Interaction of an electron with a multimode quantized radiation field

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Photon operators that obey commutation relations form Lie algebras. By identifying the Hamiltonian of an interacting system of one electron and a multimode many-photon field in a momentum representation as an element of a Cartan subalgebra, we are able to find the automorphism that maps the Hamiltonian on to that of the noninteracting system. By this method we find the exact solutions as new solutions of the Schrödinger equation for the case of one electron in a multimode many-photon field. The solutions show the existence of both ponderomotive energy and momentum, which obey the rules of linear superposition among the photon modes.

I. INTRODUCTION

The interaction between electrons and photons is one of the most basic interactions in nature. Studying this interaction by a method leading to exact solutions is of importance for understanding many basic laws. In traditional quantum electrodynamics (QED) the electron-photon interaction is treated by perturbation theory in which the order of the expansion is equal to the number of photons involved in the transition. Even though the nonperturbative study of an electron interacting with an electromagnetic wave has a long history since Gordon's[1] and Volkov's[2] pioneering work, it has gained renewed importance[3–8] with the development of modern laser techniques for producing superintense light for experiments on multiphoton ionization (MPI) and other electron-phonon scattering processes. In MPI experiments there are millions or billions of photons in the background, while tens, hundreds, or thousands of photons are absorbed in ionizing the atom. It is formidable to calculate, say, 20-photon ionization by evaluating Feynman's graph of 20th order; even then the convergence of the perturbative expansion remains uncertain. Keldysh[3] applied the nonrelativistic Volkov solution for an electron in an electromagnetic plane wave as a final state to construct a transition-rate formula for MPI in a theory called the Keldysh-Faisal-Reiss (KFR) model.[3–5] Even though the KFR model caused some controversy,[9,10] it is attractive due to the nice feature that it expresses the MPI rate to any arbitrary order of photon numbers. The Volkov solution is time dependent, since the electromagnetic wave is treated as an external classical field. Thus, in the Volkov solution, the electrons is not a truly isolated system; hence, formal scattering theory cannot be applied directly. In our recent work[11,12] we obtained a set of exact solutions of the Dirac equation for a relativistic electron in a quantized, elliptically polarized monochromatic electromagnetic field as energy eigenstates of the Hamiltonian. Using this set of exact solutions, we applied formal scattering theory to treat MPI (Ref. 13) and found that the final state in the large-photon-number and nonrelativistic limits vanishes when the ponderomotive energy is not equal to the photon energy multiplied by an integer; this result is due to balancing the four-momentum of the interacting system. It indicates that the single-mode interaction may not be sufficient for describing an MPI process even when the incident laser beam is in single mode. Strong scattering effects due to the ponderomotive energy and momentum have to be considered together with multiphoton Compton scattering, where the extra modes with different light-cone directions are involved. On the other hand, performing experiments to study the interaction between an electron and a multimode radiation field is possible. Thus, exact solutions of the Dirac and Schrödinger equations for an electron in multimode photon fields would have immediate application.

In a recent work[14] we have solved the Schrödinger equation for a nonrelativistic electron interacting with a single-mode photon field, with solutions that can be considered exact. We considered the ponderomotive momentum as well as ponderomotive energy. In the present paper we developed a Lie-algebra method to obtain exact solutions of the Schrödinger equation:

$$\left\{\frac{(-i\nabla)^2}{2m_e}-\frac{e}{2m_e}[(-i\nabla)\cdot\mathbf{A}^{(m)}+\mathbf{A}^{(m)}\cdot(-i\nabla)]+\frac{e^2(\mathbf{A}^{(m)})^2}{2m_e}+\sum_{i=1}^{m}\omega_i N_{a_i}\right\}\Psi(\mathbf{r})=\mathcal{E}\Psi(\mathbf{r}) , \quad (1)$$

where

$$\begin{aligned}\mathbf{A}^{(m)}&=\sum_{i=1}^{m}\mathbf{A}_i(-\mathbf{k}_i\cdot\mathbf{r})\\&=\sum_{i=1}^{m}g_i(\boldsymbol{\epsilon}_i e^{i\mathbf{k}_i\cdot\mathbf{r}}a_i+\boldsymbol{\epsilon}_i^* e^{-i\mathbf{k}_i\cdot\mathbf{r}}a_i^\dagger) ,\\ g_i&=(2V_\gamma\omega_i)^{-1/2} ,\\ N_{a_i}&=\tfrac{1}{2}(a_ia_i^\dagger+a_i^\dagger a_i) \quad (i=1,2,\ldots,m)\end{aligned} \quad (2)$$

for a nonrelativistic electron interacting with a multimode photon field. The solutions are applicable to small-cavity and few-photon cases with strong coupling as well as to cases with large normalization volume and large photon number in a strong radiation field.

In Volkov's pioneering work,[2] relativistic classical time-dependent wave functions were obtained for an electron in a multimode electromagnetic field in which all modes propagate in the same direction and exhibit the same linear polarization. The present solutions differ from earlier work in four respects: (1) The modes can propagate in arbitrarily different directions. (2) All modes can have arbitrarily different elliptical polarizations. (3) The present solutions are quantum-field solutions for the electromagnetic field, thus making it possible to describe absorption and emission processes with definite transferred-photon numbers. They also enable us to treat the electron and photons as an isolated system, so that the wave functions for the electron and photons are energy eigenfunctions of the Hamiltonian with a definite energy. (4) The present solutions are nonrelativistic for the electron, but relativistic for the photons. This feature is particularly advantageous for treating strong radiation fields, in contrast to earlier nonrelativistic semiclassical approaches that are mostly in dipole approximation or in long-wave approximation[3-5] where the light-cone directions are deformed, limiting them to the cases where the radiation fields are not highly intense.

This paper deals with both fundamental theory and practical calculation. Readers who only care about practical calculation can skip the descriptions of Lie algebra and directly follow the solving steps which are usually the calculations in linear algebra. In the description of Lie algebra, we try to include necessary basic definitions and concepts in the context so that readers unfamiliar with Lie algebra can follow without much difficulty.

II. LIE ALGEBRA STRUCTURE FOR PHOTON OPERATORS

Photon operators form a Lie algebra known as Heisenberg algebra. Generally, a Lie algebra $\mathscr{g}$ is a finite-dimensional linear space, which satisfies[15,16]

$$\begin{aligned} &[x_i,x_j]=-[x_j,x_i]\in\mathscr{g} ,\\ &[x_i,[x_j,x_k]]+[x_j,[x_k,x_i]]+[x_k,[x_i,x_j]]=0 , \end{aligned} \tag{3}$$

for any $x_i,x_j,x_k\in\mathscr{g}$. Alternatively, it can be defined by using structure constants C_{ij}^k,

$$\begin{aligned} &[x_i,x_j]=C_{ij}^k x_k ,\\ &C_{ij}^k=-C_{ji}^k ,\\ &C_{ij}^r C_{rk}^s+C_{jk}^r C_{ri}^s+C_{ki}^r C_{rj}^s=0 , \end{aligned} \tag{4}$$

where a sum is implied by the repeated indices.

It is easy to verify that in the single-mode-photon case, the creation operator $a^\dagger$, the annihilation operator a, and the identity operator I, which satisfies

$$[a,a^\dagger]=I \tag{5}$$

and

$$[a,a]=[a^\dagger,a^\dagger]=[I,a]=[I,a^\dagger]=0 , \tag{6}$$

form a Lie algebra.

We introduce the following notation that expresses general properties of a Lie algebra:

$$\begin{aligned} &h_0=I ,\\ &e_{-1}=a ,\\ &e_{+1}=a^\dagger . \end{aligned} \tag{7}$$

They form a Lie algebra $\mathscr{g}_0$ with commutation relations

$$\begin{aligned} &[h_0,e_{-1}]=0 ,\\ &[h_0,e_{+1}]=0 ,\\ &[e_{-1},e_{+1}]=h_0 . \end{aligned} \tag{8}$$

We write

$$\mathscr{g}_0=\{h_0,e_{-1},e_{+1}\} , \tag{9}$$

which means that the Lie algebra $\mathscr{g}_0$ is made up of h_0,e_{-1},e_{+1} and their linear combinations.

The algebra $\mathscr{g}_0$ has few direct extensions. The most immediate one is to include the number operator N_a,

$$h_1=N_a=\tfrac{1}{2}(aa^\dagger+a^\dagger a) , \tag{10}$$

to make a Lie algebra $\mathscr{g}_1$,

$$\mathscr{g}_1=\{h_0,h_1,e_{-1},e_{+1}\} . \tag{11}$$

The additional commutation relations are

$$\begin{aligned} &[h_0,h_1]=0 ,\\ &[h_1,e_{-1}]=-e_{-1} ,\\ &[h_1,e_{+1}]=e_{+1} . \end{aligned} \tag{12}$$

Thus we see that

$$\mathscr{h}=\{h_0,h_1\} \tag{13}$$

is a subalgebra of $\mathscr{g}_1$, i.e., it is a subset of the Lie algebra $\mathscr{g}_1$ and a Lie algebra itself.

A Cartan subalgebra $\mathscr{h}$ of the Lie algebra $\mathscr{g}$ is a subalgebra of $\mathscr{g}$, whose element h in the eigenvalue equation

$$[h,x]=\rho x \quad (x\in\mathscr{g}) \tag{14}$$

has a maximum number of different eigenelements x. As an example, we see that $\mathscr{h}=\{h_0,h_1\}$ is a Cartan subalgebra of $\mathscr{g}_1$.

The next extension is Lie algebra $\mathscr{g}_2$, which includes all quadratic photon operators. We define

$$\begin{aligned} &e_{-2}=a^2 ,\\ &e_{+2}=a^{\dagger 2} , \end{aligned} \tag{15}$$

and

$$\mathscr{g}_2=\{h_0,h_1,e_{-1},e_{+1},e_{-2},e_{+2}\} . \tag{16}$$

The additional commutation relations are

$$[h_0,e_{-2}]=[h_0,e_{+2}]=[e_{-1},e_{-2}]=[e_{+1},e_{+2}]=0\ ,$$
$$[h_1,e_{-2}]=-2e_{-2}\ ,$$
$$[h_1,e_{+2}]=2e_{+2}\ ,$$
$$[e_{-1},e_{+2}]=2e_{+1}\ , \tag{17}$$
$$[e_{+1},e_{-2}]=-2e_{-1}\ ,$$
$$[e_{-2},e_{+2}]=4h_1+2h_0\ .$$

Here we can see that $\mathscr{h}$ is still a Cartan subalgebra of $\mathscr{g}_2$.

We say that $\mathscr{g}_0$ is an ideal of $\mathscr{g}_2$, which means that $\mathscr{g}_0$ is a subalgebra of $\mathscr{g}_2$, such that

$$[\mathscr{g}_2,\mathscr{g}_0]\subset\mathscr{g}_0\ . \tag{18}$$

According to this relation, we can define a quotient algebra of $\mathscr{g}_2$ by $\mathscr{g}_0$,

$$\mathscr{g}_2'=\mathscr{g}_2/\mathscr{g}_0\ . \tag{19}$$

The mapping $x\ (\in g_2)\rightarrow\bar{x}\ (\in\mathscr{g}_2')$, where $\bar{x}$ denotes the residue class, mod$\mathscr{g}_0$, is a mapping of $\mathscr{g}_2$ onto $\mathscr{g}_2'$, a natural homomorphism. The set $\mathscr{g}_2'$ is also a Lie algebra, and the homomorphic image of $\mathscr{g}_2$ under the natural homomorphism. The kernel of this homomorphism is $\mathscr{g}_0$. Thus, we have

$$\mathscr{g}_2'=\{h_1,e_{-2},e_{+2}\}\ , \tag{20}$$

where the notation used for the elements corresponds to the elements under the natural homomorphism. The elements of $\mathscr{g}_2'$ satisfy the following commutation relations:

$$[h_1,e_{-2}]=-2e_{-2}\ ,$$
$$[h_1,e_{+2}]=2e_{+2}\ , \tag{21}$$
$$[e_{-2},e_{+2}]=4h_1\ ,$$

which show that the subalgebra $\mathscr{h}'=\{h_1\}$ is also a Cartan subalgebra of $\mathscr{g}_2$

Now $\mathscr{g}_2'$ is a simple Lie algebra, which means that there is no ideal in $\mathscr{g}_2'$ other than $\{0\}$ and $\mathscr{g}_2'$ itself.

All the above Lie algebra properties can be easily generalized into multimode photon field cases. If, on the Lie-algebra notation, we use the superscripts $(i),(j)$ to denote the photons in different modes, and m to denote the number of the modes, then we have

$$\mathscr{g}_0=\{h_0,e_{-1}^{(i)},e_{+1}^{(i)};i=1,2,\ldots,m\}\ , \tag{22}$$

where

$$e_{-1}^{(i)}=a_i\ ,$$
$$e_{+1}^{(i)}=a_i^\dagger\ . \tag{23}$$

The commutation relations for the elements are

$$[e_{-1}^{(i)},e_{+1}^{(j)}]=[a_i,a_j^\dagger]=\delta_{ij}h_0\ ,$$
$$[e_{-1}^{(i)},e_{-1}^{(j)}]=[e_{+1}^{(i)},e_{+1}^{(j)}]=[h_0,e_{-1}^{(i)}] \tag{24}$$
$$=[h_0,e_{+1}^{(i)}]=0$$
$$(i,j=1,2,\ldots,m)\ .$$

The algebra $\mathscr{g}_0$ can be extended to Lie algebra $\mathscr{g}_1$ by including the number operators N_{a_i}, $i=1,2,\ldots,m$. We define

$$h_1^{(i)}=\tfrac{1}{2}(a_ia_i^\dagger+a_i^\dagger a_i)\quad(i=1,2,\ldots,m) \tag{25}$$

and

$$\mathscr{g}_1=\{h_0,h_1^{(i)},e_{-1}^{(i)},e_+1^{(i)};\quad i=1,2,\ldots,m\}\ . \tag{26}$$

The additional nonvanishing commutation relations are

$$[h_1^{(i)},e_{-1}^{(i)}]=-e_{-1}^{(i)}\ ,$$
$$[h_1^{(i)},e_{+1}^{(i)}]=e_{+1}^{(i)}\quad(i=1,2,\ldots,m)\ . \tag{27}$$

A further extension can be made by including all the quadratic operators. By defining

$$e_{-2}^{(i)}=a_ia_i\ ,$$
$$e_{+2}^{(i)}=a_i^\dagger a_i^\dagger\ ,$$
$$e_{-1,-1}^{(i,j)}=a_ia_j\ ,$$
$$e_{+1,+1}^{(i,j)}=a_i^\dagger a_j^\dagger\ , \tag{28}$$
$$e_{-1,+1}^{(i,j)}=a_ia_j^\dagger\ ,$$
$$e_{+1,-1}^{(i,j)}=a_i^\dagger a_j\quad(i,j=1,2,\ldots,m;\ i<j)$$

we have Lie algebra $\mathscr{g}_2$,

$$\mathscr{g}_2=\{h_0,h_1^{(i)},e_{-1}^{(i)},e_{+1}^{(i)},e_{-2}^{(i)},e_{+2}^{(i)},e_{-1,-1}^{(i,j)},e_{+1,+1}^{(i,j)},e_{-1,+1}^{(i,j)},e_{+1,-1}^{(i,j)}\quad i,j=1,2,\ldots,m;\ i<j\}\ . \tag{29}$$

The additional commutation relations can easily be determined by those for $\mathscr{g}_1$, so we do not write them out here.

A trivial Cartan subalgebra for both $\mathscr{g}_1$ and $\mathscr{g}_2$ is

$$\mathscr{h}=\{h_0,h_1^{(i)};\ i=1,2,\ldots,m\}\ . \tag{30}$$

We can define the quotient algebra $\mathscr{g}_2'$,

$$\mathscr{g}_2'=\mathscr{g}_2/\mathscr{g}_0\ , \tag{31}$$

which is a simple algebra, expressed by

$$\mathscr{g}_2'=\{h_1^{(i)},e_{-2}^{(i)},e_{+2}^{(i)},e_{-1,-1}^{(i,j)},e_{+1,+1}^{(i,j)},e_{-1,+1}^{(i,j)},e_{+1,-1}^{(i,j)};\quad i,j=1,2,\ldots,m;\ i<j\}\ . \tag{32}$$

Thus, $\mathscr{h}'$ is a Cartan subalgebra of $\mathscr{g}_2'$, expressed by

$$\mathscr{h}'=\{h_1^{(i)};\ i=1,2,\ldots,m\}\ . \tag{33}$$

III. ELECTRON INTERACTING WITH A SINGLE-MODE PHOTON FIELD

In this paper we consider only a nonrelativistic electron moving in a multimode radiation field. The simplest case is that of an electron in a single-mode photon field. The general features of the Lie-algebra structure of photon operators in the single-mode case and in multimode case are not qualitatively different. By taking advantage of this fact, the Lie-algebra method that works for solving the Schrödinger equation in the single-mode case can be generalized directly for multimode cases. In this section, our intention is to show the details of the Lie-algebra method to reobtain the exact solution for nonrelativistic electron in a single-mode radiation field, which has been obtained elsewhere[14] by using a more intuitive method.[11] In the following sections we apply directly the method developed in this section to obtain multimode solutions without too much further analysis.

The key idea of the Lie-algebra method for solving the Schrödinger equation is based on the fact that the Hamiltonian H' of the interacting system of an electron with photons in a momentum representation is an element of the photon Lie algebra. In the noninteracting system H' is an element of the Cartan subalgebra $\mathcal{h}=\{N_a,I\}$, since H' only contains free energies and zero-point energies of photons and the electron. If we can find an automorphism, such that H' can be identified as an element of a Cartan subalgebra $\mathcal{h}'=\{N_c,I\}$, then we can solve the Schrödinger equation just as in the noninteracting case.

The Hamiltonian for a system consisting of a nonrelativistic electron and many photons in an elliptically polarized single-mode field interacting with one another can be written as

$$H=\frac{(-i\nabla)^2}{2m_e} -\frac{e}{2m_e}[(-i\nabla)\cdot\mathbf{A}(-\mathbf{k}\cdot\mathbf{r})+\mathbf{A}(-\mathbf{k}\cdot\mathbf{r})\cdot(-i\nabla)] +\frac{e^2\mathbf{A}^2(-\mathbf{k}\cdot\mathbf{r})}{2m_e}+\omega N_a\ , \tag{34}$$

where

$$\mathbf{A}(-\mathbf{k}\cdot\mathbf{r})=g(\epsilon e^{i\mathbf{k}\cdot\mathbf{r}}a+\epsilon^* e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger)\ ,\quad g=(2V_\gamma\omega)^{-1/2}\ . \tag{35}$$

Here V_γ is the normalization volume for the photon field, and N_a is the same with the operator defined in Eq. (10). We use natural units throughout, where $\hbar=c=1$.

The polarization vectors for the photon fields are

$$\epsilon=[\epsilon_x\cos(\xi/2)+i\epsilon_y\sin(\xi/2)]e^{i\Theta/2}\ ,\quad \epsilon^*=[\epsilon_x\cos(\xi/2)-i\epsilon_y\sin(\xi/2)]e^{i\Theta/2}\ ; \tag{36}$$

they satisfy

$$\epsilon\cdot\epsilon^*=1\ ,\quad \epsilon\cdot\epsilon=\cos\xi e^{i\Theta}\ ,\quad \epsilon^*\cdot\epsilon^*=\cos\xi e^{-i\Theta}\ . \tag{37}$$

We intend to find eigenvalues and the corresponding eigenfunctions of the Hamiltonian, i.e., to solve the equation

$$H\Psi(\mathbf{r})=\mathcal{E}\Psi(\mathbf{r})\ . \tag{38}$$

By making a transformation

$$\Psi(\mathbf{r})=e^{i(\mathbf{p}-\mathbf{k}N_a)\cdot\mathbf{r}}\phi\ , \tag{39}$$

we obtain a coordinate-independent equation,

$$\left[\frac{1}{2m_e}(\mathbf{p}-\mathbf{k}N_a)^2-\frac{e}{2m_e}[(\mathbf{p}-\mathbf{k}N_a)\cdot\mathbf{A}+\mathbf{A}\cdot(\mathbf{p}-\mathbf{k}N_a)] +\frac{e^2\mathbf{A}^2}{2m_e}+\omega N_a\right]\phi=\mathcal{E}\phi\ , \tag{40}$$

where

$$\mathbf{A}=g(\epsilon a+\epsilon^* a^\dagger)\ . \tag{41}$$

The difficulty in solving this equation is due to the square term N_a, which places the Hamiltonian out of the Lie algebra discussed in Sec. II. If we define that a solution of a relativistic equation should be an equivalent class, up to any higher-order relativistic effect, then we can make the following ansatz to simplify the equation, but after we solve the equation, we must check to see if the ansatz is really valid.

Ansatz. There is a real number κ, such that $(\kappa\mathbf{k}-N_a\mathbf{k})$ belongs to higher-order relativistic effects in the Schrödinger equation. This ansatz allows us to replace the operator $\mathbf{k}N_a$ in the equation by $\kappa\mathbf{k}$. The $\kappa\mathbf{k}$ will be determined later.

We introduce a vector $\mathbf{P}$, such that

$$\mathbf{p}=\mathbf{P}+\kappa\mathbf{k}\ . \tag{42}$$

In this notation the Schrödinger equation is simply reduced to

$$\left[\frac{\mathbf{P}^2}{2m_e}-\frac{e\mathbf{P}\cdot\mathbf{A}}{m_e}+\frac{e^2\mathbf{A}^2}{2m_e}+\omega N_a\right]\phi=\mathcal{E}\phi\ . \tag{43}$$

If we define

$$H'=\left[\frac{\mathbf{P}^2}{2m_e}-\frac{e\mathbf{P}\cdot\mathbf{A}}{m_e}+\frac{e^2\mathbf{A}^2}{2m_e}+\omega N_a\right]\ , \tag{44}$$

which can be regarded as the Hamiltonian in a momentum representation, Eq. (43) becomes

$$H'\phi=\mathcal{E}\phi\ . \tag{45}$$

Our task is finding all the eigenstates of H'.

By noticing that H' is an element of the Lie algebra $\mathcal{g}_2$, we may want to use Lie-algebraic relations in searching for the eigenstates of H'. Instead of solving the eigenstate equation (45) in an infinite-dimensional linear space, we may just solve a Lie-algebraic eigenelement equation in a finite-dimensional linear space,

$$[H',e']=\rho e'\ , \tag{46}$$

and solve for only one state $|0\rangle'$ with the lowest energy $\mathcal{E}_0$,

$$H'|0\rangle'=\mathcal{E}_0|0\rangle' . \quad (47)$$

Subsequently, the whole sequence of eigenstates can be generated by e' with positive eigenvalue ρ from the "vacuum" state $|0\rangle'$. H' can be recognized as an element of the Cartan subalgebra $\mathscr{h}'=\{h'_0,h'_1\}$ after solving Eq. (46). The component form of Eq. (46) is just an eigenvector equation in the linear space of the Lie algebra $\mathscr{g}_2$,

$$\sum_j \left[\sum_i H'^{(i)}C_{ij}^k-\rho\delta_j^k\right]e'^{(j)}=0 , \quad (48)$$

where $e'^{(j)}$ are the components for e'. We may just consider e' in the ideal $\mathscr{g}_0$ first, by setting $j=0,-1,+1$ in Eq. (48).

Equations (48) or (46) can be solved by the following steps: First, we write H' in the form of an element of the Lie algebra $\mathscr{g}_2$,

$$\begin{aligned} H'= & A_0h_0+A_{-1}e_{-1}+A_{+1}e_{+1}+2A_1h_1 \\ & +A_{-2}e_{-2}+A_{+2}e_{+2} , \end{aligned} \quad (49)$$

where

$$\begin{aligned} A_0 &= \frac{\mathbf{P}^2}{2m_e} , \\ A_{-1} &= -\frac{eg}{m_e}\mathbf{P}\cdot\epsilon , \\ A_{+1} &= -\frac{eg}{m_e}\mathbf{P}\cdot\epsilon^* , \\ A_1 &= \frac{\omega}{2}+\frac{e^2g^2}{2m_e} , \\ A_{-2} &= \frac{e^2g^2}{2m_e}\cos\xi e^{i\Theta} , \\ A_{+2} &= \frac{e^2g^2}{2m_e}\cos\xi e^{-i\Theta} . \end{aligned} \quad (50)$$

Second, we calculate the commutators of H' with h_0,e_-,e_+,

$$\begin{aligned} &[H',h_0]=0 , \\ &[H',e_{-1}]=-A_{+1}h_0-2A_1e_{-1}-2A_{+2}e_{+1} , \\ &[H',e_{+1}]=A_{-1}h_0+2A_{-2}e_{-1}+2A_1e_{+1} . \end{aligned} \quad (51)$$

Third, we transpose the coefficient matrix of equation set (51) and use the transposed matrix to form an eigenvector equation,

$$\begin{bmatrix} -\rho & -A_{+1} & A_{-1} \\ 0 & -2A_1-\rho & 2A_{-2} \\ 0 & -2A_{+2} & 2A_1-\rho \end{bmatrix} \begin{bmatrix} e'^{(0)} \\ e'^{(-1)} \\ e'^{(+1)} \end{bmatrix}=0 . \quad (52)$$

Three eigenvalues obtained from the equation are

$$\begin{aligned} &\rho_0=0 , \\ &\rho_{-1}=-2\left[A_1^2-|A_{+2}|^2\right]^{1/2} , \\ &\rho_{+1}=2\left[A_1^2-|A_{+2}|^2\right]^{1/2} . \end{aligned} \quad (53)$$

By choosing proper normalization constants, the three corresponding eigenvectors are

$$\begin{aligned} h'_0 &= h_0 , \\ e'_{-1} &= [(\sinh\chi)e^{-i\Theta}B_{-1}+(\cosh\chi)B_{+1}]h_0 \\ &\quad +(\cosh\chi)e_{-1}-(\sinh\chi)e^{-i\Theta}e_{+1} , \\ e'_{+1} &= [(\sinh\chi)e^{i\Theta}B_{+1}+(\cosh\chi)B_{-1}]h_0 \\ &\quad -(\sinh\chi)e^{i\Theta}e_{-1}+(\cosh\chi)e_{+1} , \end{aligned} \quad (54)$$

where

$$\begin{aligned} B_{-1} &= \frac{A_{-1}}{\rho_{+1}} , \\ B_{+1} &= \frac{A_{+1}}{\rho_{+1}} , \end{aligned} \quad (55)$$

and

$$\begin{aligned} \cosh2\chi &= \frac{2A_1}{\rho_{+1}} , \\ \sinh2\chi &= -\frac{2A_{-2}e^{-i\Theta}}{\rho_{+1}} . \end{aligned} \quad (56)$$

The normalization constants are chosen such that

$$\begin{aligned} &[e'_{-1},e'_{+1}]=h'_0=h_0 , \\ &e'_{+1}=e'^{\dagger}_{-1} . \end{aligned} \quad (57)$$

So far, we have obtained three eigenelements of H' in $\mathscr{g}_0$. The others can be found by a constructive method without solving Eqs. (46) and (48) directly. We construct

$$\begin{aligned} h'_1 &= \tfrac{1}{2}(e'_{-1}e'_{+1}+e'_{+1}e'_{-1}) , \\ e'_{-2} &= e'_{-1}e'_{-1} , \\ e'_{+2} &= e'_{+1}e'_{+1} . \end{aligned} \quad (58)$$

One can see by the definitions in Eq. (58) that

$$\begin{aligned} &[H',h'_1]=0 , \\ &[H',e'_{-2}]=-2\rho_{+1}e'_{-2} , \\ &[H',e'_{+2}]=2\rho_{+1}e'_{+2} . \end{aligned} \quad (59)$$

Thus, we have Lie algebra $\mathscr{g}'_2$,

$$\mathscr{g}'_2=\{h_0,h'_1,e'_{-1},e'_{+1},e'_{-2},e'_{+2}\} , \quad (60)$$

which is an automorphic image of $\mathscr{g}_2$. All the corresponding commutation relations of $\mathscr{g}_2$ shown in Eqs. (12) and (17) can easily be verified for $\mathscr{g}'_2$.

In $\mathscr{g}_2$, H' is an element of a Cartan subalgebra $\mathscr{h}'$, which can be seen from Eqs. (59). Thus we can express H' as

$$H'=2A'_1h'_1+A'_0h'_0 , \quad (61)$$

where the coefficient A'_1 is determined by

$$\begin{aligned} &[h'_1,e'_{-1}]=-e'_{-1} , \\ &[h'_1,e'_{+1}]=e'_{+1} , \end{aligned} \quad (62)$$

and Eq. (59), while the coefficient A'_0 is determined by substituting Eq. (54) into (61) and comparing with (49). Thus we have

$$A'_1=\frac{\rho_{+1}}{2} ,\quad A'_0=A_0-|(\sinh\chi)e^{-i\Theta}B_{-1}+(\cosh\chi)B_{+1}|^2\rho_{+1} . \tag{63}$$

Now we return to the real physical system by introducing the following notation, which reflects the physical meaning better:

$$\begin{aligned} I&=h_0 ,\\ N_b&=h'_1=\tfrac{1}{2}(bb^\dagger+b^\dagger b) ,\\ b&=e'_{-1} ,\\ b^\dagger&=e'_{+1} ,\\ b^2&=e'_{-2} ,\\ b^{\dagger 2}&=e'_{+2} . \end{aligned} \tag{64}$$

The vacuum state for the b-photon system $|0\rangle_b$ is defined by the equation

$$b|0\rangle_b=0 . \tag{65}$$

From the expression for the operator b, Eq. (54), we see that it is not easy to express $|0\rangle_b$ in terms of a-photon-number states, since $\langle m|0\rangle_b$ is coupled with $\langle m-1|0\rangle_b$ and $\langle m+1|0\rangle_b$. A more convenient way is to decompose the transformation (54) into two steps. We define the following displacement operator;

$$D=\exp(-\delta b^\dagger+\delta^* b) , \tag{66}$$

by choosing

$$\delta=-[(\sinh\chi)e^{-i\Theta}B_{-1}+(\cosh\chi)B_{+1}] , \tag{67}$$

which is unitary and shifts b and $b^\dagger$ to c and $c^\dagger$, respectively:

$$\begin{aligned} c&=DbD^\dagger=b+\delta I=(\cosh\chi)a-(\sinh\chi)e^{-i\Theta}a^\dagger ,\\ c^\dagger&=Db^\dagger D^\dagger=b^\dagger+\delta^* I=(\cosh\chi)a^\dagger-(\sinh\chi)e^{i\Theta}a , \end{aligned} \tag{68}$$

which is a type of transformation of squeezed light.[17]

The vacuum state for the c-photon system $|0\rangle_c$ can be solved easily from the equation

$$c|0\rangle_c=[(\cosh\chi)a-(\sinh\chi)e^{-i\Theta}a^\dagger]|0\rangle_c=0 . \tag{69}$$

It turns out that

$$|0\rangle_c=(\cosh\chi)^{-1/2}\sum_{s=0}^{\infty}(\tanh\chi)^s\left[\frac{(2s-1)!!}{(2s)!!}\right]^{1/2}\times e^{-is\Theta}|2s\rangle , \tag{70}$$

with definition $(-1)!!\equiv 1$, where the notation $|n\rangle$ means a number state in the a-photon representation. Thus, the eigenstate ϕ of H' is found to be

$$\phi=\frac{b^{\dagger n}}{\sqrt{n!}}|0\rangle_b \tag{71}$$

or

$$\phi=D^\dagger|n\rangle_c=D^\dagger\frac{c^{\dagger n}}{\sqrt{n!}}|0\rangle_c , \tag{72}$$

which is a coherent state.

The energy eigenvalues can be evaluated by using Eqs. (61) and (63),

$$\mathcal{E}=\frac{\mathbf{P}^2}{2m_e}+\frac{C(n+\frac{1}{2})}{m_e}-\frac{e^2g^2(\mathbf{P}\cdot\boldsymbol{\epsilon}_c)(\mathbf{P}\cdot\boldsymbol{\epsilon}_c^*)}{m_eC} , \tag{73}$$

where

$$C=[(m_e\omega+e^2g^2)^2-e^4g^4\cos^2\xi]^{1/2} \tag{74}$$

and

$$\begin{aligned} \boldsymbol{\epsilon}_c&=(\cosh\chi)\boldsymbol{\epsilon}+(\sinh\chi)e^{i\Theta}\boldsymbol{\epsilon}^* ,\\ \boldsymbol{\epsilon}_c^*&=(\cosh\chi)\boldsymbol{\epsilon}^*+(\sinh\chi)e^{-i\Theta}\boldsymbol{\epsilon} . \end{aligned} \tag{75}$$

Finally, we have the wave function $\Psi(\mathbf{r})$ as an eigenfunction of the original Hamiltonian H,

$$\Psi(\mathbf{r})=V_e^{-1/2}\exp[i(-\mathbf{k}N_a+\mathbf{P}+\kappa\mathbf{k})\cdot\mathbf{r}]D^\dagger|n\rangle_c , \tag{76}$$

where V_e is the normalization constant for this wave function and κ is a constant determined as follows.

We can define an energy term for the nonrelativistic electron,

$$E=\frac{\mathbf{P}^2}{2m_e} , \tag{77}$$

which makes the four-momentum $(E+m_e,\mathbf{P})$ on the electron-mass shell. By this definition, the energy eigenvalue shown in Eq. (73) can be written as

$$\mathcal{E}+m_e=(E+m_e)+\kappa'\omega , \tag{78}$$

where the constant κ' is defined by

$$\kappa'=\frac{C(n+\frac{1}{2})}{m_e\omega}-\frac{e^2g^2(\mathbf{P}\cdot\boldsymbol{\epsilon}_c)(\mathbf{P}\cdot\boldsymbol{\epsilon}_c^*)}{Cm_e\omega} . \tag{79}$$

Comparing Eq. (78) with Eq. (42), the four-coordinate covariance and the light-cone invariance requires

$$\kappa=\kappa'=\frac{C(n+\frac{1}{2})}{m_e\omega}-\frac{e^2g^2(\mathbf{P}\cdot\boldsymbol{\epsilon}_c)(\mathbf{P}\cdot\boldsymbol{\epsilon}_c^*)}{Cm_e\omega} . \tag{80}$$

Here we can see that the nonrelativistic property of an electron only means $v\ll c$, but we still can keep the photon relativistic. This is a great advantage for the quantum nonrelativistic solution compared with the solution in classical fields, where one does not know clearly which part of the wave function belongs to the photon field, on which one should not make an approximation in the nonrelativistic version of the solution in cases where photon fields are strong.

A very important parameter Z_n can be defined as

$$Z_n=\kappa-(n+\tfrac{1}{2}) , \tag{81}$$

with the interpretation that $Z_n\omega$ is the ponderomotive potential energy.

The validity of the ansatz can be checked in the version of the wave function in the a-photon representation,

$$\Psi(\mathbf{r})=V_e^{-1/2}\sum_{j\,(\geq -n)} e^{i[\mathbf{P}+(Z_n-j)\mathbf{k}]\cdot\mathbf{r}}|n+j\rangle \mathcal{J}_j^*(n) \times e^{-ij(\phi_\xi+\Theta/2)} , \quad (82)$$

where

$$\mathcal{J}_j^*(n)e^{-ij(\phi_\xi+\Theta/2)}=\langle n+j|D^\dagger|n\rangle_c . \quad (83)$$

From Eqs. (42) and (81), we have

$$\mathbf{p}=\mathbf{P}+(n+\tfrac{1}{2})\mathbf{k}+Z_n\mathbf{k} . \quad (84)$$

By comparing Eqs. (76) and (82), we know that when $N_a\mathbf{k}$ acts on $|n+j\rangle$, it produces $(n+j+\frac{1}{2})\mathbf{k}$ in each term of the sum. Thus we have

$$\mathbf{p}-N_a\mathbf{k}\approx\mathbf{P}+(Z_n-j)\mathbf{k} . \quad (85)$$

Z_n is a small number compared with the background photon number n, and j is the transferred-photon number. The terms contributing significantly in transitions are only those with $j \ll n$. The main part of $N_a\mathbf{k}$, $n\mathbf{k}$, has been taken care of by the ansatz; the remaining part of $N_a\mathbf{k}$ contributes insignificantly, which can be seen by noticing the inequality

$$|(Z_n-j)\omega| \ll m_e . \quad (86)$$

If we simply replace $\mathbf{p}-N_a\mathbf{k}$ by $\mathbf{p}$ in Eq. (40), there will be no difference between the total momentum $\mathbf{p}$ and the on-mass-shell momentum $\mathbf{P}$, and the ponderomotive energy and momentum will not show up in the solution. This can be true only in a weak radiation field.

This solution agrees with the nonrelativistic limit of our relativistic solution[11,13] and also can be obtained by other methods with the same ansatz.[14] Since the other methods are more specific for the single-mode case and so far we have no way to generalize them for multimode cases, we are forced to try the Lie-algebra method, which may not be the simplest one in the single-mode case, but has the generality allowing it to be applied readily to multimode cases. In large-photon-number limit the single-mode solution agrees with earlier results.[13]

It is easy to see that the operators c and $c^\dagger$ satisfy the equations

$$[\bar{H}',c]=-\rho_{+1}c , \quad [\bar{H}',c^\dagger]=\rho_{+1}c^\dagger , \quad (87)$$

where $\bar{H}'$ is the truncated form of H' obtained by keeping only quadratic terms of a and $a^\dagger$, which means that to solve c-photon operators just in $\mathcal{G}_2'$ is much simpler than that in $\mathcal{G}_2$. In the next section we shall solve for c-photon operators first, then carry out the displacement operation to get the more complicated solutions for the two-mode case.

IV. ELECTRON INTERACTING WITH A TWO-MODE PHOTON FIELD

The case which we consider in this section is of one electron moving in a two-mode photon field. The Hamiltonian for an electron and a two-mode photon field with arbitrary and different polarizations and propagating directions is

$$H=\frac{(-i\nabla)^2}{2m_e}-\frac{e}{2m_e}\{(-i\nabla)\cdot[\mathbf{A}_1(-\mathbf{k}_1\cdot\mathbf{r})+\mathbf{A}_2(-\mathbf{k}_2\cdot\mathbf{r})]+[\mathbf{A}_1(-\mathbf{k}_1\cdot\mathbf{r})+\mathbf{A}_2(-\mathbf{k}_2\cdot\mathbf{r})]\cdot(-i\nabla)\} + \frac{e^2}{2m_e}[\mathbf{A}_1(-\mathbf{k}_1\cdot\mathbf{r})+\mathbf{A}_2(-\mathbf{k}_2\cdot\mathbf{r})]^2+\omega_1N_{a_1}+\omega_2N_{a_2} , \quad (88)$$

where

$$\begin{aligned}
&\mathbf{A}_1(-\mathbf{k}_1\cdot\mathbf{r})=g_1(\epsilon_1e^{i\mathbf{k}_1\cdot\mathbf{r}}a_1+\epsilon_1^*e^{-i\mathbf{k}_1\cdot\mathbf{r}}a_1^\dagger) ,\\
&\mathbf{A}_2(-\mathbf{k}_2\cdot\mathbf{r})=g_2(\epsilon_2e^{i\mathbf{k}_2\cdot\mathbf{r}}a_2+\epsilon_2^*e^{-i\mathbf{k}_2\cdot\mathbf{r}}a_2^\dagger) ,\\
&g_1=(2V_\gamma\omega_1)^{-1/2} ,\\
&g_2=(2V_\gamma\omega_2)^{-1/2} ,\\
&N_{a_1}=\tfrac{1}{2}(a_1a_1^\dagger+a_1^\dagger a_1) ,\\
&N_{a_2}=\tfrac{1}{2}(a_2a_2^\dagger+a_2^\dagger a_2) .
\end{aligned} \quad (89)$$

We will solve the Schrödinger equation with the Hamiltonian (88),

$$H\Psi(\mathbf{r})=\mathcal{E}\Psi(\mathbf{r}) . \quad (90)$$

After applying the transformation

$$\Psi(\mathbf{r})=e^{i(\mathbf{p}-\mathbf{k}_1N_{a_1}-\mathbf{k}_2N_{a_2})\cdot\mathbf{r}}\phi , \quad (91)$$

the eigenstate equation (90) becomes

$$\left[\frac{1}{2m_e}(\mathbf{p}-\mathbf{k}_1N_{a_1}-\mathbf{k}_2N_{a_2})^2 - \frac{e}{2m_e}[(\mathbf{p}-\mathbf{k}_1N_{a_1}-\mathbf{k}_2N_{a_2})\cdot(\mathbf{A}_1+\mathbf{A}_2) +(\mathbf{A}_1+\mathbf{A}_2)\cdot(\mathbf{p}-\mathbf{k}_1N_{a_1}-\mathbf{k}_2N_{a_2})] + \frac{e^2}{2m_e}(\mathbf{A}_1+\mathbf{A}_2)^2+\omega_1N_{a_1}+\omega_2N_{a_2}\right]\phi=\mathcal{E}\phi , \quad (92)$$

where

$$\mathbf{A}_1=g_1(\epsilon_1a_1+\epsilon_1^*a_1^\dagger) , \quad \mathbf{A}_2=g_2(\epsilon_2a_2+\epsilon_2^*a_2^\dagger) . \quad (93)$$

To simplify this equation, we need the following ansatz:

Ansatz. There are two real numbers κ_1 and κ_2, such that

$$(\kappa_1\mathbf{k}_1-N_a\mathbf{k}_1+\kappa_2\mathbf{k}_2-N_{a_2}\mathbf{k}_2)$$

belongs to a higher-order relativistic effect in the Schrödinger equation.

We define a vector $\mathbf{P}$ such that

$$\mathbf{p}=\mathbf{P}+\kappa_1\mathbf{k}_1+\kappa_2\mathbf{k}_2\ . \tag{94}$$

By this definition and the ansatz, Eq. (92) reduces to

$$\left[\frac{1}{2m_e}\mathbf{P}^2-\frac{e}{m_e}\mathbf{P}\cdot(\mathbf{A}_1+\mathbf{A}_2)+\frac{e^2}{2m_e}\mathbf{A}_1^2+\frac{e^2}{2m_e}\mathbf{A}_2^2 + \frac{e^2}{m_e}\mathbf{A}_1\cdot\mathbf{A}_2+\omega_1N_{a_1}+\omega_2N_{a_2}\right]\phi=\mathcal{E}\phi\ . \tag{95}$$

The polarization vectors of each photon mode have the same form as that shown in Eq. (36), but are indexed by subscripts 1 or 2 to distinguish the modes. They satisfy

$$\begin{aligned}
&\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_1^*=\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_2^*=1\ ,\\
&\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_1=(\cos\xi_1)e^{i\Theta_1}\ ,\\
&\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_2=(\cos\xi_2)e^{i\Theta_2}\ ,\\
&\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2=\cos[\tfrac{1}{2}(\xi_1+\xi_2)]e^{i(1/2)(\Theta_1+\Theta_2)}\ ,\\
&\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*=\cos[\tfrac{1}{2}(\xi_1-\xi_2)]e^{i(1/2)(\Theta_1-\Theta_2)}\ .
\end{aligned}\tag{96}$$

Now we switch to the notation of Lie algebra and write the Hamiltonian in Eq. (95), H', as

$$\begin{aligned}
H'={}&2\mathcal{B}_1h_1^{(1)}+\mathcal{A}_1e_{-2}^{(1)}+\mathcal{A}_1^*e_{+2}^{(1)}+2\mathcal{B}_2h_1^{(2)}+\mathcal{A}_2e_{-2}^{(2)}\\
&+\mathcal{A}_2^*e_{+2}^{(2)}+2\mathcal{C}e_{-1,-1}^{(1,2)}+2\mathcal{C}^*e_{+1,+1}^{(1,2)}\\
&+2\mathcal{D}e_{-1,+1}^{(1,2)}+2\mathcal{D}^*e_{+1,-1}^{(1,2)}+\mathcal{F}h_0+\mathcal{G}_1e_{-1}^{(1)}\\
&+\mathcal{G}_1^*e_{+1}^{(1)}+\mathcal{G}_2e_{-1}^{(2)}+\mathcal{G}_2^*e_{+1}^{(2)}\ .
\end{aligned}\tag{97}$$

The coefficients denoted by the script letters are

$$\begin{aligned}
&\mathcal{B}_1=\tfrac{1}{2}\omega_1+\frac{e^2}{2m_e}g_1^2\ ,\\
&\mathcal{B}_2=\tfrac{1}{2}\omega_2+\frac{e^2}{2m_e}g_2^2\ ,\\
&\mathcal{A}_1=\frac{e^2}{2m_e}g_1^2\cos\xi_1e^{i\Theta_1}\ ,\\
&\mathcal{A}_2=\frac{e^2}{2m_e}g_2^2\cos\xi_2e^{i\Theta_2}\ ,\\
&\mathcal{C}=\frac{e^2}{2m_e}g_1g_2\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2\ ,\\
&\mathcal{D}=\frac{e^2}{2m_e}g_1g_2\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2^*\ ,\\
&\mathcal{F}=\frac{\mathbf{P}^2}{2m_e}\ ,\\
&\mathcal{G}_1=-\frac{eg_1}{m_e}\mathbf{P}\cdot\boldsymbol{\epsilon}_1\ ,\\
&\mathcal{G}_2=-\frac{eg_2}{m_e}\mathbf{P}\cdot\boldsymbol{\epsilon}_2\ .
\end{aligned}\tag{98}$$

We define $\bar{H}'$ as an element of $\mathscr{G}_2$,

$$\begin{aligned}
H'={}&2\mathcal{B}_1h_1^{(1)}+\mathcal{A}_1e_{-2}^{(1)}+\mathcal{A}_1^*e_{+2}^{(1)}\\
&+2\mathcal{B}_2h_1^{(2)}+\mathcal{A}_2e_{-2}^{(2)}+\mathcal{A}_2^*e_{+2}^{(2)}\\
&+2\mathcal{C}e_{-1,-1}^{(1,2)}+2\mathcal{C}^*e_{+1,+1}^{(1,2)}+2\mathcal{D}e_{-1,+1}^{(1,2)}+2\mathcal{D}^*e_{+1,-1}^{(1,2)}\ .
\end{aligned}\tag{99}$$

Following the idea shown in Sec. III, we will find the c-photon solution directly. The commutators we need are

$$\begin{aligned}
&[\bar{H}',e_{-1}^{(1)}]=-2\mathcal{B}_1e_{-1}^{(1)}-2\mathcal{D}^*e_{-1}^{(2)}-2\mathcal{A}_1^*e_{+1}^{(1)}-2\mathcal{C}^*e_{+1}^{(2)}\ ,\\
&[\bar{H}',e_{-1}^{(2)}]=-2\mathcal{D}e_{-1}^{(1)}-2\mathcal{B}_2e_{-1}^{(2)}-2\mathcal{C}^*e_{+1}^{(1)}-2\mathcal{A}_2^*e_{+1}^{(2)}\ ,\\
&[\bar{H}',e_{+1}^{(1)}]=2\mathcal{A}_1e_{-1}^{(1)}+2\mathcal{C}e_{-1}^{(2)}+2\mathcal{B}_1e_{+1}^{(1)}+2\mathcal{D}e_{+1}^{(2)}\ ,\\
&[\bar{H}',e_{-1}^{(1)}]=2\mathcal{C}e_{-1}^{(1)}+2\mathcal{A}_2e_{-1}^{(2)}+2\mathcal{D}^*e_{+1}^{(1)}+2\mathcal{B}_2e_{+1}^{(2)}\ .
\end{aligned}\tag{100}$$

After transposing the coefficient matrix of the above equation set, we have the matrix that we are going to solve for its eigenvalues and eigenvectors,

$$2\begin{vmatrix}
-\mathcal{B}_1 & -\mathcal{D} & \mathcal{A}_1 & \mathcal{C}\\
-\mathcal{D}^* & -\mathcal{B}_2 & \mathcal{C} & \mathcal{A}_2\\
-\mathcal{A}_1^* & -\mathcal{C}^* & \mathcal{B}_1 & \mathcal{D}^*\\
-\mathcal{C}^* & \mathcal{A}_2^* & \mathcal{D} & \mathcal{B}_2
\end{vmatrix}\ . \tag{101}$$

The four eigenvalues are

$$\begin{aligned}
&\rho_{-1}^{(1)}=-\rho_{+1}^{(1)}\ ,\\
&\rho_{+1}^{(1)}=2[\tfrac{1}{2}(\mathcal{B}_1'^2+\mathcal{B}_2'^2)|\mathcal{C}'|^2-|\mathcal{D}'|^2-\mathcal{Y}]^{1/2}\ ,\\
&\rho_{-1}^{(2)}=-\rho_{+1}^{(2)}\ ,\\
&\rho_{+1}^{(2)}=2[\tfrac{1}{2}(\mathcal{B}_1'^2+\mathcal{B}_2'^2)+|\mathcal{C}'|^2-|\mathcal{D}'|^2+\mathcal{Y}]^{1/2}\ ,
\end{aligned}\tag{102}$$

where

$$\begin{aligned}
&\mathcal{Y}=\left[\frac{(\mathcal{B}_2'^2-\mathcal{B}_1'^2)^2}{4}-(\mathcal{B}_2'-\mathcal{B}_1')^2|\mathcal{D}'|^2+(\mathcal{B}_1'+\mathcal{B}_2')^2|\mathcal{C}'|^2\right]^{1/2}\ ,\\
&\mathcal{B}_1'=\mathcal{B}_1\cosh(2\chi_1)+\mathcal{A}_1\sinh(2\chi_1)e^{-i\Theta_1}\\
&\quad=\frac{1}{2m_e}[(m_e\omega_1+e^2g_1^2)^2-e^4g_1^4\cos^2\xi_1]^{1/2}\ ,\\
&\mathcal{B}_2'=\mathcal{B}_2\cosh(2\chi_2)+\mathcal{A}_2\sinh(2\chi_2)e^{-i\Theta_2}\\
&\quad=\frac{1}{2m_e}[(m_e\omega_2+e^2g_2^2)^2-e^4g_2^4\cos^2\xi_2]^{1/2}\ ,\\
&\mathcal{C}'=C\cosh(\chi_1+\chi_2)+\mathcal{D}\sinh(\chi_1+\chi_2)e^{i\Theta_2}\ ,\\
&\mathcal{D}'=C\sinh(\chi_1+\chi_2)e^{-i\Theta_2}+\mathcal{D}\cosh(\chi_1+\chi_2)\ ,
\end{aligned}\tag{103}$$

and

$$\tanh(2\chi_1) = -\frac{|\mathcal{A}_1|}{\mathcal{B}_1} = -\frac{e^2 g_1^2 \cos\xi_1}{m_e\omega_1 + e^2 g_1^2} ,$$
$$\tanh(2\chi_2) = -\frac{|\mathcal{A}_2|}{\mathcal{B}_2} = -\frac{e^2 g_2^2 \cos\xi_2}{m_e\omega_2 + e^2 g_2^2} . \qquad (104)$$

In ordinary cases the following inequalities always hold:

$$m_e\omega_1 \gg e^2 g_1^2 ,$$
$$m_e\omega_2 \gg e^2 g_2^2 . \qquad (105)$$

Hence, we can see that χ_1 and χ_2 are small numbers, and $|\mathcal{B}_1|$ and $|\mathcal{B}_2|$ are much larger than $|\mathcal{C}|$ and $|\mathcal{D}|$. These conditions lead to $|\mathcal{B}_1'|$ and $|\mathcal{B}_2'|$ being much larger than $|\mathcal{C}'|$ and $|\mathcal{D}'|$. Thus, all four roots are real numbers.

The four corresponding vectors in photon-operator notation expressed in determinant form are

$$c_1 = t_1 \begin{vmatrix} a_1 & a_2 & a_1^\dagger & a_2^\dagger \\ -2\mathcal{D}^* & -2\mathcal{B}_2+\rho_{+1}^{(1)} & 2\mathcal{C} & 2\mathcal{A}_2 \\ -2\mathcal{A}_1^* & -2\mathcal{C}^* & 2\mathcal{B}_1+\rho_{+1}^{(1)} & 2\mathcal{D}^* \\ -2\mathcal{C}^* & -2\mathcal{A}_2^* & 2\mathcal{D} & 2\mathcal{B}_2+\rho_{+1}^{(1)} \end{vmatrix} ,$$

$$c_2 = t_2 \begin{vmatrix} a_1 & a_2 & a_1^\dagger & a_2^\dagger \\ -2\mathcal{B}_1+\rho_{+1}^{(2)} & -2\mathcal{D} & 2\mathcal{A}_1 & 2\mathcal{C} \\ -2\mathcal{A}_1^* & -2\mathcal{C}^* & 2\mathcal{B}_1+\rho_{+1}^{(2)} & 2\mathcal{D}^* \\ -2\mathcal{C}^* & -2\mathcal{A}_2^* & 2\mathcal{D} & 2\mathcal{B}_2+\rho_{+1}^{(2)} \end{vmatrix} , \qquad (106)$$

$$c_1^\dagger = t_1^* \begin{vmatrix} a_1 & a_2 & a_1^\dagger & a_2^\dagger \\ 2\mathcal{C}^* & 2\mathcal{A}_2^* & -2\mathcal{D} & -2\mathcal{B}_2+\rho_{+1}^{(1)} \\ 2\mathcal{B}_1+\rho_{+1}^{(1)} & 2\mathcal{D} & -2\mathcal{A}_1 & -2\mathcal{C} \\ 2\mathcal{D}^* & 2\mathcal{B}_2+\rho_{+1}^{(1)} & -2\mathcal{C} & -2\mathcal{A}_2 \end{vmatrix} ,$$

$$c_2^\dagger = t_2^* \begin{vmatrix} a_1 & a_2 & a_1^\dagger & a_2^\dagger \\ 2\mathcal{A}_1^* & 2\mathcal{C}^* & -2\mathcal{B}_1+\rho_{+1}^{(2)} & -2\mathcal{D}^* \\ 2\mathcal{B}_1+\rho_{+1}^{(2)} & 2\mathcal{D} & -2\mathcal{A}_1 & -2\mathcal{C} \\ 2\mathcal{D}^* & 2\mathcal{B}_2+\rho_{+1}^{(2)} & -2\mathcal{C} & -2\mathcal{A}_2 \end{vmatrix} ,$$

where the normalization constants t_1 and t_2 are chosen such that

$$[c_1, c_1^\dagger] = [c_2, c_2^\dagger] = I ,$$
$$(c_1)^\dagger = c_1^\dagger ,$$
$$(c_2)^\dagger = c_2^\dagger . \qquad (107)$$

The other commutators are zero:

$$[c_1, c_2] = [c_1^\dagger, c_2^\dagger] = [c_1, c_2^\dagger] = [c_2, c_1^\dagger] = 0 , \qquad (108)$$

which can be proven by Lie-algebraic relations, such as Eqs. (3) and (14), and also can be verified by direct computation. By the automorphism from a-photon algebra to the c-photon algebra we have

$$\overline{H}' = \rho_{+1}^{(1)} N_{c_1} + \rho_{+1}^{(2)} N_{c_2} , \qquad (109)$$

where

$$N_{c_1} = \tfrac{1}{2}(c_1 c_1^\dagger + c_1^\dagger c_1) ,$$
$$N_{c_2} = \tfrac{1}{2}(c_2 c_2^\dagger + c_2^\dagger c_2) . \qquad (110)$$

For convenience we write the transformation (106) as

$$c_1 = \alpha_{11} a_1 + \alpha_{12} a_2 + \beta_{11} a_1^\dagger + \beta_{12} a_2^\dagger ,$$
$$c_2^\dagger = \beta_{21}^* a_1 + \beta_{22}^* a_2 + \alpha_{21}^* a_1^\dagger + \alpha_{22}^* a_2^\dagger ,$$
$$c_1^\dagger = \beta_{11}^* a_1 + \beta_{12}^* a_2 + \alpha_{11}^* a_1^\dagger + \alpha_{12}^* a_2^\dagger ,$$
$$c_2^\dagger = \beta_{21}^* a_1 + \beta_{22}^* a_2 + \alpha_{21}^* a_1^\dagger + \alpha_{22}^* a_2^\dagger . \qquad (111)$$

The term of linear operators in Eq. (97) can be written in terms of c-photon operators,

$$\mathcal{G}_1 a_1 + \mathcal{G}_1^* a_1^\dagger + \mathcal{G}_2 a_2 + \mathcal{G}_2^* a_2^\dagger$$
$$= \mathcal{G}_1' c_1 + \mathcal{G}_1'^* c_1^\dagger + \mathcal{G}_2' c_2 + \mathcal{G}_2'^* c_2^\dagger , \qquad (112)$$

Thus,

$$\begin{bmatrix} \mathcal{G}_1' \\ \mathcal{G}_2' \\ \mathcal{G}_1'^* \\ \mathcal{G}_2'^* \end{bmatrix} = (L^{-1})^T \begin{bmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \\ \mathcal{G}_1^* \\ \mathcal{G}_2^* \end{bmatrix} , \qquad (113)$$

where

$$L = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \\ \beta_{11}^* & \beta_{12}^* & \alpha_{11}^* & \alpha_{12}^* \\ \beta_{21}^* & \beta_{22}^* & \alpha_{21}^* & \alpha_{22}^* \end{bmatrix} . \qquad (114)$$

The inverse of L can be obtained by evaluating the mixed commutators between c-photon operators and a-photon operators. Thus, we have

$$L^{-1} = \begin{bmatrix} \alpha_{11}^* & \alpha_{12}^* & -\beta_{11} & -\beta_{21} \\ \alpha_{12}^* & \alpha_{22}^* & -\beta_{12} & -\beta_{22} \\ -\beta_{11}^* & -\beta_{21}^* & \alpha_{11} & \alpha_{21} \\ -\beta_{12}^* & -\beta_{22}^* & \alpha_{12} & \alpha_{22} \end{bmatrix} . \qquad (115)$$

Thus, the Hamiltonian H' is expressed in terms of c-photon operators as

$$H' = \rho_{+1}^{(1)} N_{c_1} + \rho_{+1}^{(2)} N_{c_2} + \mathcal{G}_1' c_1 + \mathcal{G}_1'^* c_1^\dagger$$
$$+ \mathcal{G}_2' c_2 + \mathcal{G}_2'^* c_2^\dagger + \mathcal{F} I . \qquad (116)$$

Now we can introduce the displacement operator D to eliminate the linear terms of the annihilation and creation operators,

$$D=\exp(-\delta_1 c_1^\dagger+\delta_1^* c_1-\delta_2 c_2^\dagger+\delta_2^* c_2) . \quad (117)$$

It is easy to verify that

$$\begin{aligned} Dc_1D^\dagger &= c_1+\delta_1 I ,\\ Dc_1^\dagger D^\dagger &= c_1^\dagger+\delta_1^* I ,\\ Dc_2D^\dagger &= c_2+\delta_2 I ,\\ Dc_2^\dagger D^\dagger &= c_2^\dagger+\delta_2^* I . \end{aligned} \quad (118)$$

By choosing

$$\begin{aligned} \delta_1 &= -\frac{\mathcal{G}_1'^*}{\rho_{+1}^{(1)}} ,\\ \delta_2 &= -\frac{\mathcal{G}_2'^*}{\rho_{+1}^{(2)}} , \end{aligned} \quad (119)$$

the Hamiltonian H' in Eq. (116) becomes

$$H'=\rho_{+1}^{(1)}N_{c_1}+\rho_{+1}^{(2)}N_{c_2}+\mathcal{F}'I , \quad (120)$$

where

$$\mathcal{F}'=\frac{\mathbf{P}^2}{2m_e}-\frac{|\mathcal{G}_1'|^2}{\rho_{+1}^{(1)}}-\frac{|\mathcal{G}_2'|^2}{\rho_{+1}^{(2)}} . \quad (121)$$

Now H' is in a solvable form. The remaining task is to solve for the vacuum state $|0,0\rangle_e$, which satisfies the equations

$$\begin{aligned} c_1|0,0\rangle_c &= 0 ,\\ c_2|0,0\rangle_c &= 0 . \end{aligned} \quad (122)$$

The explicit expression for $|0,0\rangle_c$ and its derivation are presented in the Appendix. The number states of the c-photon system are defined as

$$|n_1,n_2\rangle_c=\frac{(c_1^\dagger)^{n_1}}{\sqrt{n_1!}}\frac{(c_2^\dagger)^{n_2}}{\sqrt{n_2!}}|0,0\rangle_e \quad (123)$$

and satisfy

$$\begin{aligned} N_{c_1}|n_1,n_2\rangle_c &= (n_1+\tfrac{1}{2})|n_1,n_2\rangle_c ,\\ N_{c_2}|n_1,n_2\rangle_c &= (n_2+\tfrac{1}{2})|n_1,n_2\rangle_c . \end{aligned} \quad (124)$$

The eigenstate of H' is

$$\phi=D^\dagger|n_1,n_2\rangle_c . \quad (125)$$

We finally have the solutions as an eigenstate of the Hamiltonian H,

$$\Psi(\mathbf{r})=V_e^{-1/2}\exp[i(-\mathbf{k}_1N_{a_1}-\mathbf{k}_2N_{a_2}+\mathbf{P} +\kappa_1\mathbf{k}_1+\kappa_2\mathbf{k}_2)\cdot\mathbf{r}]D^\dagger|n_1,n_2\rangle_c . \quad (126)$$

The corresponding energy is

$$\mathcal{E}=\frac{\mathbf{P}^2}{2m_e}+\rho_{+1}^{(1)}(n_1+\tfrac{1}{2})+\rho_{+1}^{(2)}(n_2+\tfrac{1}{2}) -\frac{|\mathcal{G}_1'|^2}{\rho_{+1}^{(1)}}-\frac{|\mathcal{G}_2'|^2}{\rho_{+1}^{(2)}} . \quad (127)$$

Analogous to the single-mode case, we define an energy E as

$$E=\frac{\mathbf{P}^2}{2m_e} , \quad (128)$$

by which the nonrelativistic electron four-momentum $(E+m_e,\mathbf{P})$ is on the electron mass shell. The total energy, i.e., the energy eigenvalue of the system, is related to E in the form

$$\mathcal{E}+m_e=(E+m_e)+\kappa_1'\omega_1+\kappa_2'\omega_2 , \quad (129)$$

where

$$\begin{aligned} \kappa_1' &\equiv \frac{\rho_{+1}^{(1)}(n_1+\frac{1}{2})}{\omega_1}-\frac{|\mathcal{G}_1'|^2}{\rho_{+1}^{(1)}\omega_1} ,\\ \kappa_2' &\equiv \frac{\rho_{+1}^{(2)}(n_2+\frac{1}{2})}{\omega_2}-\frac{|\mathcal{G}_2'|^2}{\rho_{+1}^{(2)}\omega_2} . \end{aligned} \quad (130)$$

By a comparison and arguments similar to that carried out in the single-mode case, we identify

$$\begin{aligned} \kappa_1=\kappa_1' &= \frac{\rho_{+1}^{(1)}(n_1+\frac{1}{2})}{\omega_1}-\frac{|\mathcal{G}_1'|^2}{\rho_{+1}^{(1)}\omega_1} ,\\ \kappa_2=\kappa_2' &= \frac{\rho_{+1}^{(2)}(n_2+\frac{1}{2})}{\omega_2}-\frac{|\mathcal{G}_2'|^2}{\rho_{+1}^{(2)}\omega_2} . \end{aligned} \quad (131)$$

A pair of important parameters Z_{n_1} and Z_{n_2} can be defined as

$$\begin{aligned} Z_{n_1} &= \kappa_1-(n_1+\tfrac{1}{2}) ,\\ Z_{n_2} &= \kappa_2-(n_2+\tfrac{1}{2}) , \end{aligned} \quad (132)$$

with the interpretation that $Z_{n_1}\omega_1$ and $Z_{n_2}\omega_2$ are the ponderomotive potential energies for each mode, while $Z_{n_1}\omega_1+Z_{n_2}\omega_2$ is the total ponderomotive potential. The additivity of the ponderomotive potentials in multimode cases is a reasonable conclusion, which can be realized in a classical picture. Since the ponderomotive energies originate from the square term in the field's vector potential $\mathbf{A}$, the nonexistence of cross terms in the ponderomotive potential is due to the fact that time averages of the cross terms of the field $\mathbf{A}_i \cdot \mathbf{A}_j$ among different photon modes are zero.[18] The same kind of additive property exists among ponderomotive momenta of the modes, which will be seen more clearly in the next section.

The validity of the ansatz can be checked by noticing that $\kappa_1\mathbf{k}_1+\kappa_2\mathbf{k}_2$ will take care of the major contributions of $N_{a_1}\mathbf{k}_1+N_{a_2}\mathbf{k}_2$ in the first term of the Hamiltonian in Eq. (92), while the remaining contributions belong to relativistic higher-order effects.

V. DISCUSSION

A. Generalization

The result obtained for the two-mode case can be formally generalized into cases with arbitrary numbers, even an infinite number, of modes. For m-mode cases, the wave function for the system of one electron and many photons should be

$$\Psi(\mathbf{r})=V_e^{-1/2}\exp[i(-\mathbf{k}_1N_{a_1}+\cdots-\mathbf{k}_mN_{a_m}+\mathbf{P}+\kappa_1\mathbf{k}_1+\cdots+\kappa_m\mathbf{k}_m)\cdot\mathbf{r}]D^\dagger|n_1,\ldots,n_m\rangle_c , \tag{133}$$

with the energy eigenvalue

$$\mathcal{E}=\frac{\mathbf{P}^2}{2m_e}+\rho_{+1}^{(1)}(n_1+\tfrac{1}{2})+\cdots+\rho_{+1}^{(m)}(n_m+\tfrac{1}{2})-|\mathcal{G}_1'|^2/\rho_{+1}^{(1)}+\cdots-|\mathcal{G}_m'|^2/\rho_{+1}^{(m)} , \tag{134}$$

where the displacement operator D is

$$D=\exp(-\delta_1c_1^\dagger+\delta_1^*c_1+\cdots-\delta_mc_m^\dagger+\delta_m^*c_m) \tag{135}$$

and the number states are

$$|n_1,\ldots,n_m\rangle_c=\frac{(c_1^\dagger)^{n_1}}{\sqrt{n_1!}}\cdots\frac{(c_m^\dagger)^{n_m}}{\sqrt{n_m!}}|0,\ldots,0\rangle_c . \tag{136}$$

The δ's are

$$\delta_i=-\frac{\mathcal{G}_i'^*}{\rho_{+1}^{(i)}} \quad (i=1,2,\ldots,m) , \tag{137}$$

and $\mathcal{G}_i$ and $\mathcal{G}_i^*$ $(i=1,2,\ldots,m)$ are the coefficients of the linear terms of c_i and $c_i^\dagger$ in H'. The vacuum state is subject to the equation set

$$c_i|0,\ldots,0\rangle_c=0 \quad (i=1,\ldots,m) , \tag{138}$$

which is solvable in principle. Thus we have exact solutions as shown by Eq. (133) for the general cases of an arbitrary number of modes.

B. Physical interpretation

The solution (133) can be written in the a-photon representation,

$$\Psi(\mathbf{r})=V_e^{-1/2}\sum_{j_1\geq -n_1,\ldots,j_m\geq -n_m}\exp\{i[\mathbf{P}+(Z_{n_1}-j_1)\mathbf{k}_1+\cdots+(Z_{n_m}-j_m)\mathbf{k}_m]\cdot\mathbf{r}\}|n_1+j_1,\ldots,n_m+j_m\rangle\times\mathcal{J}^*_{j_1,\ldots,j_m}(n_1,\ldots,n_m)e^{-ij(\phi_{\xi_1}+\Theta_1/2+\cdots+\phi_{\xi_m}+\Theta_m/2)} , \tag{139}$$

where

$$\mathcal{J}^*_{j_1,\ldots,j_m}(n_1,\ldots,n_m)e^{-ij(\phi_{\xi_1}+\Theta_1/2+\cdots+\phi_{\xi_m}+\Theta_m/2)}\equiv\langle n_1+j_1,\ldots,n_m+j_m|D^\dagger|n_1,\ldots,n_m\rangle_c , \tag{140}$$

$$\phi_{\xi_i}=\tan^{-1}[(P_y/P_x)\tan(\xi_i/2)] ,$$

and the states without subscript c are in a-photon representation. The function $\mathcal{J}_{j_1,\ldots,j_m}(n_1,\ldots,n_m)$ can be called a discrete generalized Bessel function, which approaches an ordinary Bessel function in the circularly polarized single-mode case in the large-photon-number limit.[11,13]

From Eq. (139) we can see that there are two types of photons involved in the wave function in the a-photon representation. The photons enumerated by $n_1,n_2,\ldots,n_m$ do not carry the momentum phase factor in the wave function in the current Schrödinger picture. We call this type of photon the background photons. If there is no interaction between electron and photons, the wave function for those free photons will be $|n_1,n_2,\ldots,n_m\rangle$, without any momentum phase factor. Thus we see that in the interacting system, the background photons more or less look like free photons. The photons enumerated by $j_1,j_2,\ldots,j_m$, which are dummy variables, carry momentum phase factors in the wave function and will be called the transferred photons. Those momentum phase factors originate from the interaction part of the Hamiltonian where each creation and annihilation operator is accompanied by a momentum phase factor. In summary we can say that the background photons are those originally there around the electron while the transferred photons are those emitted or absorbed by the electron.

In a strong radiation field, the ponderomotive potential energy plays an important role. Our solutions (139) show the existence of ponderomotive energies in their energy eigenvalue, and ponderomotive momenta $Z_{n_i}\mathbf{k}_i$ $(i=1,\ldots,m)$ in their momentum phase factor. The total ponderomotive four-vector, the energy and the momentum, is equal to the linear superposition (in an exact mathematical sense) of the ponderomotive four-vectors of the present modes. This conclusion is far from the knowledge of the linear superposition property of the vector-potential **A** fields.

If there are no background photons, the ponderomotive energy and momentum parts of the wave function might be neglected. The wave function in this case is

$$\Psi(\mathbf{r})=V_e^{-1/2}\sum_{j_1\geq 0,\ldots,j_m\geq 0}\exp\{i(\mathbf{P}+j_1\mathbf{k}_1+\cdots+j_m\mathbf{k}_m)\cdot\mathbf{r}\}|j_1,\ldots,j_m\rangle \times \mathcal{J}^*_{j_1,\ldots,j_m}(0,\ldots,0)e^{-ij(\phi_{\xi_1}+\Theta_1/2+\cdots+\phi_{\xi_m}+\Theta_m/2)}, \qquad (141)$$

which is the wave function for a free electron with its photon cloud where the photons around the electron are emitted by the electron itself and are of squeezed light in multimode cases. The expression $\mathcal{J}^*_{j_1,\ldots,j_m}(0,\ldots,0)$ is defined as the same in Eq. (140) by setting $n_1=\cdots=n_m=0$.

There are many remaining interesting questions to be examined, such as conversions between different photon modes. The multimode solutions should offer strong means to treat photon-mode conversions in photon-electron scattering processes.

In the single-mode case there is a quite simple expression for the solution in the large-photon-number limit. It is desirable to find out a simpler approach to get multimode solutions in large-photon-number limit directly, so that the solutions can be easily applied to the cases when radiation fields are intense.

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APPENDIX

We want to express the c-photon vacuum state $|0,0\rangle_c$ in terms of number states of a photons. Equation (122) is equivalent to

$$\alpha_{11}a_1|0,0\rangle_c+\alpha_{12}a_2|0,0\rangle_c=-\beta_{11}a_1^\dagger|0,0\rangle_c-\beta_{12}a_2^\dagger|0,0\rangle_c\,, \qquad (A1)$$

$$\alpha_{21}a_1|0,0\rangle_c+\alpha_{22}a_2|0,0\rangle_c=-\beta_{21}a_1^\dagger|0,0\rangle_c-\beta_{22}a_2^\dagger|0,0\rangle_c\,,$$

which we can rewrite in matrix form,

$$\begin{bmatrix} a_1|0,0\rangle_c \\ a_2|0,0\rangle_c \end{bmatrix}=-\alpha^{-1}\beta\begin{bmatrix} a_1^\dagger|0,0\rangle_c \\ a_2^\dagger|0,0\rangle_c \end{bmatrix}, \qquad (A2)$$

where α and β are 2×2 matrices formed from the coefficients in (A1).

By multiplying the number state of a photons, $\langle m,n|$, from the left on both sides of (A2), we obtain a recursion relation,

$$\begin{bmatrix} \langle m+1,n|0,0\rangle_c \\ \langle m,n+1|0,0\rangle_c \end{bmatrix}=F(m,n)\begin{bmatrix} \langle m,n-1|0,0\rangle_c \\ \langle m-1,n|0,0\rangle_c \end{bmatrix}, \qquad (A3)$$

where the states with negative occupation numbers mean zero, and the 2×2 matrix $F(m,n)$ is defined as

$$F(m,n)=\begin{bmatrix} (m+1)^{-1/2} & 0 \\ 0 & (n+1)^{-1/2} \end{bmatrix}\alpha^{-1}\beta \times\begin{bmatrix} 0 & m^{1/2} \\ n^{1/2} & 0 \end{bmatrix}. \qquad (A4)$$

By setting $m=0$ and $n=0$ in (A3), we are able to prove that $\langle 1,0,|0,0\rangle_c$ and $\langle 0,1|0,0\rangle_c$ are zero. By further using (A3), we find that all $\langle m,n|0,0\rangle_c$ with $m+n=$odd are zero. Thus, $|0,0\rangle_c$ should be of the form

$$
\begin{aligned}
|0,0\rangle_c &= \sum_{m,n;m+n=\text{even}} |m,n\rangle\langle m,n|0,0\rangle_c \\
&= \sum_{p,q;p,q\geq 0} (|2p+1,2q-1\rangle,|2p,2q\rangle)\begin{bmatrix}\langle 2p+1,2q-1|0,0\rangle_c \\ \langle 2p,2q|0,0\rangle_c\end{bmatrix} \\
&= \sum_{p,q;p,q\geq 0} (|2p+1,2q-1\rangle,|2p,2q\rangle)F(2p,2q-1)F(2p-1,2q-2)\cdots \\
&\quad \times \begin{cases} F(2p-2q+1,0)\begin{bmatrix}0 \\ \langle 2p-2p,0|0,0\rangle_c\end{bmatrix} & (\text{for } p\geq q) \\ F(0,2p-2q-1)\begin{bmatrix}\langle 0,2p-2q|0,0\rangle_c \\ 0\end{bmatrix} & (\text{for } p<q)\ . \end{cases}
\end{aligned} \tag{A5}
$$

The coefficients $\langle 2m,0,|0,0\rangle$ and $\langle 0,2n|0,0\rangle$ obtained by using $\langle 2m-1,0|$ to multiply both sides of Eqs. (A1) from the left are

$$
\begin{aligned}
\langle 2m,0|0,0\rangle_c &= \left[\frac{\alpha_{12}\beta_{21}-\alpha_{22}\beta_{11}}{\alpha_{11}\alpha_{22}-\alpha_{21}\alpha_{12}}\right]^m \left[\frac{(2m-1)!!}{(2m)!!}\right]^{1/2} \langle 0,0|0,0\rangle_c\ , \\
\langle 0,2n|0,0\rangle_c &= \left[\frac{\alpha_{21}\beta_{12}-\alpha_{11}\beta_{22}}{\alpha_{11}\alpha_{22}-\alpha_{21}\alpha_{12}}\right]^n \left[\frac{(2n-1)!!}{(2n)!!}\right]^{1/2} \langle 0,0|0,0\rangle_c\ ,
\end{aligned} \tag{A6}
$$

where $\langle 0,0|0,0\rangle_c$ can be determined by the normalization condition

$$\langle 0,0|_c\ |0,0\rangle_c = 1\ . \tag{A7}$$

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