eq. (3.14)]. Another important difference between Eq. (3.8) and the similar one for $P_n(t)$ is that Eq. (3.8) is a finite-dimensional matrix equation, as was pointed out by the explicit notation $\mathcal{H}_D$ and Θ_D. As before, instead of solving this finite system, we will solve the associated infinite system with the artificial boundary conditions $\langle F_n(t)\rangle=0$ if $n\notin D$. Using the explicit form of Θ we can write for each component of the vector $\langle F(t)\rangle$ in the Laplace representation the following equation:

$$z\langle \hat{F}_n(z)\rangle-1=\mu\mathcal{H}\langle \hat{F}_n(z)\rangle+\sum_{p=0}^{\infty}\sum_{n_1,\dots,n_p\in[-L,L]}\langle \xi_n\xi_{n_1}\cdots\xi_{n_p}\rangle_T J_{nn_1}(z)J_{n_1n_2}(z)\cdots J_{n_{p-1}n_p}(z)\mathcal{H}\langle \hat{F}_{n_p}(z)\rangle\ , \tag{3.16}$$

where $\langle \xi_n\xi_{n_1}\cdots\xi_{n_p}\rangle_T\equiv\mathcal{P}\xi_n(1-\mathcal{P})\xi_{n_1}\cdots(1-\mathcal{P})\xi_{n_p}$ are Terwiel's[15] cumulants of the random variables $\{\xi_{n_i}\}$ (see Appendix C) and $J_{nn'}(z)$ is the Laplace transform of $\sum_m(\mathcal{H}_D)_{nm}G_{mn'}(t|t')$. Noting that the indices n_i in the sum of Eq. (3.16) run from $-L$ to L, and that the boundary conditions for G ensure that $G_{L+1,n}=G_{-L-1,n}=0$, we see that we can use $J_{nn'}(t|t')=\mathcal{H}G_{nn'}(t|t')$, where $\mathcal{H}$ is the nondisordered part of the master operator in the infinite chain $[\mathcal{H}=(E^++E^--2)]$ and acts on the first index of $G_{nn'}$. Explicit expressions for $J_{nn'}$ are given in Appendix B.

The solution of (3.16) has to satisfy the boundary condition

$$\langle \hat{F}_n(z)\rangle=0,\quad \forall n\notin D\ . \tag{3.17}$$

We will call Eq. (3.16) the effective backwards master equation. We will study the behavior of the survival probability and the FPTD in the short- and long-time limits from Eq. (3.16) for several statistical models for the ξ_n's. We show in Sec. III B that (3.16) predicts results which cannot be obtained if we try to obtain the FPTD simply by taking the adjoint of the kernel in the GME for $P_n(t)$.

B. Solution of the effective backwards master equation

The short-time behavior of $\langle F\rangle$ can be obtained from (3.16) considering the $z\to\infty$ limit of expression (B9) for the propagator

$$J_{nn'}(z)\sim-\frac{2}{z}\delta_{nn'}+\frac{1}{\mu}(\mu/z)^{|n-n'|}(1-\delta_{nn'})\ . \tag{3.18}$$

We then see that Eq. (3.16) is a good perturbative series for short times in the sense that the p contribution in the sum is of order z^{-p}. The dominant contributions to the survival probability and to the FPTD at short times turn out to be independent of disorder. So we can use for the FPTD the result obtained for a nondisordered chain (A10) for $z\to\infty$: $\langle \hat{f}(z)\rangle\sim z^{-(L+1)+n}$. Using the Tauberian theorem,[16] we see that, for all kinds of disorder, the short-time behavior is

$$\langle f_n(t)\rangle\sim t^{L-n}\ \text{ if } t\to 0\ . \tag{3.19}$$

This result can also be understood by using heuristic arguments.

More interesting is the study of the survival probability in the limit of long times. Taking in (3.16) the limit $z\to 0$ and using that $J_{nn'}(z)$ behaves [see Appendix B, Eq. (B10)] as

$$J_{nm}(z\to 0)\sim-\frac{1}{\mu}\delta_{nm}+\frac{1}{\mu}\frac{z}{\mu}\mathcal{T}_{nm}+\cdots\ , \tag{3.20}$$

where $\mathcal{T}$ is a z- and μ-independent matrix, we see that the leading contribution arises from

$$\mu\mathcal{H}\langle \hat{F}_n(z\to 0)\rangle+\sum_{p=1}^{\infty}\left\langle\left[-\frac{1}{\mu}\right]^p[\xi_n(1-\mathcal{P})]^p\xi_n\right\rangle \times\mathcal{H}\langle \hat{F}_n(z\to 0)\rangle=-1\ . \tag{3.21}$$

We can sum up the contributions from the second term on the right-hand side of (3.21) maintaining the arbitrariness in the statistics of ξ_n. To this end we define

$$\mathcal{D}_n\equiv\sum_{p=1}^{\infty}\left[-\frac{1}{\mu}\right]^p[\xi_n(1-\mathcal{P})]^p\xi_n \tag{3.22}$$

and use that $\langle\mathcal{D}\rangle+(1-\mathcal{P})\mathcal{D}_n=\mathcal{D}_n$. Note that the equivalence between all the random variables $\{\xi_n\}$ ensures that $\langle\mathcal{D}\rangle$ is independent of n. Using that

$$\sum_{p=1}^{\infty}\left[-\frac{1}{\mu}\right]^p[\xi_n(1-\mathcal{P})]^p=\left[1+\frac{\xi_n}{\mu}(1-\mathcal{P})\right]^{-1}-1\ , \tag{3.23}$$

and that $\mathcal{P}\xi_n=0$, we can formally write Eq. (3.22) in the form

$$\mathcal{D}_n=-[\mu+\xi_n(1-\mathcal{P})]^{-1}(\xi_n)^2\ , \tag{3.24}$$

which can be put as

$$[\mu+\xi_n(1-\mathcal{P})]\mathcal{D}_n=\mu\mathcal{D}_n+\xi_n(\mathcal{D}_n-\langle\mathcal{D}\rangle)=-(\xi_n)^2\ . \tag{3.25}$$

Solving for $\mathcal{D}_n$ and applying the projector $\mathcal{P}$ we obtain

$$\langle\mathcal{D}\rangle=\left\langle-\frac{(\xi_n)^2}{\mu+\xi_n}\right\rangle+\left\langle\frac{\xi_n}{\mu+\xi_n}\right\rangle\langle\mathcal{D}\rangle\ , \tag{3.26}$$

from which we can obtain $\langle\mathcal{D}\rangle$. Due to the structure of Eq. (3.21) we are interested in the quantity $\mu_{\text{eff}}\equiv\mu+\langle\mathcal{D}\rangle$, which, using (3.26), can be written as

$$\mu_{\text{eff}}=\left\langle\frac{1}{\mu+\xi_n}\right\rangle^{-1}=\left\langle\frac{1}{w_n}\right\rangle^{-1}\ . \tag{3.27}$$

This result for the sum in (3.21) is a general one valid for any statistics of the independent random variables ξ_n. Of course, they must satisfy the positivity condition $\mu+\xi_n>0$ to ensure that the probability transitions are well defined. Weak disorder is characterized by the finiteness of inverse moments like (3.27). In this case the leading contribution in (3.21) becomes

$$-1=\left\langle\frac{1}{w_n}\right\rangle^{-1}\mathcal{H}\langle\hat{F}_n(z=0)\rangle\ . \tag{3.28}$$

This equation must be solved with the boundary condition $\langle\hat{F}_n(z)\rangle=0$ if $n\notin D$. The solution can be read off immediately from the analysis of the survival probability in a nondisordered chain (see Appendix A) by changing μ to μ_{eff}. Remembering that the MFPT can be obtained from the survival probability as $\langle T_n\rangle=\langle\hat{F}_n(z=0)\rangle$, we use (3.28) to obtain the exact expression for the MFPT in weakly disordered media:

$$\langle T_n\rangle=\frac{(L+1)^2-n^2}{2\mu_{\text{eff}}}\ . \tag{3.29}$$

The effect of weak disorder is to introduce an effective time scale through the occurrence of an effective diffusion coefficient $\mu_{\text{eff}}\equiv\langle 1/w_n\rangle^{-1}<\mu$.

Strong disorder is defined by probability distributions for the ξ_n such that $\langle 1/w_n\rangle$ is infinity. In this case, our first-order contribution (3.27) vanishes. The next z-dependent contribution from $J_{nn'}(z)$ is needed to obtain information about the small-z behavior of $\langle\hat{F}\rangle$. Concerning the MFPT we can see physically that it cannot be a finite quantity, in order to respect the equality in Eq. (3.28). We will show in Sec. IV how to relate the divergence of the MFPT with the statistics of the random variables ξ_n, characterizing the disorder.

IV. STRONG DISORDER

A. Finite-effective-medium approximation

In the strong disorder case the quantity $\mu_{\text{eff}}=\langle 1/w_n\rangle^{-1}$ vanishes. Then, our first approximation, Eq. (3.28) is not enough. We need to take into account the z-dependent element of $J_{nn'}(z)$ in the exact equation (3.16). By direct substitution of the expression (3.20) for $J_{nn'}(z)$ into (3.16) a problem appears: there is an infinity of terms contributing to a given order in z. Some method must be devised to sum up these contributions. Following past experience,[13] we find it convenient to write Eq. (3.16) as

$$z\langle\hat{F}_n(z)\rangle-1=\mu\mathcal{H}\langle\hat{F}_n(z)\rangle+\sum_{p=0}^{\infty}\sum_{\substack{n_1\neq n,\\ n_2\neq n_1,\\ \dots,\\ n_p\neq n_{p-1}}}\sum_{i_0,i_1,\dots,i_p=1}\langle\xi_n\cdots\xi_n\xi_{n_1}\cdots\xi_{n_1}\cdots\xi_{n_p}\cdots\xi_{n_p}\rangle_T$$
$$\times(J_{nn})^{i_0-1}J_{nn_1}(J_{n_1n_1})^{i_1-1}J_{n_1n_2}\cdots(J_{n_pn_p})^{i_p-1}\mathcal{H}\langle\hat{F}_{n_p}(z)\rangle\ . \tag{4.1}$$

The Terwiel's cumulant $\langle\ \rangle_T$ contains the random variables $\xi_n,\xi_{n_1},\dots$, and ξ_{n_p} $i_0,i_1,\dots$, and i_p times, respectively. The indices $\{n_i\}$ take values in D. Now we define the random operator $\psi_k(z)$ by

$$\psi_k(z)\equiv\sum_{i_k=1}^{\infty}[J_{kk}(z)\xi_k(1-\mathcal{P})]^{i_k-1}\xi_k\ . \tag{4.2}$$

It must be understood as acting on any disorder-dependent quantity at its right. The geometric sum in (4.2) can be evaluated, resulting in

$$\psi_k(z)=M_k(z)-\frac{M_k(z)J_{kk}(z)}{1+\langle M_k(z)\rangle J_{kk}(z)}\mathcal{P}M_k(z)\ , \tag{4.3a}$$

where

$$M_k(z) \equiv \frac{\xi_k}{1-\xi_k J_{kk}(z)} . \tag{4.3b}$$

By using the definition of Terwiel's cumulants and (4.2), Eq. (4.1) can be written in the form

$$z\langle \hat{F}_n(z)\rangle - 1 = \mu\mathcal{K}\langle \hat{F}_n(z)\rangle + \sum_{p=0}^{\infty} \sum_{\substack{n_1\neq n,\\ n_2\neq n_1,\\ \ldots,\\ n_p\neq n_{p-1}}} \langle \psi_n \psi_{n_1} \cdots \psi_{n_p}\rangle_T J_{nn_1} J_{n_1 n_2} \cdots J_{n_{p-1}n_p} \mathcal{K}\langle \hat{F}_{n_p}(z)\rangle . \tag{4.4}$$

Equation (4.4) offers us an alternative starting point to study the long-time limit of $\langle F_n(t)\rangle$. Formally, it is the same as (4.1), but, through the introduction of the random operator $\psi_k(z)$, we have summed up all the terms containing the diagonal parts of $J_{nn'}(z)$. Our previous result for weak disorder is reobtained by taking into account the first term in the sum of Eq. (4.4) [that is, the $p=0$ contribution giving $\mu_{\text{eff}}=\mu+\langle \psi_n(z=0)\rangle$]. In this case (4.4) is strictly a perturbative expansion in the sense that $\langle \hat{F}_n(z)\rangle$ can be constructed order by order in z by successively calculating terms with increasing value of p in (4.4). This comes from the fact that $\langle \psi_{n_0}\psi_{n_1} \cdots \psi_{n_p}\rangle_T \sim O(z^0)$ for weak disorder. Nevertheless, Eq. (4.4) cannot be used for strong disorder because of the appearance of the quantities $\langle w_n^{-k}\rangle$, which are infinite in this case. It was shown in another context[13] that strong disorder introduces nonperturbative effects which ask for further rearrangement of Eq. (4.4).

We propose to do a sort of perturbative analysis around an effective homogeneous medium to study the z dependence of Eq. (4.4). First of all, we rewrite Eq. (3.2) in Laplace representation adding and subtracting a mean-field term: $\Gamma(z)(E^+ + E^- - 2)$, $\Gamma(z)$ being an arbitrary effective rate to be determined below:

$$z\hat{F}_n(z) - F_n(t=0) = \Gamma(z)(E^+ + E^- - 2)\hat{F}_n(z) + (E^+ + E^- - 2)[\mu + \xi_n - \Gamma(z)] \times \hat{F}_n(z) . \tag{4.5}$$

By defining the quantities $\eta_n(z) \equiv w_n - \Gamma(z)$, we see that the role played by μ and ξ_n in Eq. (3.2) is played by $\Gamma(z)$ and $\eta_n(z)$, respectively, in Eq. (4.5). As before we can introduce a projection operator $\mathcal{P}$ averaging over the random variables $\eta_n(z)$. Using that $F(t) = \langle F(t)\rangle + (1-\mathcal{P})F(t)$, we can follow from Eq. (3.2) to (3.8) in the same way as before. We only need to replace μ by $\Gamma(z)$ and ξ_n by $\eta_n(z)$ in Eq. (3.8) to obtain an equation describing perturbatively the effects of the random variables $\eta_n(z)$ on the mean effective rate $\Gamma(z)$:

$$z\langle \hat{F}_n(z)\rangle - 1 = \Gamma(z)\mathcal{K}\langle \hat{F}_n(z)\rangle + \sum_{p=0}^{\infty} \sum_{n_1,\ldots,n_p} \langle \eta_n \eta_{n_1} \cdots \eta_{n_p}\rangle_T \mathcal{J}_{nn_1}(\Gamma,z)\mathcal{J}_{n_1 n_2}(\Gamma,z) \cdots \mathcal{J}_{n_{p-1}n_p}(\Gamma,z)\mathcal{K}\langle \hat{F}_{n_p}(z)\rangle . \tag{4.6}$$

A new propagator

$$\mathcal{J}_{nm}(\Gamma,z) \equiv \mathcal{K}\hat{G}_{nm}(\Gamma,z) = G_{n+1,m} + G_{n-1,m} - 2G_{n,m} \tag{4.7}$$

has been introduced. $\hat{G}_{n,m}(\Gamma z,)$ is the solution of

$$z\hat{G}_{nm}(\Gamma,z) - \delta_{nm} = \Gamma(z)\mathcal{K}\hat{G}_{nm}(\Gamma,z) , \tag{4.8}$$

satisfying $\hat{G}_{L+1,n}(\Gamma,z) = \hat{G}_{-L-1,n}(\Gamma,z) = 0$. The next step is to write the analog of Eq. (4.1) with μ replaced by $\Gamma(z)$, ξ_n by $\eta_n(z)$, and $J_{n,m}(z)$ by $\mathcal{J}_{n,m}(z)$.

In order to obtain a meaningful perturbative analysis around the mean-field term we define a new random operator $\Psi_k(\Gamma,z)$ as follows:

$$\Psi_k(\Gamma,z) \equiv \sum_{i_k=1}^{\infty} [\mathcal{J}_{kk}(\Gamma,z)\eta_k(z)(1-\mathcal{P})]^{i_k-1} \eta_k(z) . \tag{4.9}$$

It is clear that equations similar to (4.3) and (4.4) follow in the same way as before:

$$\Psi_k(\Gamma,z) = \mathcal{M}_k(\Gamma,z) - \frac{\mathcal{M}_k(\Gamma,z)\mathcal{J}_{kk}(\Gamma,z)}{1+\langle \mathcal{M}_k(\Gamma,z)\rangle \mathcal{J}_{kk}(\Gamma,z)} \mathcal{P}\mathcal{M}_k(\Gamma,z) , \tag{4.10a}$$

where now

$$\mathcal{M}_k(\Gamma,z) \equiv \frac{\eta_k(z)}{1-\eta_k(z)\mathcal{J}_{kk}(\Gamma,z)} \tag{4.10b}$$

and

$$z\langle \hat{F}_n(z)\rangle - 1 = \Gamma(z)\mathcal{K}\langle \hat{F}_n(z)\rangle + \sum_{p=0}^{\infty} \sum_{\substack{n_1\neq n,\\ n_2\neq n_1,\\ \ldots,\\ n_p\neq n_{p-1}}} \langle \Psi_n \Psi_{n_1} \cdots \Psi_{n_p}\rangle_T \times \mathcal{J}_{nn_1}\mathcal{J}_{n_1 n_2} \cdots \mathcal{J}_{n_{p-1}n_p} \times \mathcal{K}\langle \hat{F}_{n_p}(z)\rangle . \tag{4.11}$$

The steps made so far are formally the same realized in Ref. 13 for the problem of averaging Eq. (2.11) to obtain the moments of the position of the walker. In that context, a diagrammatic analysis of the perturbation series analogous to the second term on the right-hand side of (4.11) showed that the best election for $\Gamma(z)$ (the mean-field rate, still undefined) was that given by the solution of $\langle \Psi_n(\Gamma,z)\rangle=0$. This was the best choice in the sense that it allowed to vanish an infinity of terms in the evolution equation for $\langle P_n(t)\rangle$, pertaining to a particular diagrammatic class. In the present problem it is impossible to have $\langle \Psi_n(\Gamma,z)\rangle=0$ for all the values of n because the boundary conditions implicit in the construction of $\hat{G}(\Gamma,z)$ destroyed average translational invariance. Nevertheless, we can tentatively define $\Gamma(z)$ by $\langle \Psi_n(\Gamma,z)\rangle=0$ at some particular site n, and explore the consequences of such an election. In the following, $\Gamma(z)$ will be taken as the solution of

$$\langle \Psi_{n=0}(\Gamma,z)\rangle=0\ . \tag{4.12}$$

It will turn out that this election is a very convenient one because, although it does not produce a drastic simplification of the right-hand side of (4.11), it makes useful an approach related to the EMA. It was demonstrated in Ref. 13 that the EMA consists, within this formalism, in neglecting all the terms containing Terwiel's cumulants in the evolution equation for $\langle P_n(t)\rangle$ analogous to (4.11). The analogous approximation in the present problem would consist in reducing (4.11) to

$$z\langle \hat{F}_n(z)\rangle-1=\Gamma(z)(E^{+}+E^{-}-2)\langle \hat{F}_n(z)\rangle\ , \tag{4.13}$$

with $\Gamma(z)$ defined by (4.12) or, more explicitly, using (4.10) and $\eta_0=w_0-\Gamma$,

$$\left\langle \frac{w_0-\Gamma(z)}{1-[w_0-\Gamma(z)]\mathcal{G}_{00}(\Gamma,z)}\right\rangle=0\ . \tag{4.14}$$

Equations (4.13) and (4.14) define an approximation to the averaged survival probability $\langle \hat{F}_n(z)\rangle$ for all the values of z. The solution of (4.13) is obtained from (2.17) by replacing μ with the $\Gamma(z)$ obtained from (4.14).

We note that, although Eq. (4.14) has a structure very similar to the equation for the kernel of GME for $\langle P_n(t)\rangle$ in the EMA approach, there is an important difference: the propagator $\mathcal{G}_{nn'}(\Gamma,z)$ has a completely different z dependence. Then the $\Gamma(z)$ obtained here is not the same as the diffusion coefficient obtained in the EMA, and Eq. (4.13) does not coincide with the adjoint or backwards equation of the GME given by the EMA, as would be the case if the RW considered would be Markovian. Physically speaking, we have taken into account the correct realizations of the walker paths in the construction of the non-Markovian contribution appearing in the exact effective backwards master equation (3.8). To stress these differences, coming from the finiteness of the interval $[-L,L]$ used to define the FPT, we call the approximation defined by (4.13) and (4.14) the finite EMA (FEMA). It will be demonstrated in Appendix C that the FEMA gives the exact exponent for the small-z behavior of $\langle \hat{F}_n(z)\rangle$ for models (a), (b), (c), and (d). It can be also seen that the large-z behavior given by the FEMA is the same as the exact one obtained previously [Eq. (3.19)]. Then the FEMA can be taken as a good approximation for all the values of z.

B. Survival probability in the finite-effective-medium approximation

The solution of Eq. (4.13) follows immediately from our considerations in Sec. II, by using (2.17), where now μ must be substituted by the $\Gamma(z)$ obtained from (4.14):

$$z\langle \hat{F}_n(z)\rangle=1-\frac{(x_1)^n+(x_2)^n}{(x_1)^{(L+1)}+(x_2)^{(L+1)}}\ , \tag{4.15a}$$

with

$$x_{1,2}(z)\equiv 1+\frac{s}{2}\pm\left[s\left[1+\frac{s}{4}\right]\right]^{1/2} \tag{4.15b}$$

and $s\equiv z/\Gamma(z)$. The long-time limit can be obtained from (4.15a) with $z\to 0$. For models (a), (b), and (c) of disorder, $z/\Gamma(z)\to 0$, as can be seen in what follows, so that we obtain

$$\langle \hat{F}_n(z\to 0)\rangle\sim\frac{(L+1)^2-n^2}{2\Gamma(z\to 0)}\ . \tag{4.16}$$

The z dependence arises from the behavior of $\Gamma(z)$ for each particular model of disorder.

In order to solve (4.14) for small z we need to find the z dependence of the propagator $\mathcal{G}_{00}(\Gamma(z),z)$. This can be obtained for models (a), (b), and (c) from our general expression (B10) as

$$\mathcal{G}_{00}(\Gamma(z),z)\sim-\frac{1}{\Gamma(z)}+\frac{1}{\Gamma(z)}\frac{L+1}{2}\frac{z}{\Gamma(z)}+\cdots\ . \tag{4.17}$$

If we define the quantity

$$R(z)\equiv-\Gamma(z)-\{\mathcal{G}_{00}[\Gamma(z),z]\}^{-1}\ , \tag{4.18}$$

we can write Eq. (4.14) in a simpler form:

$$\left\langle\frac{w}{w+R(z)}\right\rangle=\Gamma(z)\left\langle\frac{1}{w+R(z)}\right\rangle\ . \tag{4.19}$$

$R(z)$ behaves as $R(z)\sim(L+1)z/2$ for models (a), (b), and (c) and for small z.

As the first application of the FEMA we will reobtain our result for the MFPT in the case of weak disorder. In this case the inverse moments $\beta_k\equiv\langle w_n^{-k}\rangle$ are finite quantities and we can expand (4.19) as

$$1-R\beta_1+R^2\beta_2-\cdots=\Gamma(z)(\beta_1-R\beta_2+R^2\beta_3-\cdots)\ . \tag{4.20}$$

Solving for $\Gamma(z)$ and using the small-z behavior of $R(z)$, we obtain

$$\Gamma(z)\sim\frac{1}{\beta_1}\left[1+\frac{\beta_2-(\beta_1)^2}{\beta_1}\frac{L+1}{2}z+\cdots\right]\ . \tag{4.21}$$

At this point an important difference between the EMA and our FEMA appears: the analog of $\Gamma(z)$ in the EMA for the probability $\langle P_n(t)\rangle$ has the meaning of a z-dependent diffusion coefficient and, for weak disorder, behaves for small z as a power series in $z^{1/2}$. Equation (4.21) contains no $z^{1/2}$ term, showing that the problem of FPT in random media cannot be simply reduced to that of the study of $\langle P_n(t)\rangle$ or the frequency-dependent diffusion coefficient. The substitution of (4.21) for $z=0$ into (4.16) gives the MFPT in weakly disordered chains, a result previously obtained by a different method in Sec. II:

$$\langle T_n\rangle=\langle \hat{F}_n(z=0)\rangle=\frac{(L+1)^2-n^2}{2\Gamma(z=0)}=\frac{(L+1)^2-n^2}{2}\left\langle\frac{1}{w}\right\rangle . \quad (4.22)$$

The second case that we want to study is model (b) of strong disorder. We need to solve Eq. (4.19) where the average has to be performed with the probability distribution for the w_n's given by (1.3). From (4.9) we arrive at

$$1-R(z)\ln[1+R(z)]+R(z)\ln R(z)=\Gamma(z)\{\ln[1+R(z)]-\ln R(z)\} , \quad (4.23)$$

and, solving for $\Gamma(z)$,

$$\Gamma(z\to 0)\sim|\ln z|^{-1}\left[1+|\ln z|^{-1}\ln\frac{L+1}{2}+\cdots\right] . \quad (4.24)$$

Then the small-z behavior of the survival probability is given, from (4.16) and (4.24), by

$$\langle \hat{F}_n(z)\rangle\sim\frac{(L+1)^2-n^2}{2}|\ln z| . \quad (4.25)$$

This result shows that the MFPT is a divergent quantity for all the values of L and n. This divergence is characterized by the singularity law of the survival probability in the limit $z\to 0$, $\langle\hat{F}_n(z)\rangle\sim|\ln z|$. The result (4.25), although obtained in the FEMA, is an exact result. This is demonstrated in Appendix C by an analysis of the corrections to the FEMA.

Case (c) of strong disorder is characterized by the probability distribution for the w_n's given by (1.4). Evaluating the averages in (4.19) for small z:

$$\left\langle\frac{1}{w+R(z)}\right\rangle=\frac{(1-\alpha)\pi}{\sin(\pi\alpha)}[R(z)]^{-\alpha}-\frac{1-\alpha}{\alpha}+\frac{1-\alpha}{1+\alpha}R(z)-\frac{1-\alpha}{2+\alpha}R(z)^2+O(R(z))^3 \quad (4.26)$$

and

$$\left\langle\frac{w}{w+R(z)}\right\rangle=1-R\left\langle\frac{1}{w+R(z)}\right\rangle , \quad (4.27)$$

the solution of (4.19) gives

$$\Gamma(z)\sim\frac{\sin(\pi\alpha)}{\pi(1-\alpha)}[R(z)]^{\alpha}\sim\left[\frac{L+1}{2}\right]^{\alpha}\frac{\sin(\pi\alpha)}{\pi(1-\alpha)}z^{\alpha} . \quad (4.28)$$

The survival probability behaves for small z in the following way:

$$\langle\hat{F}_n(z)\rangle\sim\frac{(L+1)^2-n^2}{2}\frac{2^{\alpha}\pi(1-\alpha)}{(L+1)^{\alpha}\sin(\pi\alpha)}z^{-\alpha} , \quad (4.29)$$

showing the announced divergence of the MFPT for case (c) of strong disorder. It will be shown in Appendix C that the law $\langle\hat{F}_n(z)\rangle\sim z^{-\alpha}$ is an exact result. The coefficient in (4.29) is not the exact one, but the numerical results of Appendix C and Ref. 8 show that it is an accurate approximation, especially for small α.

We remark that the law $z^{-\alpha}$ is a consequence of having used the $\Gamma(z)$ solution of (4.17)–(4.19). As was previously pointed out, this result cannot be obtained from the adjoint equation of the GME for the probability $\langle P_n(t)\rangle$ given by the EMA, which predicts the incorrect result $\langle\hat{F}(z)\rangle\sim z^{-\alpha/(2-\alpha)}$.

The last example we want to work out is that of diffusion in presence of randomly distributed absorbing sites. The probability distribution for the rates w_n is given in (1.5), so that sites can be normal with probability p or perfect traps with probability $1-p$. Substituting (1.5) into (4.19) we obtain

$$\frac{pw_0}{w_0+R(z)}=\Gamma\frac{R(z)+w_0(1-p)}{[w_0+R(z)]R(z)} , \quad (4.30)$$

so that

$$\Gamma(z)\sim\frac{p}{1-p}R(z)\sim-\frac{p}{\mathcal{J}_{00}[\Gamma(z),z]} \quad (4.31)$$

for small z. We have used (4.18) and (B9). Equations (4.31) and (B9) give for $z\to 0$ and $p\neq 1$

$$\Gamma(z)\sim Cz . \quad (4.32)$$

The coefficient C is a function of p and L satisfying a complicated implicit equation. It can be solved in some limits; for example, for $L\to\infty$,

$$\lim_{L\to\infty} C=\frac{p(2-p)}{4(1-p)^2} . \quad (4.33)$$

The survival probability is obtained from the replacement $\mu\to\Gamma(z)$ in (2.17). The result predicts a behavior $\langle\hat{F}(z)\rangle\sim z^{-1}$, which could also be obtained simply by noting that there is a finite probability for the walker to become trapped in any finite interval. The coefficient of the law z^{-1} is the probability q of no escape from the interval D. The FEMA predicts

$$q=\lim_{z\to 0} z\langle\hat{F}_n(z)\rangle=1-\frac{(x_1)^n+(x_2)^n}{(x_1)^{L+1}+(x_2)^{L+1}} , \quad (4.34a)$$

with

$$x_{1,2}=1+\frac{1}{2C}\pm\left[\frac{1}{2C}\left[1+\frac{1}{4C}\right]\right]^{1/2} . \quad (4.34b)$$

In Appendix C we show that the behavior z^{-1} is correct, but the coefficient is not the exact one. Exact results for q were obtained in Ref. 8 for $L=1$, 2, and 3 by directly solving the $(2L+1)\times(2L+1)$ matrix problem (2.7) and then averaging by exact enumeration of all the configurations.

V. CONCLUSIONS

The question addressed here has been the study of the random walk in a disordered chain by first-passage time techniques, characterizing anomalous diffusion and the degree of disorder by the long-time behavior of a survival probability. Disorder was represented by random variables appearing in the master-equation matrix H. This matrix was split into a disordered and an ordered part. A fairly general method, based in a projection-operator formalism, has been presented. It has been applied in detail to the random-trap model, but our approach is rather general and similar steps can be followed to study other models of disorder.

We have been able to derive a general evolution equation for the survival probability for arbitrary statistics of the random rates characterizing the disordered medium. A generalized Dynkin's equation has also been obtained. We remark that these are exact results and that the correct backflow exclusion has been automatically taken into account by averaging the backwards equation for each realization of the disorder.

For the case of weak disorder the expression for the MFPT has been exactly obtained [Eq. (3.29)]. It is surprisingly simple when compared to the behavior of the moments of the position of the walker, which are infinite series in decreasing powers of $t^{1/2}$. A new approximation, the FEMA, has been introduced to take into account nonperturbative effects, of crucial importance in cases of strong disorder. The long-time behavior of the survival probability for cases (b) and (c) of strong disorder has been obtained [Eqs. (4.25) and (4.29)]. The following asymptotic behavior for the FPTD follows as $t \to \infty$ from $\hat{f} = 1 - z\hat{F}$ and Tauberian theorems:

$$f_n(t) \sim \begin{cases} \dfrac{(L+1)^2 - n^2}{2} \dfrac{1}{t^2} \quad [\text{model (b)}] & (5.1) \\ \dfrac{(L+1)^2 - n^2}{(L+1)^\alpha} \dfrac{2^{\alpha-1}(1-\alpha)^2\Gamma(1-\alpha)}{t^{2-\alpha}} \quad [\text{model (c)}]. & (5.2) \end{cases}$$

The long-time tails predict a divergence in the MFPT for both models of disorder. We can associate the concept of strong disorder to a divergent MFPT and characterize the degree of disorder by the long-time behavior in (5.1) and (5.2). The exponents of t in (5.1) and (5.2) are exact results. They are independent of the size of the interval and of the initial condition, so that FPTD and survival probabilities show universal behavior for different values of L, n, and small z, if they are adequately normalized, as was numerically shown in a previous work.[8] The coefficient of the power law in (5.1) is also an exact result. The one in (5.2) is a good approximation. For the case of diffusion in presence of traps, the correct exponent z^{-1} giving the small-z divergence in $\langle \hat{F}(z) \rangle$ has been obtained, and the FEMA gives approximate expressions for its coefficient, which is a residence probability.

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APPENDIX A: FIRST-PASSAGE-TIME DISTRIBUTION IN A NONDISORDERED CHAIN

We want to obtain the survival probability and the FPTD to leave the domain $D = [-L, L]$ for a RW in an homogeneous chain. Equation (2.14) can be written in Laplace representation as

$$z\hat{F}_n(z) - 1 = \mu(E^+ + E^- - 2)\hat{F}_n(z) \ . \tag{A1}$$

We must solve this equation with the *artificial* boundary conditions

$$\hat{F}_{-L-1}(z) = \hat{F}_{L+1}(z) = 0 \ . \tag{A2}$$

Equation (A1) can be written as

$$[(E^+)^2 - (2+r)E^+ + 1]A_n = -1 \ , \tag{A3}$$

where $r \equiv z/\mu$ and the vector A is defined by

$$A_{n+1} \equiv \mu\hat{F}_n(z) \ . \tag{A4}$$

Equation (A3) is a second-order linear inhomogeneous difference equation which can be solved by adding a particular solution to the general solution of the corresponding homogeneous equation. The general solution of the homogeneous part is

$$A_n^h = c_1(x_1)^n + c_2(x_2)^n \ . \tag{A5}$$

c_1 and c_2 are arbitrary constants and $x_{1,2}$ are the roots of the equation $x^2 - (2+r)x + 1 = 0$, given by

$$x_{1,2}(z) \equiv 1 + \frac{r}{2} \pm \left[r \left(1 + \frac{r}{4} \right) \right]^{1/2} . \tag{A6}$$

A particular solution is the constant

$$A_n^p = 1/r \ . \tag{A7}$$

Then, using (A4) and the solutions A^h and A^p, we obtain

$$\hat{F}_n(z) = \frac{1}{z} + c_1(x_1)^{n+1} + c_2(x_2)^{n+1} \ . \tag{A8}$$

c_1 and c_2 are obtained by imposing the boundary conditions (A2):

$$c_1 x_1 = -\frac{(x_1)^{L+1}}{z} \frac{1 - (x_2)^{2(L+1)}}{(x_1)^{2(L+1)} - (x_2)^{2(L+1)}} \ , \tag{A9}$$

$$c_2x_2=\frac{(x_2)^{L+1}}{z}\frac{1-(x_1)^{2(L+1)}}{(x_1)^{2(L+1)}-(x_2)^{2(L+1)}} .$$

Equation (2.17) follows from (A8) and (A9). The FPTD $\hat{f}_n(z)$ is related to $\hat{F}_n(z)$ by $\hat{f}_n(z)=1-z\hat{F}_n(z)$, so that

$$\hat{f}_n(z)=\frac{(x_1)^n+(x_2)^n}{(x_1)^{(L+1)}+(x_2)^{(L+1)}} . \quad \text{(A10)}$$

We see from (A10) that the normalization condition

$$\int_0^\infty f_n(t)dt=\hat{f}_n(z=0)=1, \quad \forall n \quad \text{(A11)}$$

is satisfied. From (A10) we can obtain the known results for the Wiener process: defining a lattice spacing l, a variable $x_0\equiv nl$, and a domain $\mathcal{D}\equiv[-X,X]$ with $X\equiv Ll$, we can take the continuous limit as $L\to\infty$, $\mu\to\infty$, $l\to 0$, with $\Delta\equiv\mu l^2$ and X finite. The FPTD to leave the domain $\mathcal{D}$ having started at point x_0 is

$$\hat{f}(x_0,s)=\frac{\cosh[x_0(z/\Delta)^{1/2}]}{\cosh[X(z/\Delta)^{1/2}]} , \quad \text{(A12)}$$

where we have used that $x_{1,2}=1\pm l(z/\Delta)^{1/2}+\cdots$ in (A10) for $l\to 0$. It is easy to see that the MFPT is

$$T(x_0)=-\partial_z\hat{f}(x_0,z\to 0)=\frac{X^2-x_0^2}{2\Delta} . \quad \text{(A13)}$$

APPENDIX B: GREEN'S FUNCTION IN A FINITE DOMAIN

The problem of solving the finite-dimensional matrix equation (3.7) is reduced to that of solving Eq. (3.10) with the boundary conditions Eq. (3.11). This is equivalent to finding the Green's function for a RW in the finite domain $D=[-L,L]$ with absorbing boundary conditions at the domain ends. It is known[14] that the method of the images is a good technique to solve this problem. It consists in summing to the Green's function in the absence of boundaries (G_{nm}^0), with indices in D, terms of the form $\pm G_{km}^0$, with k being the specular image of n with respect to the boundary considered. In the case of absorbing boundaries the image must have a negative sign. Due to the fact that we are in a closed domain, each boundary reflects the other boundary. This introduces an infinite set of images leading to the result in (3.12). From this result it is easy to see that the initial condition is satisifed: for any $n,m\in D$ we have

$$G_{nm}(t|t)=\sum_{k=-\infty}^{\infty}\delta_{n-m,-4k(L+1)}-\delta_{n+m,-2(L+1)(2k+1)}=\delta_{n,m} \quad \text{(B1)}$$

because $n-m\le 2L+1$. The boundary conditions are also satisfied, as can be seen from

$$G_{-L-1,m}(t|t')=\sum_{k=-\infty}^{\infty}G^0_{(L+1)(4k-1),m}-\sum_{k=-\infty}^{\infty}G^0_{(L+1)(4k-1),m}=0 \quad \text{(B2)}$$

and

$$G_{L+1,m}(t|t')=\sum_{k=-\infty}^{\infty}G^0_{(L+1)(4k-1),m}-\sum_{k=-\infty}^{\infty}G^0_{(L+1)(-4k-3),m}=0 . \quad \text{(B3)}$$

We are interested in the expression of the Laplace transform of $G_{nm}(t|t')$. Laplace transforming Eq. (3.13) we can write (3.12) as

$$\hat{G}_{nm}(z)=B(z)\sum_{k=-\infty}^{\infty}(A^{|n+4(L+1)k-m|}-A^{|n+2(L+1)(2k+1)+m|}) , \quad \text{(B4a)}$$

where

$$A(z)\equiv 1+\frac{z}{2\mu}-\left[\frac{z}{\mu}+\frac{z^2}{4\mu^2}\right]^{1/2} ,$$
$$B(z)\equiv\frac{1}{2\mu}\left[\frac{z}{\mu}+\frac{z^2}{4\mu^2}\right]^{-1/2} . \quad \text{(B4b)}$$

We will sum each term separately. If $n-m>0$ and using that $n-m<2(L+1)$, we can write the first sum as

$$A^{m-n}\sum_{k=-\infty}^{-1}A^{-4(L+1)k}+A^{n-m}\sum_{k=0}^{\infty}A^{4(L+1)k}$$
$$=A^{m-n}\frac{A^{4(L+1)}}{1-A^{4(L+1)}}+A^{n-m}\frac{1}{1-A^{4(L+1)}} . \quad \text{(B5)}$$

The case $n-m<0$ can be considered in a similar way. The general expression for the first sum is

$$\sum_{k=-\infty}^{\infty}A^{|n+4(L+1)k-m|}=\frac{A^{m-n}A^{-4(L+1)}A^{4(L+1)H(n-m)}}{A^{-4(L+1)}-1}+\frac{A^{n-m}A^{4(L+1)H(m-n)}}{1-A^{4(L+1)}} , \quad \text{(B6)}$$

where $H(n-m)$ is the Heaviside step function. In the same way, using $n+m<2(L+1)$, we can write for the second sum the following general expression:

$$\sum_{k=-\infty}^{\infty}A^{|n+2(L+1)(2k+1)+m|}=\frac{A^{-n-m-2(L+1)}}{A^{-4(L+1)}-1}+\frac{A^{n+m+2(L+1)}}{1-A^{4(L+1)}} . \quad \text{(B7)}$$

Putting (B6) and (B7) into (B4) we obtain the exact result (3.14). From this equation we can see that $\hat{G}_{nm}(z)$ is a function that is symmetric in the indices n and m. However, it does not depend only on the absolute value $|n-m|$. This can be seen, for example, from a large-L expansion of (3.14). The first correction to the *free* Green's function $\hat{G}^{\,0}_{nm}(z)$ is

$$-2(A^{L+1})^2\cosh[(n+m)\ln A] . \quad \text{(B8)}$$

We now analyze the propagator $J_m(z)\equiv(E^+ + E^- - 2)\hat{G}_{nm}(z)$ in the limit $z\to 0$. We remark that we will not perform an expansion in the parameter L, which gives the size of the system. We will keep the exact dependence in L given in (3.14) and study the small-z limit. In this way we conserve the correct L dependence in the limit of long times. Applying $(E^+ + E^- - 2)$ to (3.14), the following expression is found ($\epsilon\equiv A^L$):

$$J_{nm}(z)=B(z)(1-\epsilon^4A^4)^{-1}[(A^{|n+1-m|}+A^{|n-1-m|}-2A^{|n-m|}) -\epsilon^2A^2(A^{-(n+1+m)}+A^{n+1-m}+A^{-(n-1+m)}+A^{n-1+m}-2A^{-(n+m)}-2A^{n+m}) +\epsilon^4A^4(A^{-|n+1-m|}+A^{-|n-1-m|}-2A^{-|n-m|})] \ . \tag{B9}$$

After a straightforward but tedious algebra we obtain

$$J_{nm}(z)\sim -\frac{1}{\mu}\delta_{nm} + \frac{[L+1+\min(n,m)][L+1-\max(n,m)]}{2\mu(L+1)} \times\frac{z}{\mu}+\cdots \tag{B10}$$

for all n, m, L, and small z. From this expression we can see that the first z-dependent contribution is different from that which would be obtained by using the usual *free* Green's function. Namely, the first z-dependent term in $J^0_{nm}(z)\equiv(E^+ + E^- - 2)G^0_{nm}(z)$ is of order $z^{1/2}$. This difference is an example of the effects coming from the new ingredients needed in the study of FPT statistics in random media, absent in the study of the position of the walker in disordered media.

APPENDIX C: ANALYSIS OF THE CORRECTIONS TO THE FINITE-EFFECTIVE-MEDIUM APPROXIMATION

In this appendix we study the exact equation (4.11) in order to establish whenever the FEMA gives the exact result for the small-z behavior of $\langle \hat{F}_n(z)\rangle$. To this end we need some properties[13(b),15] of Terwiel's cumulants appearing in (4.11). The first one, easily obtained from its definition

$$\langle x_1x_2\cdots x_n\rangle_T\equiv\mathcal{P}x_1(1-\mathcal{P})x_2(1-\mathcal{P})\cdots(1-\mathcal{P})x_n \ , \tag{C1}$$

is the expansion of Terwiel's cumulants in moments:

$$\langle x_1x_2\cdots x_n\rangle_T =\sum_{i=0}^{n-1}(-1)^i\sum_{1\le l_1<\ldots<l_i<n}\langle x_1\cdots x_{l_1}\rangle \times\langle x_{l_1+1}\cdots x_{l_2}\rangle\cdots \times\langle x_{l_i+1}\cdots x_n\rangle \ . \tag{C2}$$

Explicit examples of this formula are

$$\begin{aligned}\langle x_1\rangle_T&=\langle x_1\rangle \ ,\\ \langle x_1x_2\rangle_T&=\langle x_1x_2\rangle-\langle x_1\rangle\langle x_2\rangle \ ,\\ \langle x_1x_2x_3\rangle_T&=\langle x_1x_2x_3\rangle-\langle x_1\rangle\langle x_2x_3\rangle -\langle x_1x_2\rangle\langle x_3\rangle+\langle x_1\rangle\langle x_2\rangle\langle x_3\rangle \ .\end{aligned} \tag{C3}$$

Note that the order of the x's on the right-hand side of (C2) must be the same as in Terwiel's cumulant. The same result (C2) is obtained when the x's are random variables as when they are random operators.

Another important property of Terwiel's cumulant $\langle x_1x_2\cdots x_n\rangle_T$ is that, if it is possible to split it into two sets $\{x_1\cdots x_k\}$ and $\{x_{k+1}\cdots x_n\}$ without altering the order of the x's in such a way that the variables in one of the sets are statistically independent of those in the other set, the cumulant vanishes.

Equation (C2) can be applied to the cumulants $\langle\Psi_n\Psi_{n_1}\cdots\Psi_{n_p}\rangle_T$ appearing in (4.11). Then an expression in terms of the moments $\langle\Psi_{l_1}\cdots\Psi_{l_k}\rangle$ is obtained. These moments can be further manipulated to obtain an expression in terms of averages of random variables, involving no random operators. This can be done by using that [Eq. (4.10)]

$$\Psi_n=\mathcal{M}_n(1-\mathcal{N}_n\mathcal{P}\mathcal{M}_n) \ , \tag{C4}$$

where

$$\mathcal{M}_n\equiv\frac{\eta_n(z)}{1-\eta(z)\mathcal{J}_{nn}[\Gamma(z),z]}, \quad \eta_n(z)\equiv w_n-\Gamma(z) \ , \tag{C5}$$

and

$$\mathcal{N}_n=\frac{\mathcal{J}_{nn}[\Gamma(z),z]}{1+\langle\mathcal{M}_n\rangle\mathcal{J}_{nn}[\Gamma(z),z]} \ . \tag{C6}$$

The algebraical steps needed to evaluate

$$\langle\Psi_1\cdots\Psi_n\rangle=\mathcal{P}\mathcal{M}_1(1-\mathcal{N}_1\mathcal{P}\mathcal{M}_1)\mathcal{M}_2(1-\mathcal{N}_2\mathcal{P}\mathcal{M}_2)\cdots \times\mathcal{M}_n(1-\mathcal{N}_n\langle\mathcal{M}_n\rangle) \tag{C7}$$

are very similar to those needed to obtain (C2) from (C1). The final result is

$$\langle \Psi_1\Psi_2\cdots\Psi_N\rangle=\sum_{i=0}^{n}(-1)^i\sum_{1\le l_1<\cdots<l_i\le n}\langle \mathcal{M}_1\mathcal{M}_2\cdots\mathcal{M}_{l_1}\rangle \times\mathcal{N}_{l_1}\langle \mathcal{M}_{l_1}\mathcal{M}_{l_1+1}\cdots\mathcal{M}_{l_2}\rangle\mathcal{N}_{l_2}\langle \mathcal{M}_{l_2}\cdots\rangle\cdots\mathcal{N}_{l_i}\langle \mathcal{M}_{l_i}\cdots\mathcal{M}_n\rangle\ . \quad \text{(C8)}$$

As an example of Eq. (C8) we can see that

$$\langle \Psi_1\Psi_2\rangle=\langle \mathcal{M}_1\mathcal{M}_2\rangle-\langle \mathcal{M}_1\rangle\mathcal{N}_1\langle \mathcal{M}_1\mathcal{M}_2\rangle -\langle \mathcal{M}_1\mathcal{M}_2\rangle\mathcal{N}_2\langle \mathcal{M}_2\rangle +\langle \mathcal{M}_1\rangle\mathcal{N}_1\langle \mathcal{M}_1\mathcal{M}_2\rangle\mathcal{N}_2\langle \mathcal{M}_2\rangle\ . \quad \text{(C9)}$$

A convenient expression for $\mathcal{M}_n$ in (C5) is

$$\mathcal{M}_n=[\Gamma(z)+R_n]\frac{w_n-\Gamma(z)}{w_n+R_n}\ , \quad \text{(C10)}$$

where we have used $\eta_n(z)=w_n-\Gamma(z)$ and the definition

$$R_n\equiv-\Gamma(z)-1/\mathcal{J}_{nn}\ . \quad \text{(C11)}$$

The small-z behavior of R_n for cases (a), (b), and (c) of disorder [for which $z/\Gamma(z)\to 0$] is

$$R_n(z)\sim\frac{(L+1)^2-n^2}{2(L+1)}z+\cdots\equiv g_n z+\cdots\ . \quad \text{(C12)}$$

Up to this order, $R_n(z)$ is Γ independent.

After these preliminaries we are in the disposition of analyzing the corrections to the FEMA from Eq. (4.11), which can be written as

$$z\langle \hat{F}_n(z)\rangle-1=\Gamma(z)\mathcal{H}\langle \hat{F}_n(z)\rangle +\sum_{p=0}^{\infty}\sum_{\substack{n_1\ne n,\\ n_2\ne n_1,\\ \dots,\\ n_p\ne n_{p-1}}}A_{nn_1\cdots n_p}(p,z)\mathcal{H}\langle \hat{F}_{n_p}(z)\rangle\ , \quad \text{(C13)}$$

where we have defined

$$A_{nn_1\cdots n_p}(p,z)\equiv\begin{cases}\langle \Psi_n\Psi_{n_1}\cdots\Psi_{n_p}\rangle_T\mathcal{J}_{nn_1}\mathcal{J}_{n_1n_2}\cdots\mathcal{J}_{n_{p-1}n_p}\ , & p\ge 0\\ \langle \Psi_n\rangle,\ \ p=0\ . & \end{cases} \quad \text{(C14)}$$

Because the sums in (C13) are all finite, taking the indices values in D, the small-z results for $\langle \hat{F}_n(z)\rangle$ obtained in the FEMA will be the exact ones whenever $\Gamma(z)$ is more important than $A_{n\cdots n_p}(p,z)$ for all p and small z. Otherwise, the FEMA would need corrections from $A_{n\cdots n_p}(p,z)$ in order to give the exact small-z behavior of $\langle \hat{F}_n(z)\rangle$.

Let us begin analyzing the order in z of (C14) for weak disorder and $p\ne 0$. In this case, by (4.21), $\Gamma(z)$ is of order z^0, so that, using (C10) and (C12) and the fact that all the moments of the form $\langle (1/w_j)^N\rangle$ are finite, $\langle \mathcal{M}_j(z)^N\rangle$ is of order z^0 for all N. From (C6) it can also be seen that $\mathcal{N}_j(z)\sim O(z^0)$. Combining this result with (C2) and (C8) we see that $\langle \Psi_n\Psi_{n_1}\cdots\Psi_{n_p}\rangle_T$ is of order z^0. We note that only the nondiagonal parts of the $\mathcal{J}$'s enter in (C13), so that according to (B10), $\mathcal{J}_{nm}(z)\sim O[z/\Gamma(z)^2]\sim O(z)$ and from (C14), $A_{n\cdots n_p}(p,z)\sim O(z^p)$. The smaller value of $p\ne 0$ entering in (C13) is $p=2$ because the cumulant in (C13) for $p=1$ consists of two independent random operators, so it vanishes. Then the most important contribution from $A_{n\cdots n_p}(p,z)$ for $p\ne 0$ is of the order z^2. For the case $p=0$, we can see that

$$\langle \Psi_n\rangle\sim\frac{\beta_2-(\beta_1)^2}{(\beta_1)^2}\frac{n^2}{2(L+1)}z+O(z^2)\ . \quad \text{(C15)}$$

Remember that $\beta_K\equiv\langle 1/w^k\rangle$. Then the contributions from the sum in (C13) are negligibly small when compared with $\Gamma(z)$ for small z. This demonstrates that the FEMA gives the exact result for the small-z behavior of $\langle \hat{F}_n(z)\rangle$ in the weak disorder case.

The first correction to the FEMA term $\Gamma(z)\mathcal{H}\langle \hat{F}_n(z)\rangle$ in (C13) is of order z and can be obtained from (C15). The next correction, of order z^2, comes from the next term in the expansion of $\langle \Psi_n\rangle$ and from the $p=2$ term:

$$\sum_{\substack{n_1\ne n\\ n_2\ne n_1}}A_{nn_1n_2}(p=2,z)\mathcal{H}\langle \hat{F}_n(z)\rangle =\sum_{\substack{n_1\ne n\\ n_2\ne n_1}}\langle \Psi_n\Psi_{n_1}\Psi_{n_2}\rangle_T\mathcal{J}_{nn_1}\mathcal{J}_{n_1n_2}\mathcal{H}\langle \hat{F}_n(z)\rangle =\sum_{n_1\ne n}\langle \Psi_n\Psi_{n_1}\Psi_n\rangle_T\mathcal{J}_{nn_1}\mathcal{J}_{n_1n}\mathcal{H}\langle \hat{F}_n(z)\rangle\ . \quad \text{(C16)}$$

In writing the last equality we have used the fact that, unless $n_2=n$, the cumulant vanishes because it can be split into independent pieces. The equation for $\langle \hat{F}_n(z)\rangle$ becomes more complicated when successive higher-order terms are added. Equations (C15) and (C16) introduce explicit n dependence in the coefficients of this equation, and higher-order terms break down its nearest-neighbor structure. Nevertheless, standard approximation schemes[11] can be used in these cases, so that Eq. (C13) provides a systematic method to calculate $\langle \hat{F}_n(z)\rangle$ order by order in z.

We analyze now (C14) for model (b) of disorder, for which $\Gamma(z)\sim|\ln z|^{-1}$. The integrals defining the moments of $\mathcal{M}_n$ behave for small z as

$$\langle \mathcal{M}_n(z)^N\rangle \sim \Gamma(z)^N\int_0^1 dw\frac{[w-\Gamma(z)]^N}{[w+R_n(z)]^N} \sim \frac{(-1)^N\Gamma(z)^{2N}}{(N-1)R_n(z)^{N-1}} \sim \left[-\frac{\Gamma(z)^2}{z}\right]^N\frac{z}{(N-1)g_n^{N-1}} \tag{C17}$$

if $N\geq 2$. g_n was defined in (C12). If $N=1$, the average reads

$$\langle \mathcal{M}_n(z)\rangle \sim \Gamma(z)^2\ln\frac{g_n}{g_0} . \tag{C18}$$

In any case, we have

$$\langle \mathcal{M}_n(z)^N\rangle \sim O\left[z\left[\frac{\Gamma(z)^2}{z}\right]^N\right],\quad N=1,2,3,\ldots . \tag{C19}$$

Then, for averages of the form $\langle \mathcal{M}_{n_0}\mathcal{M}_{n_1}\cdots\mathcal{M}_{n_p}\rangle$ in which I of the indices $\{n_0,n_1,\ldots,n_p\}$ are different, $\{i_1,i_2,\ldots,i_I\}$, being the i_j repeated m_j times ($\sum_{j=1}^I m_j=p+1$), we find

$$\langle \mathcal{M}_{n_0}\mathcal{M}_{n_1}\cdots\mathcal{M}_{n_p}\rangle=\langle \mathcal{M}_{i_1}^{m_1}\rangle\langle \mathcal{M}_{i_2}^{m_2}\rangle\cdots\langle \mathcal{M}_{i_I}^{m_I}\rangle \sim O\left[z^I\left[\frac{\Gamma(z)^2}{z}\right]^{p+1}\right]. \tag{C20}$$

Here we have used the fact that the $\{\mathcal{M}_{n_j}\}$ are independent random variables for different n_j. In the expansion of $\langle \Psi_{n_0}\Psi_{n_1}\cdots\Psi_{n_p}\rangle$ in moments of $\mathcal{M}_j$ [Eq. (C8)], we see that the dominant term for $z\to 0$ is precisely (C20). This is so because (a) all other terms are split at least once more [this introduces a factor $O(z)$], (b) at least one repetition of some $\mathcal{M}$ is added somewhere [this introduces a factor $O(\Gamma^2z^{-2})$], and (c) a factor $\mathcal{N}$ [$\sim O(\Gamma^{-1})$] appears with each split in (C8). These three circumstances introduce at least a factor $O(z\Gamma^2z^{-1}\Gamma^{-1})\sim O(\Gamma)\sim O(|\ln z|^{-1})$ with respect to (C20), showing that (C20) is the most important term in $\langle \Psi_{n_0}\Psi_{n_1}\cdots\Psi_{n_p}\rangle$ for $z\to 0$. Using these results, we see that in the expansion of $\langle \Psi_{n_0}\Psi_{n_1}\cdots\Psi_{n_p}\rangle_T$ in moments of Ψ_j [Eq. (C2)], the term with $i+1$ moments is of order $(\Gamma^2z^{-1})^{p+1}z^X$, where $X=\sum_{k=0}^i I_k$. $\{I_k\}$ are the numbers of different subindexes of $\mathcal{M}$ present in the moments. Since $\sum_{k=0}^i I_k>I$, the dominant term for small z is that containing only one moment, $\langle \Psi_{n_0}\Psi_{n_1}\cdots\Psi_{n_p}\rangle \sim O((\Gamma^2z^{-1})^{p+1}z^I)$. Combining this result with the fact that the contribution in (C14) coming from the $\mathcal{J}$'s is of order $(z\Gamma^{-2})^p$, it follows that the dominant contribution to the sum in (C13) comes from the $A_{nn_1\cdots n_p}$ containing the minimum number I of different random operators Ψ_n, that is, $I=1$:

$$\max[A_{n\cdots n_p}(p,z)]\sim O((z\Gamma^{-2})^p(\Gamma^2z^{-1})^{p+1}z^I)_{I=1} \sim O(\Gamma^2)\sim O(|\ln z|^{-2}) . \tag{C21}$$

This contribution, which is simply $\langle \Psi_n\rangle_T$, is less important for small z than $\Gamma(z)$, so that the FEMA also gives for this case the exact result for $\langle \hat{F}_n(z\to 0)\rangle$.

In general, the order of $A_{n\cdots n_p}(p,z)$ is Γ^2z^{I-1}, independent of p, so that in order to obtain systematic corrections to the FEMA, an infinity of terms with different values of $p>0$ must be summed up. A calculation of this kind was done in Ref. 13 to calculate the frequency-dependent diffusion coefficient beyond the EMA.

Next, we analyze (C14) for model (c) of disorder [$\Gamma(z)\sim z^\alpha$]. In this case care must be taken in evaluating the small-z singularity in the moments of $\mathcal{M}_n$. We find, for $N\geq 2$,

$$\langle \mathcal{M}_n(z)^N\rangle=[\Gamma(z)+R_n(z)]^N\int_0^1 dw\frac{1-\alpha}{w^\alpha}\frac{[w-\Gamma(z)]^N}{[w+R_n(z)]^N} \sim \left[-\frac{\Gamma(z)^2}{R_n(z)}\right]^N R_n(z)^{1-\alpha}(1-\alpha) \times B(1-\alpha,N+\alpha-1) \tag{C22}$$

and, for $N=1$,

$$\langle \mathcal{M}_n(z)\rangle \sim \Gamma(z)[1-(1-\alpha)\Gamma(z)R_n(z)^{-\alpha}B(1-\alpha,\alpha)] . \tag{C23}$$

$B(x,y)$ is the beta function. Using (C12) we find, for all the values of N,

$$\langle \mathcal{M}_n(z)^N\rangle \sim O(z^{1-\alpha}[\Gamma(z)^2z^{-1}]^N) , \tag{C24}$$

so that

$$\langle \mathcal{M}_{n_0}\mathcal{M}_{n_1}\cdots\mathcal{M}_{n_p}\rangle \sim \langle \mathcal{M}_{i_1}^{m_1}\rangle\langle \mathcal{M}_{i_2}^{m_2}\rangle\cdots\langle \mathcal{M}_{i_I}^{m_I}\rangle \sim O\left[z^{I(1-\alpha)}\left[\frac{\Gamma(z)^2}{z}\right]^{p+1}\right]. \tag{C25}$$

It turns out that all the terms in the expansion of $\langle \Psi_{n_0}\Psi_{n_1}\cdots\Psi_{n_p}\rangle$ in moments of $\mathcal{M}_j$ are of the same order in z as (C25). In the expansion of $\langle \Psi_{n_0}\Psi_{n_1}\cdots\Psi_{n_p}\rangle_T$ in moments of Ψ_j, the dominant term is again $\langle \Psi_{n_0}\Psi_{n_1}\cdots\Psi_{n_p}\rangle$, so that the cumulant is also of the order of (C25). Using this result in (C14), the leading $A_{n\cdots n_p}(p,z)$ is again that one containing the cumulant with $I=1$, $\langle \Psi_n\rangle_T$:

$$\max[A_{n\cdots n_p}(p,z)]\sim O((z\Gamma^{-2})^p(\Gamma^2z^{-1})^{p+1}z^{I(1-\alpha)})_{I=1} \sim O(\Gamma) . \tag{C26}$$

We have used the fact that $\Gamma(z)\sim O(z^\alpha)$. Then the FEMA needs a correction which, although does not change the predicted divergence law $\langle \hat{F}_n(z)\rangle\sim z^{-\alpha}$, does change its coefficient.

To estimate the importance of this correction we write the equation which contains the whole dominant z dependence:

$$z\langle \hat{F}_n(z)\rangle - 1 = \Gamma(z)\mathcal{K}\langle \hat{F}_n(z)\rangle + \langle \Psi_n(z)\rangle_T \mathcal{K}\langle \hat{F}_n(z)\rangle \sim \Gamma(z)\left[\frac{(L+1)^2-n^2}{(L+1)^2}\right]^{\alpha} \mathcal{K}\langle \hat{F}_n(z)\rangle \ . \qquad \text{(C27)}$$

This last expression has been obtained by calculating $\langle \Psi_n\rangle_T = \langle \Psi_n\rangle$ from (C4)–(C6), (C12), and (C23). We see that the correction to the FEMA becomes negligible for small α. We can test the accuracy of the FEMA by considering small values of L, so that (C27) could be exactly solved as a $(2L+1)\times(2L+1)$ matrix equation. For example, for the case $L=1$, the exact solution of (C27) shows that the FEMA result (4.29) underestimates the exact coefficient of $z^{-\alpha}$ in an amount which, for the worst case $\alpha \approx 1$, is less than the 19%.

Finally, we analyze the case of diffusion in the presence of randomly placed traps. The average $\langle \mathcal{M}_n(z)^N\rangle$ can be readily evaluated by using the binary distribution (1.5). The result is of order z^N. All the terms in the expansion of $\langle \Psi_{n_0}\Psi_{n_1}\cdots\Psi_{n_p}\rangle_T$ in moments of Ψ, and in the expansion of $\langle \Psi_{n_0}\Psi_{n_1}\cdots\Psi_{n_p}\rangle$ in moments of $\mathcal{M}$, are of the same order in z, so that $\langle \Psi_{n_0}\Psi_{n_1}\cdots\Psi_{n_p}\rangle_T \sim O(z^{p+1})$. Finally, using the fact that $\Gamma(z)\sim O(z)$,

$$A_{n\cdots n_p}(p,z) \sim O(z^{p+1}[z\Gamma(z)^{-2}]^p) \sim O(z) \ . \qquad \text{(C28)}$$

Then there are corrections to the FEMA term $\Gamma(z)$ which need to be taken into account. The exponent in the FEMA result $\langle \hat{F}_n(z)\rangle \sim O(z^{-1})$ is the exact one, as can be seen from the fact that there is a finite probability for the walker to remain trapped in any finite interval, but the coefficient will not be the exact one.

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