Green's function for one-dimensional Fokker-Planck equations with exponentially dependent drift and diffusion coefficients

A. Debosscher
Afdeling Toegepaste Mechanica en Energieconversie, Katholieke Universiteit Leuven, Celestijnenlaan 300 A, B-3030 Heverlee (Leuven), Belgium
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The class of Fokker-Planck equations considered in this paper bridges the gap between the generalized Verhulst-Landau process and the Rayleigh process, which were apparently unrelated up to now. The equations are solved analytically for the transition probability density function of the underlying stochastic processes. The solution method uses operational calculus, and simultaneously produces all components of the spectral problem: spectrum, eigenfunctions, and their normalization. The general results obtained are benchmarked with the known results for the above-mentioned processes by inserting suitable parameters.

INTRODUCTION

A stochastic process $\{x(t)\}$ can formally be described by a stochastic differential equation (SDE) of the Langevin type:

$$\dot{x}(t)=f(x)+g(x)F(t)\ , \tag{1}$$

where f,g are deterministic functions of x (eventually of t), and where $F(t)$ represents the stochastic excitation. Restricting $F(t)$ to white noise, with

$$\langle F(t)\rangle=0\ , \tag{2}$$

$$\langle F(t)F(t+\tau)\rangle=\sigma_F^2\delta(\tau)\ , \tag{3}$$

the process $\{x(t)\}$ becomes a one-dimensional Markov process and is completely characterized by its transition probability density function (PDF) $w(x,t/x_0)$.

This PDF is the solution of the Fokker-Planck equation (FPE) associated with (1),[1,2]

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2}{\partial x^2}(Bw)-\frac{\partial}{\partial x}(Aw)\ , \tag{4}$$

with initial condition

$$w(x,0/x_0)=\delta(x-x_0) \tag{5}$$

and eventually subject to suitable boundary conditions. The diffusion and drift coefficients in (4) are given, respectively, by[1,2]

$$B(x)=g^2(x)\ ,$$
$$A(x)=f(x)+g\frac{dg}{dx}\ , \tag{6}$$

when the Stratonovich interpretation of (1) is accepted, considering $F(t)$ as a zero correlation-time limit of realistic continuous noise, and when a suitable rescaling of time in (1) normalizes $F(t)$ such that in (3) one has

$$\sigma_F^2=2\ . \tag{7}$$

Rules of classical calculus apply to (1), and a transformation $x\rightarrow y(x)$ can eventually be used to make $g\equiv 1$ in (1) and (6) ["additive-noise version" of (1) and (4)].

Clearly, the FPE is an appropriate tool for the study of *nonlinear* stochastic processes. In spite of the growing interest in nonlinear models, the number of cases where the FPE has been solved exactly is still small in comparison to the numerous instances where the equation has been used for approximate or numerical analyses, e.g., for moment calculations. See the extensive reference list in Ref. 1.

In general, the effort of searching for a new exact analytical solution is rewarding only when the stochastic processes underlying the FPE have a sufficient degree of universality (see, e.g., Ref. 3), or when tuning new solution methods (see, e.g., Refs. 4 and 5). To some extent, both conditions are met in the present paper.

A CLASS OF FOKKER-PLANCK EQUATIONS

The subsequent analysis will deal with FPE's of the following type:

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2}{\partial x^2}[(ae^x+b)w]-\frac{\partial}{\partial x}[(ce^x+d)w]\ ,\quad x\in[-\infty,+\infty] \tag{8}$$

where the diffusion and drift coefficients exponentially depend upon x.

Constants a and b in (8) are non-negative in order to have a valid diffusion equation. They are in fact redundant, as both can be made unity by translation of x and scaling of t, but they will be kept explicit to facilitate the identification of (8) with some known FPE's. Restrictions for constants c and d will be specified whenever appropriate.

The stochastic process $\{x(t)\}$ that generates FPE (8) can be modeled by the SDE:

$$\dot{x}=\left[c-\frac{a}{2}\right]e^x+d+(ae^x+b)^{1/2}F(t) \tag{9}$$

or, eventually, allowing for *two* normalized but mutually uncorrelated white-noise functions:

$$\dot{x}=\left[c-\frac{a}{2}\right]e^{x}+d+(ae^{x})^{1/2}F_1(t)+b^{1/2}F_2(t)\ . \tag{10}$$

Some known physically important cases of (9) or (10) are easily recognized. Setting

$$a=0,\quad b=1,\quad d=-c=2\alpha>0 \tag{11}$$

gives the SDE

$$\dot{x}=2\alpha(1-e^{x})+F(t)\ , \tag{12}$$

which describes Brownian motion in a Toda potential[1,6] and also is the additive-noise version of the generalized stochastic Verhulst-Landau models[2,7,8] which represent multiplicative stochastic processes with state-dependent feedback, e.g.,

$$\dot{y}=[\alpha+F(t)-\beta y^{\gamma}]y,\quad y\in[0,\infty],\quad \alpha,\beta,\gamma>0\ . \tag{13}$$

A second example is the well-known Rayleigh process[9,10] which can be recovered from (9) or (10) by letting

$$a=1,\quad b=0,\quad c=\frac{1-\beta}{2}<1,\quad d=2\alpha>0,\quad y=2e^{-x/2}\ , \tag{14}$$

which yields the SDE

$$\dot{y}=\frac{\beta}{y}-\alpha y+F(t),\quad y\in[0,\infty],\quad \alpha>0,\quad \beta>-1\ . \tag{15}$$

[The sign of $F(t)$ is unimportant.]

Finally, the most general version of (9), with *a and* $b\neq0$, can be cast in several forms. The additive-noise version with a minimal number of parameters however, is unique:

$$\dot{y}=\frac{\beta}{\sinh(y)}-\alpha\tanh\left[\frac{y}{2}\right]+F(t),\quad y\in[0,\infty],\quad \alpha,\beta>0 \tag{16}$$

and is retrieved from (9) by the substitutions

$$a=b=1,\quad d=\alpha,\quad c=\frac{1-\beta}{2},\quad y=2\,\mathrm{arcsinh}(e^{-x/2})\ . \tag{17}$$

Equation (16) models Brownian motion in a strongly asymmetrical potential well.

The above-mentioned stochastic processes (12), (13), (15), and (16) appear to be stable in probability if the following *sufficient* conditions are met:

$$c<0\ , \tag{18}$$

$$d>0\ . \tag{19}$$

These conditions will be accepted as a *working hypothesis;* they will be sharpened or relaxed whenever it is convenient or necessary.

CONSTRUCTIVE SOLUTION METHOD

The mathematical problem of solving an FPE such as (8):

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2}{\partial x^2}[(ae^{x}+b)w]-\frac{\partial}{\partial x}[(ce^{x}+d)w]\ ,$$
$$x\in[-\infty,+\infty],\quad a,b,d>0,\quad c<0 \tag{20}$$

subject to the initial condition

$$w(x,0/x_0)=\delta(x-x_0) \tag{21}$$

and to natural boundary conditions[1,2,9]

$$w(\pm\infty,t/x_0)=0,\quad \frac{\partial w}{\partial x}=0\ \text{ for } x=\pm\infty \tag{22}$$

classically may consist of the following steps.

(1) Elimination of the time variable from (20), either by Laplace transformation, or by an eigenfunction expansion such as

$$w(x,t/x_0)=w_s(x)\sum_{k=0}^{\infty}\varphi_k(x)\varphi_k(x_0)e^{-\lambda_k t}\ . \tag{23}$$

(2) Determining the stationary PDF $w_s(x)$ from (20) with

$$\frac{\partial w_s}{\partial t}=0\ . \tag{24}$$

(3) Normalizing $w_s(x)$.

(4) "Solving" the eigenfunction equation for the φ_k's, possibly by transformation to some known type of second-order equation, or by series solution, by further expansion in terms of suitable known eigenfunctions, or just numerically. This step usually involves a lot of trial and error.

(5) Determining the spectrum of eigenvalues λ_k, by application of the boundary and/or integrability conditions.

(6) Normalization of the φ_k's with respect to $w_s(x)$.

In contrast, with the actual solution method, all components of the expansion (23) will almost simultaneously and automatically emerge from a formal series solution, and normalization is included. The method proceeds as follows.

The Laplace and Fourier transformation, with respect to t and x, respectively, applied to (20) results in

$$(p-bz^2-dz)\overline{\Theta}(z,p)=e^{zx_0}+(az^2+cz)\overline{\Theta}(z+1,p)\ , \tag{25}$$

where

$$\overline{\Theta}(z,p)=\int_0^{\infty}dt\,e^{-pt}\Theta(z,t)$$
$$=\int_0^{\infty}dt\,e^{-pt}\int_{-\infty}^{\infty}dx\,e^{zx}w(x,t/x_0)\ ,$$
$$z=i\omega,\quad \omega\in[-\infty,+\infty] \tag{26}$$

i.e., $\overline{\Theta}$ is the Laplace transformed characteristic function for the stochastic process $\{x(t)\}$.

Equation (25) is devoid of all derivatives. It is a functional equation with a unitary z-argument shift in $\overline{\Theta}(z,p)$.

The initial condition (21) is incorporated via e^{zx_0}. Natural boundary conditions (and integrability) are implicit in the use of the Fourier transform.

Considering (25) as a "functional recurrence equation," a solution is obtained by repeated substitution of the shifted $\overline{\Theta}(z+k,p)$ functions in the right member:

$$\bar{\Theta}(z,p)=\frac{e^{zx_0}}{\lambda_0(p,z)}\left[1+\frac{az(z+c/a)}{\lambda_1(p,z)}e^{x_0}+\cdots+\frac{a^k(z)_k(z+c/a)_k}{\lambda_1\lambda_2\cdots\lambda_k}e^{kx_0}+\cdots\right]$$

$$=\frac{e^{zx_0}}{\lambda_0(p,z)}\psi(p,z,x_0)\ , \tag{27}$$

where

$$(z)_k=z(z+1)(z+2)\cdots(z+k-1)=\frac{\Gamma(z+k)}{\Gamma(z)}\ , \tag{28}$$

$$\lambda_0(p,z)=p-bz^2-dz=-b(z-z^+)(z-z^-)\ , \tag{29}$$

$$z^{\pm}(p)=-\frac{d}{2b}\pm\left[\frac{d^2}{4b^2}+\frac{p}{b}\right]^{1/2}\ , \tag{30}$$

$$\lambda_k(p,z)=\lambda_0(p,z+k)\ . \tag{31}$$

The factorization of λ_0 in (29) allows one to write the denominators in (27) as

$$\prod_{j=1}^{k}\lambda_j(p,z)=(-b)^k\frac{\Gamma(z-z^++k+1)}{\Gamma(z-z^++1)}\times\frac{\Gamma(z-z^-+k+1)}{\Gamma(z-z^-+1)}\ , \tag{32}$$

which clearly shows that $\psi(p,z,x_0)$ in (27) is a generalized hypergeometric series;[11,12]

$$\psi(p,z,x_0)={}_3F_2\left[z,z+\frac{c}{a},1;z-z^++1,z-z^-+1;-\frac{a}{b}e^{x_0}\right]\ , \tag{33}$$

with

$${}_3F_2(a,b,c;p,q;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(c)_k}{(p)_k(q)_k}\frac{x^k}{k!}\ . \tag{34}$$

The convergence of the ${}_nF_m$ series encountered in the present analysis will not be questioned. Their use is purely *formal*, as operational calculus expressions, e.g., and even with "divergent" series, sensible results will be obtained, as such expressions may constitute an asymptotic expansion of some other function, or as they may truncate to a polynomial in a further stage of the calculation.

LAPLACE INVERSION

It is clear from (27), (29), and (31) that, as a function of p, $\bar{\Theta}(z,p)$ has an infinity of simple poles at the "points":

$$p=p_k(z)=p_0(z+k)=b\left[z+k+\frac{d}{2b}\right]^2-\frac{d^2}{4b}\ ,\qquad k=0,1,2,\ldots \tag{35}$$

and thus may equivalently be represented by its partial fraction expansion:

$$\bar{\Theta}(z,p)=\sum_{k=0}^{\infty}\frac{r_k(z)}{p-p_k(z)}\ , \tag{36}$$

where the residues r_k are given by

$$r_k(z)=e^{zx_0}(ae^{x_0})^k\frac{(z)_k(z+c/a)_k}{\prod_{j=0}^{k-1}\lambda_j(p_k,z)}\psi(p_k,z+k,x_0)\ . \tag{37}$$

Here, use has been made of the self-reproducing property of ψ in (27) and (33):

$$\psi(p,z,x_0)=\sum_{j=0}^{\infty}T_j=\sum_{j=0}^{k-1}T_j+T_k\psi(p,z+k,x_0)\ , \tag{38}$$

i.e., when the kth term T_k ($k\neq0$) is factored from the rest of the series, the function is seen to repeat itself with z shifted to $z+k$. The term T_k contains the pole $p=p_k(z)$, while the partial sum *and* $\psi(p,z+k,x_0)$ are analytic there, which explains (37).

Some further development of (37) can be done:

$$\prod_{j=0}^{k-1}\lambda_j(p_k,z)=\prod_{j=0}^{k-1}[p_k(z)-p_j(z)]=b^kk!\frac{\Gamma(2z+d/b+2k)}{\Gamma(2z+d/b+k)}\ , \tag{39}$$

$$\psi(p_k,z+k,x_0)={}_3F_2\left[z+k,z+\frac{c}{a}+k,1;z+k-z_k^++1,z+k-z_k^-+1;-\frac{a}{b}e^{x_0}\right]\ , \tag{40}$$

where, from (30):

$$z_k^+=z^+[p_k(z)]=z+k\ , \tag{41}$$

$$z_k^-=z^-[p_k(z)]=-z-k-\frac{d}{b}\ . \tag{42}$$

It follows that the ${}_3F_2$ series in (40) contracts to an ${}_2F_1$ series:

$$\psi(p_k,z+k,x_0)={}_2F_1\left[z+k,z+\frac{c}{a}+k,2z+\frac{d}{b}+2k+1;-\frac{a}{b}e^{x_0}\right] \tag{43}$$

by equality of an upper and a lower parameter:

$$z+k-z_k^{+}+1=1\ . \tag{44}$$

Substituting (28), (39), and (43) into (37) yields

$$r_k(z)=e^{zx_0}\left[\frac{a}{b}e^{x_0}\right]^k\frac{1}{k!}\frac{\Gamma(z+k)}{\Gamma(z)}\frac{\Gamma(z+c/a+k)}{\Gamma(z+c/a)}\frac{\Gamma(2z+d/b+k)}{\Gamma(2z+d/b+2k)}$$
$$\times{}_2F_1\left[z+k,z+\frac{c}{a};2z+\frac{d}{b}+2k+1;-\frac{a}{b}e^{x_0}\right]\ . \tag{45}$$

So, with (36) and (45), the inverse Laplace transform of (27) is

$$\Theta(z,t)=\frac{e^{zx_0}}{\Gamma(z)\Gamma(z+c/a)}\sum_{k=0}^{\infty}e^{p_k(z)t}\frac{[(a/b)e^{x_0}]^k}{k!}\frac{\Gamma(z+k)\Gamma(z+c/a+k)\Gamma(2z+d/b+k)}{\Gamma(2z+d/b+2k)}\,{}_2F_1(\ )\ , \tag{46}$$

where $p_k(z)$ is given by (35), and ${}_2F_1$ has arguments as in (45).

This still is a formal expression and the "operators" $e^{p_k(z)t}$ are not yet related to the decaying time exponentials $e^{-\lambda_k t}$ in an eigenfunction expansion like (23). A rearrangement of (46) is necessary and may be performed by contour integration.

CONTOUR INTEGRATION

Considering (46) as the sum of residues of a suitable complex function $G(s)$ at the poles $s=s_k=k$, $k=0,1,2,\ldots$ of the summation function $\Gamma(-s)$, one has[13]

$$\Theta(z,t)=\frac{e^{zx_0}}{2\pi i\Gamma(z)\Gamma(z+c/a)}\int_{C_1}ds\,\exp[p_0(z+s)t]$$
$$\times\left[-\frac{a}{b}e^{x_0}\right]^s\frac{\Gamma(z+s)\Gamma(z+c/a+s)\Gamma(2z+d/b+s)\Gamma(-s)}{\Gamma(2z+d/b+2s)}$$
$$\times{}_2F_1\left[z+s,z+\frac{c}{a}+s;2z+\frac{d}{b}+2s+1;-\frac{a}{b}e^{x_0}\right]\ , \tag{47}$$

where the contour C_1 in the s plane circles all poles of $\Gamma(-s)$ clockwise (Fig. 1), without enclosing any of the *other* poles of the integrand. These constitute three descending pole chains, originating, respectively, from

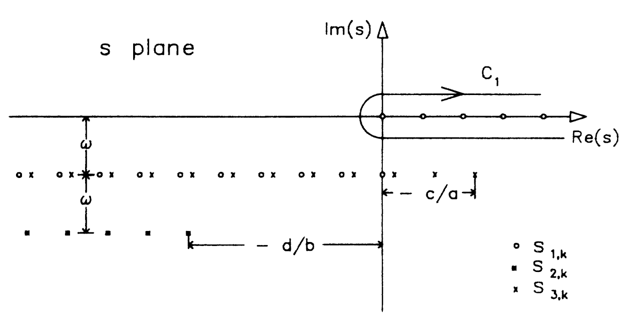

FIG. 1. Original pole configuration and contour in the complex s plane.

$$\Gamma(z+s):\quad s=s_{1,k}=-z-k\ ,$$
$$\Gamma\left[2z+\frac{d}{b}+s\right]:\quad s=s_{2,k}=-2z-\frac{d}{b}-k\ , \tag{48}$$
$$\Gamma\left[z+\frac{c}{a}+s\right]:\quad s=s_{3,k}=-z-\frac{c}{a}-k\ ,$$
$$z=i\omega,\quad k=0,1,2,\ldots\ .$$

The interference (Fig. 1) of the sets $s_{1,k}$ and $s_{3,k}$ on the same horizontal is undesirable in view of the subsequent contour transformations, as it will appear that the set $s_{1,k}$ which starts on the imaginary axis is the proper candidate for generating the eigenvalues. The set $s_{3,k}$ can, however, be avoided by a "flip-over" in (46) before the contour integration is introduced:

$$\frac{\Gamma(z+c/a+k)}{\Gamma(z+c/a)}=(-1)^k\frac{\Gamma(1-z-c/a)}{\Gamma(1-z-c/a-k)}\ , \quad (49)$$

which is based on the reflection formula of the Γ function:[14]

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}\ . \quad (50)$$

The new suitable version of (47) becomes

$$\Theta(z,t)=\frac{e^{zx_0}}{2\pi i}\frac{\Gamma(1-z-c/a)}{\Gamma(z)}\int_{C_1}ds\exp[p_0(z+s)t]\left[\frac{a}{b}e^{x_0}\right]^s\frac{\Gamma(z+s)\Gamma(2z+d/b+s)\Gamma(-s)}{\Gamma(1-z-c/a-s)\Gamma(2z+d/b+2s)}\,{}_2F_1(\ )\ , \quad (51)$$

with the pole configuration and the contour C_1 as in Fig. 2.

A variety of pole-separating s contours, equivalent to C_1, is possible (e.g., C_2 in Fig. 2). For real non-negative eigenvalues to emerge from (51), a contour should be chosen such that

$$p_0(z+s)=b(z+s+d/2b)^2-d^2/4b=-\lambda\ , \quad (52)$$

where λ is *real* and *non-negative*. The locus where this condition is met can be found from

$$\begin{aligned}&\mathrm{Im}(z+s+d/2b)^2=0\ ,\\&(z+s+d/2b)^2\le d^2/4b^2\ ,\end{aligned} \quad (53)$$

and consists of two connected parts (Fig. 2): the vertical line L,

$$\mathrm{Re}(s)=-\frac{d}{2b}\ , \quad (54)$$

and the horizontal line segment

$$\mathrm{Im}(s)=-\omega,\quad -\frac{d}{b}\le\mathrm{Re}(s)\le 0,\quad z=i\omega \quad (55)$$

intersecting in the point $O(-d/2b,-i\omega)$. A contour closely approaching this locus and still equivalent to C_1 or C_2 is given by C_3 in Fig. 2.

In the limit, one gets an integral over the line L (54) and a finite set of $N+1$ "pinched-off" residues at the poles (48) $s_{1,0},s_{1,1},\ldots,s_{1,N}$ of the integrand, with

$$N=\mathrm{int}(d/2b)\ . \quad (56)$$

This corresponds, respectively, to a continuous and a discrete part of the spectrum of eigenvalues in the eigenfunction representation of the time-dependent characteristic function:

$$\Theta(z,t)=S+I\ , \quad (57)$$

with

$$\begin{aligned}S&=\sum_{k=0}^{N}\mathrm{Res}(s=s_{1,k})\\&=\left[\frac{a}{b}\right]^{-z}\frac{\Gamma(1-z-c/a)}{\Gamma(z)}\sum_{k=0}^{N}\exp[k(bk-d)t]\frac{\Gamma(z+d/b-k)\Gamma(z+k)}{\Gamma(1-c/a+k)\Gamma(d/b-2k)}\frac{[-(a/b)e^{x_0}]^{-k}}{k!}\\&\qquad\times{}_2F_1\left[-k,\frac{c}{a}-k;\frac{d}{b}+1-2k;-\frac{a}{b}e^{x_0}\right]\end{aligned} \quad (58)$$

and

$$I=\int_L ds\cdots,\quad s=-z-\frac{b}{2d}+i\mu,\quad \mu\in[-\infty,\infty]\ ,$$

$$\begin{aligned}I=&\left[\frac{a}{b}\right]^{-z}\frac{\Gamma(1-z-c/a)}{2\pi\Gamma(z)}e^{-(d^2/4b)t}\left[\frac{a}{b}e^{x_0}\right]^{-d/2b}\\&\times\int_{-\infty}^{\infty}d\mu\,e^{-b\mu^2t}\left[\frac{a}{b}e^{x_0}\right]^{i\mu}\frac{\Gamma(i\mu-d/2b)\Gamma(z+d/2b+i\mu)}{\Gamma(1+d/2b-c/a-i\mu)}\frac{\Gamma(z+d/2b-i\mu)}{\Gamma(2i\mu)}\\&\times{}_2F_1\left[-\frac{d}{2b}+i\mu,-\frac{d}{2b}+\frac{c}{a}-i\mu;1+2i\mu;-\frac{a}{b}e^{x_0}\right]\ .\end{aligned} \quad (59)$$

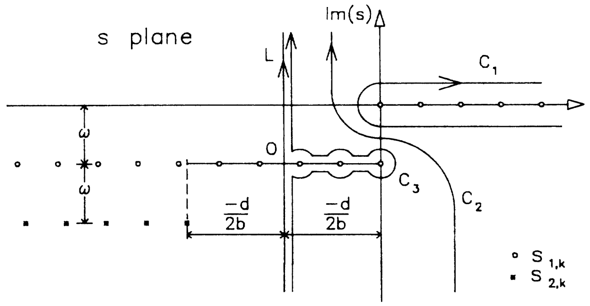

FIG. 2. Adapted pole configuration, locus of real nonnegative eigenvalues, and equivalent contours in the complex s plane.

Formulas (58) and (59) come straightforward from (51), and no atempt will be made in this paper to simplify or to symmetrize them.

The expression (57)–(59) for $\Theta(z,t)$ is of direct use for the calculation of conditional moments, e.g.,

$$M_m(t/x_0)=\langle e^{mx}/x_0\rangle = \int_{-\infty}^{\infty} dx\, e^{mx} w(x,t/x_0)=\Theta(m,t) \quad (60)$$

by mere substitution of $z=m$, $m=0,1,2,\ldots$.

The x_0-dependent parts in each term of (58) already display the *unnormalized* discrete eigenfunctions $\varphi_k(x_0)$ which, due to truncation of $_2F_1$, are kth-degree polynomials in $(b/a)e^{-x_0}$ and eventually can be identified as Jacobi polynomials $P_k^{(\eta,v)}(x)$:[11,14]

$$\frac{[-(a/b)e^{x_0}]^{-k}}{k!}\,{}_2F_1\left[-k,\frac{c}{a}-k;\frac{d}{b}+1-2k;-\frac{a}{b}e^{x_0}\right] = \frac{\Gamma(k-d/b)}{\Gamma(2k-d/b)}P_k^{(\eta,v)}\left[1+\frac{2b}{a}e^{-x_0}\right],$$

$$\eta=-\frac{c}{a}, \quad v=\frac{c}{a}-\frac{d}{b}-1 . \quad (61)$$

The discrete part of the spectrum of eigenvalues λ_k is clear from the time exponentials:

$$\lambda_k=k(d-bk),\quad k=0,1,\ldots,N=\operatorname{int}\left[\frac{d}{2b}\right]. \quad (62)$$

FOURIER INVERSION

The $k=0$ term ($\lambda_0=0$) of (58) yields the normalized stationary first-order characteristic function:

$$\Theta_s(z)=\lim_{t\to\infty}\Theta(z,t) = (a/b)^{-z}\frac{\Gamma(1-z-c/a)\Gamma(z+d/b)}{\Gamma(1-c/a)\Gamma(d/b)} \quad (63)$$

and, by Fourier inversion, the stationary first-order PDF:

$$w_s(x)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\, e^{-zx}(a/b)^{-z} \times\frac{\Gamma(1-z-c/a)\Gamma(z+d/b)}{\Gamma(1-c/a)\Gamma(d/b)} \quad (64)$$

$$=\frac{\Gamma(1+d/b-c/a)}{\Gamma(1-c/a)\Gamma(d/b)}\frac{[(a/b)e^{x}]^{d/b}}{[1+(a/b)e^{x}]^{1+d/b-c/a}} \quad (65)$$

under the conditions (see Ref. 15—Mellin transforms)

$$\frac{d}{b}>0 \quad\text{or } d>0 , \quad (66)$$

$$\frac{c}{a}<1 \quad\text{or } c<a , \quad (67)$$

which henceforth replace (18) and (19).

The normalizing constant in (65) can be written in terms of a beta function (B):[14]

$$N_s=\frac{\Gamma(1+d/b-c/a)}{\Gamma(1-c/a)\Gamma(d/b)}=\frac{1}{B(d/b,1-c/a)} . \quad (68)$$

Only the normalization of the eigenfunctions φ_k remains to be found.

Considering a general term of (58) with $k\neq0$, the Fourier inversion of the z-dependent part is expected to yield a function proportional to $w_s(x)\varphi_k(x)$ [in reference to (23)]:

$$I_1=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\, e^{-zx}(a/b)^{-z}\frac{\Gamma(1-z-c/a)\Gamma(z+d/b-k)\Gamma(z+k)}{\Gamma(z)} \propto w_s(x)\varphi_k(x) . \quad (69)$$

The integral I_1 is not readily available from tables. When compared to the integral (64) and (65), I_1 is seen to represent a generalization of Rodrigues's formula[14] for orthogonal polynomials with weight function $w_s(x)$, but more straightforwardly it can be identified as a particular case of Meijer's G function:[11,12,15]

$$I_1=G_{22}^{12}\left[\frac{b}{a}e^{-x}\,\middle|\,\begin{matrix}1-k,1+k-d/b\\ 1-c/a,1\end{matrix}\right] = G_{22}^{21}\left[\frac{a}{b}e^{x}\,\middle|\,\begin{matrix}c/a,0\\ k,d/b-k\end{matrix}\right] , \quad (70)$$

which in turn is related to a $_2F_1$ function:[11,15]

$$I_1=\frac{\Gamma(k+1-c/a)\Gamma(d/b-c/a-k+1)}{\Gamma(1-c/a)} \times\frac{[(a/b)e^{x}]^{d/b}}{[1+(a/b)e^{x}]^{1+d/b-c/a}} \times{}_2F_1\left[-k,k-\frac{d}{b};1-\frac{c}{a};-\frac{b}{a}e^{-x}\right] . \quad (71)$$

In this last expression $w_s(x)$ (65) is already apparent, and $_2F_1$ indeed is a Jacobi polynomial again:[14]

$$ {}_2F_1\left[-k,k-\frac{d}{b};1-\frac{c}{a};-\frac{b}{a}e^{-x}\right] = \frac{k!\Gamma(1-c/a)}{\Gamma(1-c/a+k)}P_k^{(\eta,v)}\left[1+\frac{2b}{a}e^{-x}\right] , \quad (72)$$

with η and v as in (61). Using now (71), (72), (61), and (65) one has

$$\varphi_k(x)\varphi_k(x_0) = B(d/b, 1-c/a) \times \frac{\Gamma(d/b-c/a+1-k)\Gamma(k-d/b)k!}{\Gamma(1-c/a+k)\Gamma(d/b-2k)\Gamma(2k-d/b)} \times P_k^{(\eta,v)}[h(x)]P_k^{(\eta,v)}[h(x_0)] ,$$

$$\eta = -c/a ,$$

$$v = c/a - d/b - 1 , \tag{73}$$

$$h(x) = 1 + \frac{2b}{a}e^{-x} ,$$

showing the normalization of the eigenfunctions $\varphi_k(x)$.

This completes the inversion for the discrete part of the spectrum. A similar treatment of the continuous spectrum (59) is possible but will not be undertaken in this paper, as the essentials of the method should be sufficiently clear from the above analysis.

SPECIAL CASES

We have the following.

(a) Results for the Verhulst-Landau stochastic process are reconstructed by choosing the parameters (11), which involves some plausible limiting processes as $a \to 0$. The stationary PDF is

$$w_s(x) = \lim_{a\to 0}\left[\frac{\Gamma(1+2\alpha+2\alpha/a)}{\Gamma(1+2\alpha/a)\Gamma(2\alpha)}\frac{(ae^x)^{2\alpha}}{(1+ae^x)^{1+2\alpha+2\alpha/a}}\right] = \frac{(2\alpha)^{2\alpha}}{\Gamma(2\alpha)}e^{-2\alpha(e^x-x)} . \tag{74}$$

The discrete spectrum is

$$\lambda_k = k(2\alpha - k), \quad k = 0,1,2,\ldots,N = \mathrm{int}(\alpha) . \tag{75}$$

The discrete eigenfunctions are, e.g., obtainable by performing a limit in (72):

$$\lim_{a\to 0} {}_2F_1\left[-k, k-2\alpha; 1+2\alpha/a; -\frac{1}{a}e^{-x}\right] = {}_2F_0\left[-k, k-2\alpha; -\frac{1}{2\alpha}e^{-x}\right] = (2\alpha e^x)^{-k}U(-k; 2\alpha+1-2k; 2\alpha e^x) = (-2\alpha e^x)^{-k}k!L_k^{2(\alpha-k)}(2\alpha e^x) , \tag{76}$$

where U is a confluent hypergeometric function, and $L_k^{\gamma}(x)$ is a generalized Laguerre polynomial.[14]

Further substitutions, as well as the treatment of the continuous spectrum, exactly reproduce the results which are available in Ref. 7, in terms of the variable $z = 2\alpha e^x$. (See also Refs. 2 and 10.)

(b) The Rayleigh-process results[9,10] can be recovered exactly by insertion of the parameter set (14). Limits are now related to $b \to 0$. The spectrum (62) now becomes *entirely discrete,* as the vertical integration path L (Fig. 2), and simultaneously the pole series $s_{2,k}$, disappears to infinity when $b \to 0$. Eigenvalues are now equidistant:

$$\lambda_k = 2\alpha k, \quad k = 0,1,2,\ldots \tag{77}$$

and "confluence" of (72) yields[11]

$$\lim_{b\to 0} {}_2F_1\left[-k, k-2\alpha/b; \frac{1+\beta}{2}; -be^{-x}\right] = {}_1F_1\left[-k; \frac{1+\beta}{2}; 2\alpha e^{-x}\right] = \frac{k!\Gamma((\beta+1)/2)}{\Gamma((\beta+1)/2+k)}L_k^{(\beta-1)/2}\left[\frac{\alpha y^2}{2}\right] , \tag{78}$$

which is a generalized Laguerre polynomial with constant, k-independent parameter $(\beta-1)/2$.

For completeness, the stationary PDF follows from (65):

$$w_s(x) = \lim_{b\to 0}\left[\frac{\Gamma(2\alpha/b+(\beta+1)/2)}{\Gamma((\beta+1)/2)\Gamma(2\alpha/b)}\frac{[(1/b)e^x]^{2\alpha/b}}{[1+(1/b)e^x]^{2\alpha/b+(\beta+1)/2}}\right] = \frac{(2\alpha)^{(\beta+1)/2}}{\Gamma((\beta+1)/2)}\exp\left[-2\alpha e^x - \frac{(\beta+1)}{2}x\right] \tag{79}$$

or, in terms of $y = 2e^{-x/2}$:

$$w_s(y) = w_s(x)\left|\frac{dx}{dy}\right| = \frac{(2\alpha)^{(\beta+1)/2}}{\Gamma((\beta+1)/2)}(y/2)^{\beta}\exp\left[-\frac{\alpha y^2}{2}\right] . \tag{80}$$

CONCLUSIONS

The spectral problem has been solved for a rather general class of Fokker-Planck equations. The considered class realizes a unification between, and a generalization of, such apparently unrelated stochastic processes as the Verhulst-Landau and the Rayleigh process. The same results, and the same unifying characteristics apply to the class of one-dimensional Schrödinger equations (with imaginary time) that can be associated[1] with the FPE's.

The solution method employed basically starts from a first-order functional recurrence for some integral-transformed function, and thus is *not* strictly limited to the equation type or the integral-transform type of the actual application. The method considerably enlarges the scope of operational calculus techniques, and the formal solution of the recurrence equation may be a good starting point in the search for alternative solution forms [such as, e.g., the closed form of the Rayleigh-process PDF (Ref. 10)].

The unification between the two-parameter Jacobi polynomials and the one-parameter generalized Laguerre polynomials as limiting cases is a spin-off which does not seem to be well documented yet in applied mathematics.

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