Amplitude-noise reduction in lasers with intracavity nonlinear elements

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We consider lasers with intracavity nonlinear elements, e.g., two-photon absorption, second-harmonic generation, as a possible means to reduce the amplitude fluctuations in a laser. While amplitude squeezing up to 37% may be obtained for the internal field, only a modest amount of noise reduction (10% for two-photon absorption, effectively zero for second-harmonic generation) is found in the output field at the laser frequency. In the second-harmonic field an amplitude-noise reduction of 50% below the shot-noise level may be achieved.

I. INTRODUCTION

In this paper we investigate stabilization of the laser amplitude by the use of intracavity nonlinear elements. The goal is to achieve subshot-noise operation of an active laser system by a feedback system internal to the laser cavity. The prototype of such a system is a laser with a two-photon absorber present. Large intensity fluctuations will be reduced by the action of the two-photon absorber and consequentially the laser intensity should be stabilized. The potential of such a system has already been recognized by Bandilla and Ritze,[1] and Herzog,[2] who calculated the intensity fluctuations of the laser internal to the laser cavity. They found a maximum reduction of intensity fluctuations in the limit where the two-photon loss mechanism is dominant over the one-photon cavity loss. The quantity of experimental interest, however, is the intensity fluctuations in the output light. To achieve maximum reduction of intensity fluctuations in the external field then involves a trade-off between the two-photon absorption and the one-photon cavity loss.

We shall first present an analysis based on the Scully-Lamb master equation[3] for the laser in the photon-number representation. We then show that similar results are obtained using Langevin equations derived from Louisell's[4] Fokker-Planck equation for the laser in the Glauber-Sudarshan P representation.[5,6] We then use this method to study the situation of intracavity second-harmonic generation. Intracavity second-harmonic generation is known to produce squeezed light[7,8] with reduced amplitude fluctuations and hence might be expected to reduce the amplitude fluctuations in an active laser cavity. We shall determine how effective intracavity second-harmonic generation is in reducing laser amplitude fluctuations.

II. LASER WITH TWO-PHOTON ABSORBER: MASTER-EQUATION APPROACH

We consider a laser with an intracavity two-photon absorber. We take as our starting point the Scully-Lamb laser master equation[3] in the photon-number representation

$$\dot{\rho}_n = -A(n+1)\left[1+(n+1)\frac{B}{A}\right]^{-1}\rho_n + A_n\left[1+\frac{nB}{A}\right]^{-1}\rho_{n-1} - \kappa n\rho_n + \kappa(n+1)\rho_{n+1}\ , \tag{2.1}$$

where $\rho_n = \langle n|\rho|n\rangle$,

$$A = 2r_a\left[\frac{g^2}{\gamma_a\gamma_{ab}}\right],$$

$$B = 8r_a\left[\frac{g^2}{\gamma_a\gamma_{ab}}\right]\left[\frac{g^2}{\gamma_a\gamma_b}\right],$$

$$\gamma_{ab} = \tfrac{1}{2}(\gamma_a+\gamma_b)\ .$$

κ is the cavity loss rate and g is the dipole coupling constant; γ_a and γ_b are the decay rates of the upper and lower levels of the lasing transition; and r_a is the rate at which excited atoms are injected into the cavity. The two-photon absorber introduces the additional term[1,2]

$$\dot{\rho}_n = \chi[(n+2)(n+1)\rho_{n+2} - n(n-1)\rho_n]\ , \tag{2.2}$$

where χ is the two photon absorption rate.

Combining Eqs. (2.1) and (2.2) we have

$$\dot{\rho}_n = \kappa C(a_n\rho_{n-1} - a_{n+1}\rho_n) + \kappa[(n+1)\rho_{n+1} - n\rho_n] + \chi[(n+2)(n+1)\rho_{n+1} - n(n-1)\rho_n]\ , \tag{2.3}$$

where we have defined

$$a_n = \frac{n}{1+\frac{n}{n_s}}\ ,$$

where n_s is the saturation photon number, $n_s = A/B = \gamma_a \gamma_b / 4g^2$. $C = (r_a / 2\kappa n_s)(\gamma_a/\gamma_{ab})$ is the modified upper-level pump rate.

The steady state of the above equation does not satisfy the usual detailed balance conditions. Hence we shall adopt a method developed by Görtz and Walls.[9] We define the quantity

$$q_n = \frac{\rho_{n+1}}{\rho_n} , \tag{2.4}$$

which is characteristically a slowly varying quantity with n. The probability distribution may be obtained from the ratios q_m as

$$\rho_m = \rho_0 \prod_{n=0}^{m-1} q_n . \tag{2.5}$$

An approximate steady state solution of Eq. (2.3) may be found in the limit of large photon numbers $(n+1 \approx n)$ and slowly varying q_n $(q_{n+1} \approx q_n)$,

$$0 = \kappa C a_n (q_n^{-1} - 1) + \kappa n (q_n - 1) + \chi n^2 (q_n^2 - 1) . \tag{2.6}$$

One root of this equation is the unphysical $q_n = 1$ for every n which is a consequence of equating the upward and downward transition rates.

Dividing out this unnormalizable root one obtains a quadratic equation

$$0 = \chi n^2 q_n^2 + n(\kappa + \chi n) q_n - \kappa C a_n . \tag{2.7}$$

The peak photon number occurs at $q_n = 1$. We shall assume the peak and mean photon number $\bar{n}$ are approximately equal. This gives for the mean photon number

$$\bar{n} = \frac{1}{2} \left\{ \left[\left[n_s + \frac{\kappa}{2\chi} \right]^2 + \frac{2\kappa n_s (C-1)}{\chi} \right]^{1/2} - \left[n_s + \frac{\kappa}{2\chi} \right] \right\} , \tag{2.8}$$

which is real and positive provided $C > 1$.

This may be compared with the solution for the usual laser where no two-photon loss is present:

$$\bar{n} = (C-1) n_s . \tag{2.9}$$

A useful scaled measure of the nonlinearity is

$$X = \frac{2 n_s \chi}{\kappa} , \tag{2.10}$$

which is just the ratio of nonlinear to linear loss at the saturation intensity. In terms of this we have

$$\frac{\bar{n}}{n_s} = \frac{1}{2X} \{ [(X+1)^2 + 4X(C-1)]^{1/2} - (X+1) \} . \tag{2.11}$$

If we consider the large pumping limit $C \gg 1, X$ (and $CX \gg 1$), we find

$$\bar{n} = \left[\frac{C}{X} \right]^{1/2} n_s . \tag{2.12}$$

The variance may be calculated by using the Gaussian approximation for the distribution

$$\rho_{\bar{n}+\delta m} = \rho_{\bar{n}} e^{-(\delta m)^2 / 2\sigma^2} . \tag{2.13}$$

Using Eq. (2.5) we may write

$$\begin{aligned} \rho_{\bar{n}+\delta m} &= \rho_{\bar{n}} \prod_{\nu=\bar{n}}^{\bar{n}+\delta m - 1} q(\nu) \\ &= \rho_{\bar{n}} \exp \left[\sum_{\nu=\bar{n}}^{\bar{n}+\delta m-1} \ln q(\nu) \right] \\ &\approx \rho_{\bar{n}} \exp \left[\int_{\bar{n}}^{\bar{n}+\delta m} d\nu \ln q(\nu) \right] . \end{aligned} \tag{2.14}$$

Making a Taylor expansion of $q(\nu)$ about the mean

$$q(\nu) \approx 1 + q'(\bar{n})(\nu - \bar{n}) , \tag{2.15}$$

we find

$$\sigma^2 = - \left[\left. \frac{\partial q}{\partial n} \right|_{n=\bar{n}} \right]^{-1} . \tag{2.16}$$

Applying this result to Eq. (2.17) we find

$$\sigma^2 = \bar{n} \frac{\frac{3}{2} \frac{\bar{n} X}{n_s} + 1}{2 \frac{\bar{n} X}{n_s} + 1 - C a'_{\bar{n}}} , \tag{2.17}$$

where

$$a'_{\bar{n}} = \left. \frac{\partial a_n}{\partial n} \right|_{n=\bar{n}} .$$

In the limit of large pumping $C a'_{\bar{n}} \rightarrow X$. This term along with 1 may be neglected in comparison to the leading term $2\bar{n}X/n_s \approx \sqrt{4CX}$, leaving us with

$$\sigma^2 \xrightarrow{C \gg 1} \tfrac{3}{4} \bar{n} . \tag{2.18}$$

This result is in agreement with that obtained by Bandilla and Ritze[1] and Herzog[2] in the limit $\chi \gg \kappa$.

Spectrum of intensity fluctuations

The above result refers to the distribution of photons in the cavity mode. The experimentally measurable quantity is the spectrum of intensity fluctuations in the output field. The boundary condition linking the output field with the input field and the cavity field is

$$b_{\text{out}}(t) = b_{\text{in}}(t) + \sqrt{\kappa} a(t) , \tag{2.19}$$

where $a(t)$ is the internal cavity mode.

For a vacuum input field the intensity correlation spectrum is related to the internal moments using

$$\begin{aligned} \langle b_{\text{out}}^\dagger(t) b_{\text{out}}(t) b_{\text{out}}^\dagger(t') b_{\text{out}}(t') \rangle &= \kappa^2 \langle \tilde{T}[a^\dagger(t') a^\dagger(t)] T[a(t) a(t')] \rangle \\ &\quad + \kappa \langle a^\dagger(t) a(t') \rangle \delta(t - t') , \end{aligned} \tag{2.20}$$

where T and $\tilde{T}$ are the time-ordering and time-antiordering operators, respectively. The steady-state intensity fluctuation spectrum may be written as

$$S(\omega)=\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\langle I_{\text{out}}(\tau),I_{\text{out}}(0)\rangle_{\text{SS}}$$
$$=2\,\text{Re}\int_0^{\infty} d\tau\, e^{-i\omega\tau}\langle I_{\text{out}}(\tau),I_{\text{out}}(0)\rangle_{\text{SS}}$$
$$=\bar{n}\kappa\left[1+2\kappa\,\text{Re}\int_0^{\infty} d\tau\, e^{-i\omega\tau}\left[\frac{g(\tau)}{\bar{n}}-\bar{n}\right]\right], \quad (2.21)$$

where $g(\tau)=\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle$, $\tau>0$.

It may be shown that under the Markovian and Gaussian approximations[10]

$$g(\tau)=\bar{n}^{\,2}+(\sigma^2-\bar{n})e^{-\delta\tau}, \quad (2.22)$$

where $\delta=4\bar{n}\chi+\kappa-\kappa C a'_{\bar{n}}$. This gives for the output spectrum of intensity fluctuations

$$S(\omega)=\bar{n}\kappa\left[1+2\kappa\,\text{Re}\int_0^{\infty} d\tau\left[\frac{\sigma^2-\bar{n}}{\bar{n}^{\,2}}\right]e^{-(\delta+i\omega)\tau}\right]$$
$$=\bar{n}\kappa\left[1+\frac{2\kappa\delta}{\delta^2+\omega^2}\left[\frac{\sigma^2}{\bar{n}}-1\right]\right]. \quad (2.23)$$

Inserting the results for σ^2 and δ the normalized spectrum

$$\bar{S}(\omega)\equiv\frac{S(\omega)}{\bar{n}\kappa}=1-\frac{2\kappa(\bar{n}-a'_n)}{\delta^2+\omega^2}, \quad (2.24)$$

which represents a Lorentzian dip of width δ below the shot-noise level.

The minimum occurs at $\omega=0$ where

$$\bar{S}(0)=1-\frac{2\kappa(\bar{n}\chi-\kappa C a'_{\bar{n}})}{(4\bar{n}\chi+\kappa-\kappa C a'_{\bar{n}})^2}. \quad (2.25)$$

For very large pumping $r, rf \gg n_s,\kappa$, $r\gg f$ we find

$$\bar{n}\to\left[\frac{C}{X}\right]^{1/2} n_s,$$
$$C a'_{\bar{n}}\to X,$$

so

$$\delta\to\frac{2\kappa\bar{n}X}{n_s}\to\sqrt{4XC}\,\kappa$$

and

$$\bar{S}(0)\to 1-\frac{n_s}{4\bar{n}X}\to 1-\frac{1}{4\sqrt{XC}}; \quad (2.26)$$

so for large pumping we will not see any significant reduction in the intensity fluctuations since $\kappa\ll\bar{n}\chi$.

In the region of somewhat lower pump strength where we may no longer neglect κ compared to $\bar{n}\chi$, we may still pump strongly enough to make $\bar{n}\gg n_s$ and $\kappa C a'_{\bar{n}}\ll\bar{n}\chi$. In this case an extremum is found at

$$4\bar{n}\chi=\kappa \quad (2.27)$$

for which

$$\bar{S}(0)=\tfrac{7}{8}. \quad (2.28)$$

So even for this optimum condition the reduction in intensity fluctuations below the shot-noise limit is only modest.

The conditions for a large reduction in photon-number fluctuations inside the cavity is that the two-photon absorption rate $\bar{n}\chi$ greatly exceeds the one-photon cavity loss rate κ. However, a large internal loss is deleterious to the amplitude squeezing in the output field. Clearly there must be a trade-off, and the result is that only a small reduction in intensity fluctuations of the output field is possible.

III. LASER WITH TWO-PHOTON ABSORBER: LANGEVIN EQUATION APPROACH

The laser with an intracavity two-photon absorber may be described entirely by the diagonal matrix elements of the density operator in the number state basis. However, for different intracavity elements, for example, an intracavity second-harmonic generator, phase-dependent terms are present and hence it is necessary to use a Langevin equation treatment of the stochastic field amplitude. We therefore use the stochastic equations for the laser derived by Louisell[4] in the Glauber-Sudarshan P representation:

$$\frac{\partial}{\partial t}\begin{bmatrix}\alpha\\ \alpha^*\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\kappa\left[\dfrac{C}{1+\dfrac{I}{n_s}}-1\right]\\ \kappa\left[\dfrac{C}{1+\dfrac{I}{n_s}}-1\right]\end{bmatrix}\begin{bmatrix}\alpha\\ \alpha^*\end{bmatrix}+\begin{bmatrix}\left[\dfrac{-\frac{1}{2}\kappa C\dfrac{\alpha^2}{n_s}}{\left[1+\dfrac{I}{n_s}\right]^2}\right] & \dfrac{\kappa C}{1+\dfrac{I}{n_s}}\left[1-\dfrac{\dfrac{I}{2n_s}}{1+\dfrac{I}{n_s}}\right]\\ \dfrac{\kappa C}{1+\dfrac{I}{n_s}}\left[1-\dfrac{\dfrac{I}{2n_s}}{1+\dfrac{I}{n_s}}\right] & \left[\dfrac{-\frac{1}{2}\kappa C\dfrac{\alpha^{*2}}{n_s}}{\left[1+\dfrac{I}{n_s}\right]^2}\right]\end{bmatrix}^{1/2}\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix}, \quad (3.1)$$

where C and n_s are defined in Sec. II and $I=\alpha^*\alpha$ and ϵ_1, and ϵ_2 are fluctuating forces with the following correlation functions:

$$\langle \epsilon_1^*(t)\epsilon_1(t')\rangle=\delta(t-t') ,$$
$$\langle \epsilon_2^*(t)\epsilon_2(t')\rangle=\delta(t-t') ,$$
$$\langle \epsilon_1^*(t)\epsilon_2(t')\rangle=0 .$$

These equations are valid far above threshold.

To demonstrate the equivalence of this approach to the master-equation approach used in Sec. II we shall analyze the intracavity two-photon absorber. The Hamiltonian for two-photon absorption is

$$H=\tfrac{1}{2}\chi a^2\Gamma^\dagger+\text{H.c.} , \tag{3.2}$$

where χ is the strength of the interaction and Γ is a reservoir operator modeling the absorbing system.

A master equation may be derived for the field density and transformed to a Fokker-Planck equation using the generalized P representation. The equivalent Langevin equations are[11]

$$\frac{\partial}{\partial t}\begin{bmatrix}\alpha\\ \alpha^\dagger\end{bmatrix}=\begin{bmatrix}-\chi\alpha^2\alpha^\dagger\\ -\chi\alpha^{\dagger 2}\alpha\end{bmatrix}+\begin{bmatrix}-\chi\alpha^2 & 0\\ 0 & -\chi\alpha^{\dagger 2}\end{bmatrix}^{1/2}\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix} , \tag{3.3}$$

where α and $\alpha^\dagger$ are independent variables.

We may then write the Langevin equations for the laser with intracavity two-photon absorber in the generalized P representation by combining Eqs. (3.3) and (3.1) with α^* replaced by $\alpha^\dagger$. This gives

$$\frac{\partial}{\partial t}\begin{bmatrix}\alpha\\ \alpha^\dagger\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\kappa\left[\dfrac{C}{1+\dfrac{I}{n_s}}-1\right]\alpha-2\chi\alpha^2\alpha^\dagger\\ \kappa\left[\dfrac{C}{1+\dfrac{I}{n_s}}-1\right]\alpha^\dagger-2\chi\alpha^{\dagger 2}\alpha\end{bmatrix}+\begin{bmatrix}-\chi\alpha^2-\dfrac{\frac{1}{2}\kappa C\dfrac{\alpha^2}{n_s}}{\left[1+\dfrac{I}{n_s}\right]^2} & \dfrac{\kappa C}{1+\dfrac{I}{n_s}}\left[1-\dfrac{\dfrac{I}{2n_s}}{1+\dfrac{I}{n_s}}\right]\\ \dfrac{\kappa C}{1+\dfrac{I}{n_s}}\left[1-\dfrac{\dfrac{I}{2n_s}}{\left[1+\dfrac{I}{n_s}\right]}\right] & -\chi\alpha^{\dagger 2}-\dfrac{\frac{1}{2}\kappa C\dfrac{\alpha^{\dagger 2}}{n_s}}{\left[1+\dfrac{I}{n_s}\right]^2}\end{bmatrix}^{1/2}\begin{bmatrix}\epsilon_1\\ \epsilon_2\end{bmatrix} . \tag{3.4}$$

As these equations stand, they do not have a deterministic steady state because of the laser phase diffusion. So we change to intensity and phase variables, defined by

$$I=\alpha^\dagger\alpha,\quad \phi=\frac{1}{2i}\ln\frac{\alpha^\dagger}{\alpha} , \tag{3.5}$$

for which the Langevin equations are

$$\begin{aligned}\frac{\partial I}{\partial t}&=I\left[\kappa\left[\frac{C}{1+I/n_s}-1\right]-2\chi I\right]-\frac{\kappa C(1+I/2n_s)}{(1+I/n_s)^2}\\&\quad+\left[\frac{2\kappa CI}{\left[1+\dfrac{I}{n_s}\right]^2}-2\chi I^2\right]^{1/2}\epsilon_I ,\\ \frac{\partial\phi}{\partial t}&=\frac{1}{2}\left[\frac{2\kappa C/I}{\left[1+\dfrac{I}{n_s}\right]^2}+2\chi\right]^{1/2}\epsilon_\phi .\end{aligned} \tag{3.6}$$

That is, the freely diffusing phase has no effect on the intensity. The second drift term in the intensity equation is very small (in the regime of interest it changes the mean intensity inside the cavity by less than one photon) and will be dropped. The deterministic steady-state solution $\bar{n}\approx\langle I\rangle$ is then given by the same relation as in Sec. II, Eq. (2.11). Linearizing about this gives

$$\partial\dot{I}=-A_I\partial I+(D_{II})^{1/2}\epsilon_I , \tag{3.7}$$

where

$$\begin{aligned}A_I&=\frac{\kappa\bar{n}/n_s}{1+\bar{n}/n_s}\left[1+X+2\frac{\bar{n}}{n_s}X\right] ,\\ D_{II}&=\kappa\bar{n}\left[\frac{2\left[1+\dfrac{\bar{n}}{n_s}X\right]}{1+\bar{n}/n_s}-\frac{\bar{n}}{n_s}X\right] .\end{aligned} \tag{3.8}$$

Remembering that the P representation gives normally ordered moments, the photon-number variance inside the cavity is

$$\begin{aligned}\frac{V(n)}{\bar{n}}&=1+\frac{V(I)}{\bar{n}}\\&=1+\frac{D_{II}}{2\bar{n}A_I}\\&=1+\frac{\left[1+\dfrac{\bar{n}}{n_s}X\right]-\frac{1}{2}\left[1+\dfrac{\bar{n}}{n_s}\right]\dfrac{\bar{n}}{n_s}X}{\dfrac{\bar{n}}{n_s}\left[1+X+2\dfrac{\bar{n}}{n_s}X\right]} .\end{aligned} \tag{3.9}$$

For sufficiently large $\bar{n}/n_s$ this becomes

$$\frac{V(n)}{\bar{n}} \approx 1-\tfrac{1}{4}=\tfrac{3}{4} , \tag{3.10}$$

as in Sec. II. The output intensity fluctuation spectrum is

$$\frac{S_I^{\text{out}}(\omega)}{\bar{I}_{\text{out}}}=1+\frac{\kappa D_{II}/\bar{n}}{A_I^2+\omega^2} . \tag{3.11}$$

Assuming large intensity ($\bar{n}/n_s \gg 1$) but small nonlinearity ($X \ll 1$) gives

$$\frac{S_I^{\text{out}}(0)}{\bar{I}_{\text{out}}}=1-\frac{\frac{\bar{n}}{n_s}X}{\left[1+2\frac{\bar{n}}{n_s}X\right]^2} . \tag{3.12}$$

This has its minimum value of $\frac{7}{8}$ when

$$2\frac{\bar{n}}{n_s}X=1 . \tag{3.13}$$

This is the same condition as Eq. (2.27), namely that the average linear loss is twice the average nonlinear loss. Thus we see that in all important respects the results from Louisell's Langevin equation laser model agree with those from the Scully-Lamb master-equation model.

IV. LASER WITH INTRACAVITY SECOND-HARMONIC GENERATION

We now consider the operation of a laser with an intracavity second-harmonic generator. It is established that second-harmonic generation in a passive cavity will squeeze the amplitude fluctuations in the output field. We shall now investigate whether it will have a similar effect on the output of an active laser system. The dynamics of a laser with intracavity second-harmonic generation has been studied by Mandel and Xiao-Guang.[12] In order to study the quantum fluctuations we shall use the Langevin equations describing second-harmonic generation which have been derived in the generalized P representation by Drummond, McNeil, and Walls.[13]

We may combine these with the Langevin equations for the laser to give the following equations for a laser with intracavity second-harmonic generation:

$$\frac{\partial}{\partial t}\boldsymbol{\alpha}=A(\boldsymbol{\alpha})+D^{1/2}\boldsymbol{\epsilon}(t) , \tag{4.1}$$

where

$$\boldsymbol{\alpha}=\begin{bmatrix}\alpha_1\\ \alpha_1^\dagger\\ \alpha_2\\ \alpha_2^\dagger\end{bmatrix}, \quad \boldsymbol{\epsilon}(t)=\begin{bmatrix}\epsilon_1(t)\\ \epsilon_1^\dagger(t)\\ \epsilon_2(t)\\ \epsilon_2^\dagger(t)\end{bmatrix} ,$$

and

$$A_{\alpha_1}=\frac{\kappa_1}{2}\left[\frac{C}{1+\frac{\alpha_1^\dagger\alpha_1}{n_s}}-1\right]\alpha_1+\chi\alpha_1^\dagger\alpha_2 ,$$

$$A_{\alpha_2}=-\frac{\kappa_2}{2}\alpha_2-\tfrac{1}{2}\chi\alpha_1^2 ,$$

$$D_{\alpha_1\alpha_1}=D^\dagger_{\alpha_1^\dagger\alpha_1^\dagger}=\chi\alpha_2-\frac{\frac{1}{2}\kappa_1 C\alpha_1^2/n_s}{\left[1+\frac{\alpha_1^\dagger\alpha_1}{n_s}\right]^2} ,$$

$$D_{\alpha_1^\dagger\alpha_1}=D_{\alpha_1\alpha_1^\dagger}=\kappa_1 C\frac{1+\frac{\alpha_1^\dagger\alpha_1}{2n_s}}{\left[1+\frac{\alpha_1^\dagger\alpha_1}{n_s}\right]^2} ,$$

and all other elements of the diffusion matrix are zero. C and n_s are defined as in the preceding sections. κ_1 and κ_2 are the cavity decay rates of the fundamental and second-harmonic modes, respectively, and χ is the strength of the second-harmonic nonlinearity.

Changing to intensity and phase variables

$$\begin{aligned}
\alpha_1&=(I_1)^{1/2}e^{-i\varphi_1} ,\\
\alpha_2&=(I_2)^{1/2}e^{-i\varphi_2} ,\\
A_{I_1}&=\kappa_1\left[\frac{C}{1+\frac{I_1}{n_s}}-1\right]I_1\\
&\quad+\frac{\kappa_1 C}{1+\frac{I_1}{n_s}}\left[1-\frac{\frac{I_1}{2n_s}}{1+\frac{I}{n_s}}\right]+2\chi I_1(I_2)^{1/2}\cos\psi ,\\
A_{I_2}&=-\kappa_2 I_2-\chi I_1(I_2)^{1/2}\cos\psi ,\\
A_\psi&=\frac{\chi}{2(I_2)^{1/2}}\left[I_1-4I_2+\frac{2I_2}{I_1}\right]\sin\psi ,\\
D_{I_1I_1}&=2I_1\left[\frac{\kappa_1 C}{\left[1+\frac{I_1}{n_s}\right]^2}+\chi(I_2)^{1/2}\cos\psi\right] ,\\
D_{I_1\psi}&=D_{\psi I_1}=-2\chi(I_2)^{1/2}\sin\psi ,\\
D_{\psi\psi}&=\frac{2}{I_1}\left[\frac{\kappa_1 C}{1+I_1/n_s}-\chi(I_2)^{1/2}\cos\psi\right] ,
\end{aligned} \tag{4.2}$$

where $\psi=2\phi_1-\phi_2$. The phases ϕ_1 and ϕ_2 separately have no steady state and play no role in the intensity fluctuations.

For $I_1 \gg 1$ the second term in A_{I_1} is small and may be neglected. The steady-state solutions may be obtained from the drift terms. Two regimes of operation may be distinguished. For pumping strengths such that

$$I_1 > \frac{\kappa_2^2}{4\chi^2} \equiv I_{\text{crit}} \ ,$$

the phase difference ψ becomes bistable. We shall concentrate our attention on the region below this transition, in which the difference phase is locked to π, and intensity and phase fluctuations decouple, making the calculation of the spectrum particularly easy. It should be pointed out that this critical point is not physical, but an artifact of the choice of variables. It is closely related to the fact that the phase distribution for a finite-amplitude squeezed state can be bimodal.[14]

We find below the transition

$$\cos\psi = -1, \quad I_2 = \left[\frac{X}{\kappa_2} I_1\right]^2 ,$$
$$\frac{I_1}{n_s} = \frac{1}{2X}\{[(X+1)^2 + 4X(C-1)]^{1/2} - (X+1)\} , \tag{4.3}$$

where $X = 2n_s\chi^2/\kappa_1\kappa_2$. We note that this equation has the same form as Eq. (2.11) for two-photon absorption since for the mean intensities the second-harmonic generation acts as an effective two-photon absorber. The fluctuations about the mean intensity will, however, be different for the two processes.

Note the diffusion coefficient

$$D_{I_1I_1} = \kappa_1 I_1 \left[\frac{2C}{\left[1+\frac{I_1}{n_s}\right]^2} - \frac{I_1 X}{n_s}\right] \tag{4.4}$$

can become negative for sufficiently large I_1/n_s (i.e., large C). Note that large X alone is not sufficient since for large X, $I_1/n_s \propto 1/X$.

The steady-state variance $V(I_1)$ may now be calculated

$$\frac{V(I_1)}{I_1} = \frac{D_{I_1I_1}}{\kappa_1 i I_1} \frac{\kappa_2 + 2\kappa_1 i \left[X + \frac{1+X_i}{i}\right]}{2\left[\kappa_2 + 2\kappa_1 i \frac{1+X_i}{1+i}\right]\left[X + \frac{1+Xi}{1+I}\right]}$$
$$= \frac{\left[2\frac{1+iX}{(1+i)} - iX\right]\left[i_0 X + iX + i\frac{1+Xi}{1+i}\right]}{2\left[i_0 X + i\frac{1+Xi}{1+i}\right]\left[iX + i\frac{1+Xi}{1+i}\right]} , \tag{4.5}$$

where $i = I_1/n_s$ and $i_0 = \kappa_2/2\kappa_1 X$ (critical value of i). The best sub-Poissonian variance in the photon number is for $i \approx i_0$, $i_0 X$ large

$$\frac{V(n_1)}{I_1} = 1 + \frac{V(I_1)}{I_1} \approx 1 - \frac{iX3iX}{2(2iX)(2iX)} = \frac{5}{8} . \tag{4.6}$$

The condition $i_0 X$ large implies $\kappa_2 \gg \kappa_1$, which means that most of the output is at the second harmonic. The variance in the photon number in the second-harmonic mode is

$$\frac{V(n_2)}{I_2} = 1 + \frac{V(I_2)}{I_2} = 1 + \frac{D_{I_1I_1}}{I_1} \frac{X}{\left[\kappa_2 + 2\kappa_1(1+Xi)\frac{i}{1+i}\right]\left[X + \frac{1+Xi}{1+i}\right]}$$
$$= 1 + \tfrac{1}{2} \frac{X\left[2\frac{1+iX}{(1+i)} - iX\right]}{\left[Xi_0 + (1+Xi)\frac{i}{1+i}\right]\left[X + \frac{1+Xi}{1+i}\right]} = \frac{7}{8} \tag{4.7}$$

for $i \approx i_0$ large.

Intensity fluctuations in the output field

We may also calculate the spectrum of fluctuations in the output field. The value of the normally ordered spectrum at $\omega = 0$ is, for the fundamental,

$$\frac{:S_{11}^{\text{out}}(0):}{I_1^{\text{out}}} = \frac{2(1+iX) - iX(1+i)}{i^2(1+X+2Xi)^2}(1+i) . \tag{4.8}$$

This again has the same form as for two-photon absorption [Eq. (3.11) with $\omega = 0$], with a minimum when $2iX = 1$ with $i \gg 1$. It has a small value when Xi is large since the width of the spectrum is $\approx \kappa_i(1+2Xi)$ in this limit. Thus the reduction in number fluctuations in the internal fundamental mode [Eq. (4.6)] is not achieved in the external field due to a power broadening.

At the second-harmonic frequency

$$\frac{:S_{22}^{\text{out}}(0):}{I_2^{\text{out}}} = \frac{2Xi :S_{11}^{\text{out}}(0):}{I_1^{\text{out}}}$$
$$\approx -\tfrac{1}{2} \tag{4.9}$$

when i and Xi are large. The intensity fluctuations are

$$S_{22}^{\text{out}}(0)=I_2^{\text{out}}\left[\frac{:S_{22}^{\text{out}}(0):}{I_2^{\text{out}}}\right]+1 = \tfrac{1}{2}I_2^{\text{out}} . \qquad (4.10)$$

In this case a reduction of fluctuations 50% below the shot-noise limit is achieved.

Two points may be made about this result. First, the reduction in output fluctuations is much greater than that in the internal variance: in the region in which the conditions $i, Xi \gg 1$ are satisfied, $V(n_2)$ has a minimum value of $\frac{7}{8}$, and may be only slightly less than 1. This results from the fact that the nonlinearity produces narrowing of the second-harmonic fluctuation spectrum. It is the opposite effect from that seen in the fundamental mode and in the two-photon absorber, where a broadening of the spectrum meant that appreciable reduction in the internal variance could give negligible reduction in output fluctuations. Physically, the narrow spectrum means that the full improvement in photon statistics is only seen over a long counting time.

Second, it is possible to give a very simple explanation of the origin of the 50% noise reduction, in terms of the rates of photon production and conversion. The condition that i is large implies that the laser is generating photons with Poisson statistics: if the rate of production is I_1^{tot}, then $V(I_1^{\text{tot}})=\langle I_1^{\text{tot}}\rangle$. The condition that Xi is large implies that nearly all of these photons are converted to the second harmonic, i.e.,

$$I_1^{\text{out}} \ll I_1^{\text{tot}}, \quad I_2^{\text{out}} \approx \tfrac{1}{2}I_1^{\text{tot}} .$$

So now the variance in the second-harmonic output is

$$V(I_2^{\text{out}})=(\tfrac{1}{2})^2 V(I_1^{\text{tot}})=\tfrac{1}{2}\langle I_2^{\text{out}}\rangle .$$

Exactly the same effect can be seen in a passive second-harmonic generator operated at the point of 100% conversion efficiency (in the notation of Ref. 7, this is when $|\epsilon_2|=\gamma_1$); that is, over a sufficiently long time, the output intensity fluctuations are 50% of Poisson, regardless of the internal statistics. We note that 50% amplitude squeezing is predicted in the signal and idler modes of a nondegenerate parametric oscillator operating well above threshold.[15,16]

V. CONCLUSIONS

We have considered intracavity nonlinear elements as ways of reducing the amplitude fluctuations in a laser. We have analyzed the cases of intracavity two-photon absorption and intracavity second-harmonic generation. For the two-photon absorber while 25% reduction in amplitude fluctuations may be achieved for the intracavity field, this occurs under different conditions from those necessary to maximize the noise reduction in the external field. To achieve maximum squeezing of the intracavity field requires the nonlinearity of the two-photon absorber to be large compared to the cavity loss. However, a large internal loss is deleterious to squeezing in the output field. Under optimum conditions the best amplitude noise reduction in the output field that may be achieved is ~12.5%.

The situation is little better for intracavity second-harmonic generation. While a noise reduction of ~37% may be achieved in the internal lasing (fundamental) mode, power broadening of the output spectrum reduces the squeezing to a negligible value in the output field.

In the second-harmonic mode, however, the spectrum is narrowed rather than broadened, so a maximum internal squeezing of 12.5% translates to a 50% noise reduction in the amplitude of the output field. This may have some important consequences since experiments demonstrating squeezing in a parametric oscillation have used intracavity second-harmonic generation to pump the oscillator.[17]

Note added. After this manuscript was submitted, we received copies of unpublished work by V. N. Gorbachev and E. S. Polzik, H. Ritsch, P. Garcia Fernandez, L. A. Lugiato, F. J. Bermejo, and P. Galatola on a similar topic. V. N. Gorbachev and E. S. Polzik have considered nth-harmonic generation inside a laser cavity and find that the nth harmonic is squeezed by a factor of n, as would be expected by the arguments given in Sec. IV. The results in our paper pertain to an incoherently pumped laser. Clearly, better amplitude noise reduction will result if a regularly pumped laser[10,18] is considered. However, this is largely independent of the effects of the nonlinearity since 100% amplitude squeezing is in principle attainable from a regularly pumped laser without any intracavity nonlinear element.

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