Low-frequency theory of multiphoton ionization

Yonatan Abranyos and Marvin H. Mittleman
Department of Physics, The City College of the City University of New York, New York, New York 10031
(Received 7 June 1989)

We evaluate the ionization rate from a recent (adiabatic) theory of multiphoton ionization for the case of a recent experiment: a circularly polarized CO_2 laser at an intensity of about 10^{14} W/cm^2. For S states we obtain a rate 10^4-10^5 times larger than the Keldysh-Faisal-Reiss theory. For P states we obtain a rate about 10^9 times larger. We also make a prediction about the angular distribution from S states.

I. INTRODUCTION

The most commonly referenced S-matrix theory of multiphonon ionization is the Keldysh-Faisal-Reiss (KFR) theory.[1–3] It has some formal difficulties that were pointed out in a recent Brief Report[4] by one of us (M.H.M.), and another theory was presented that remedied these difficulties. In this "adiabatic" theory an approximation was made on the exact wave function that evolves from the initial bound state in the distant past. The approximation improves as the laser frequency decreases relative to the energy of the initial state ($\hbar=1$).

In a recent Letter[5] an experiment on ionization by an intense ($I\sim10^{14}$ W/cm^2) CO_2 laser was reported. These conditions are ideal for the applicability of this new theory. Moreover, the rather unpleasant integral that results for the T matrix becomes analytically tractable for the parameters of the experiment for the case of circular polarization. We report on these results here and present a comparison with the KFR theory for the same parameters.

In the presentation of the adiabatic theory[4] it was stated that the numerical results were similar to those of the older theory in most cases, but not all. This is a case in which the new theory gives much larger results than the KFR theory (factors of 10^5 or larger occur) which remedies one of the difficulties in the comparison of the KFR theory with numerical experiments.

In Sec. II we present an evaluation of the T matrix and transition rate for the experimental conditions of Ref. 5 for circular polarization and for a $1S$-like state. A comparison with the KFR theory for the same case is also given. The new ionization rate is about a factor of 10^4-10^5 larger than that given in the KFR theory. A new prediction concerning angular distribution is given.

In Sec. III we outline a similar calculation for a simple P state. The ionization rate is about a factor of 10^9 larger than the previous theory. We also investigate a more general S state, given by a sum of Slater orbitals and give its T matrix. It is not significantly different from the $1S$-like result of Sec. II.

The results lead one to suspect the stability of these theories. That is, the adiabatic theory presented in Ref. 4 remedies some of the difficulties of the KFR theory but it is not expected to be a really significant improvement without the inclusion of intermediate resonances, which are now known to be important. Yet under the (admittedly extreme) conditions of the experiment[5] discussed here the results are strikingly different. One wonders what the next "slight improvement" of the theory will produce.

It would of course be useful to compare the different results with reliable calculations or experiments under similar conditions. The experiment[5] does not provide absolute reaction rates so, as pointed out in Sec. II, it is not capable of differentiating among the theories. Relevant calculations on model systems also are not yet available. The Bersons[6] model is the only three-dimensional calculation of which we are aware. It is pathological in that it is a zero-range model that is outside the domain of nonsingular quantum mechanics. Its zero-range feature makes the adiabatic-theory results essentially the same as the previous theories.

There are now several one-dimensional model calculations in the literature, but it is clear from our results stated above that angular momentum plays a critical role in the theory, and this is a concept alien to one-dimensional calculations. It would therefore seem that there are as yet no reliable results available for differentiation among the theories discussed here.

II. ADIABATIC THEORY

The derivation of the T matrix for the adiabatic theory has been presented[4] so we merely note an intermediate result for circular polarization. From Ref. 4, Eq. (3.21), we can write

$$T_N^A=\int_{-\pi}^{\pi}\frac{d\beta}{2\pi}\exp i(N\beta-\alpha_0 q_\perp\sin\beta)\times\int d^3r\, e^{-i\mathbf{Q}\cdot\mathbf{r}}V(r)u_0(r)\ , \tag{2.1}$$

where N is the number of photons absorbed, $u_0(r)$ is the initial single-particle bound state which is bound by the potential V, and

$$Q^2=q^2-\sqrt{2}m\omega\alpha_0 q_\perp\cos\beta+\tfrac{1}{2}(m\omega\alpha_0)^2\ . \tag{2.2}$$

Here $\mathbf{q}$ is the momentum of the outgoing electron whose magnitude is determined by the energy conservation condition to be

$$\frac{q^2}{2m}=\frac{q^2(N)}{2m}=N\omega-U_P-W=\omega(N-N_m)\ , \qquad (2.3)$$

where W is the dressed energy of the bound state and N_m, the minimum "number" of photons required for ionization is defined by this equation. Here α_0 has its usual definition $\alpha_0=eE/m\omega^2$, U_P is the ponderomotive potential, and $q_\perp=\sin\theta$ is the component of q perpendicular to $\hat{z}$, the laser propagation direction. The performance of the spatial integral for the $1S$ state of hydrogen allows (2.1) to be written as

$$T_N^A=-\frac{2\sqrt{\pi}e^2}{m\omega a_0^{3/2}}\left[N-\frac{|\Delta W|}{\omega}\right]^{-1}I_1\ , \qquad (2.4)$$

where we define

$$I_n=\int_{-\pi}^{\pi}\frac{d\beta}{2\pi}\frac{e^{i(N\beta-x\sin\beta)}}{(1-\eta\cos\beta)^n} \qquad (2.5)$$

and where

$$x=2^{-1/2}\alpha_0 q_\perp(N)=2\left[\frac{U_P}{\omega}(N-N_m)^{1/2}\right]\sin\vartheta\ , \qquad (2.6)$$

$$\eta=\left[N-\left|\frac{\Delta W}{\omega}\right|\right]^{-1}x\approx N^{-1}x<1\ . \qquad (2.7)$$

Here ΔW is the shift of the bound state energy W_0 due to the field, given in this model by

$$\Delta W=-\tfrac{1}{4}\alpha_s E^2\ , \qquad (2.8)$$

where α_s is the static polarizability of the state. For the experimental condition of a CO_2 laser at $I\sim 10^{14}$ W/cm^2 the relevant numbers are

$$\frac{U_P}{\omega}\approx 0.5\times 10^4,\quad \frac{|W_0|}{\omega}\approx 10^2,\quad \frac{|\Delta W|}{\omega}\approx 1\ , \qquad (2.9)$$

so that the last part of (2.7) is a good approximation since $N\geq N_m\approx\frac{1}{2}\times 10^4$. We therefore adopt this approximation.

We shall see that the part of the (N,θ) space that gives the dominant contribution is $N\approx 2N_m$, $\theta\sim\pi/2$ as has been shown previously.[7] This brings η to a value just below 1. The essential difference between this theory and the KFR result is that I_1 in (2.4) is replaced by I_0 to get the KFR result. But in I_1 the denominator is almost singular so we can expect $I_1\gg I_0$ as is indeed the case. This is the reason that $|T_N^A|>>|T_N^{\mathrm{KFR}}|$.

We evaluate I_1 by treating x and η as independent parameters and finding a differential equation for I_1 that is

$$\left[\frac{d^2}{dx^2}-q^2\right]I_1(x)=\eta^{-2}(I_0-2I_{-1})$$
$$=-\eta^{-2}\left[1+\frac{\eta N}{x}\right]J_N(x)\ , \qquad (2.10)$$

where

$$q=\eta^{-1}(1-\eta^2)^{1/2}\ . \qquad (2.10')$$

The equation is solved subject to boundary conditions

$$I_1(0)=(1-\eta^2)^{-1/2}Z^N\ ,$$
$$I_1'(0)=\tfrac{1}{2}(1-\eta^2)^{-1/2}(Z^{N+1}-Z^{N-1})\ , \qquad (2.11)$$

where $z=1/\eta-q$. The result is

$$I_1=\tfrac{1}{2}\eta^{-2}q^{-1}\int_0^\infty dx'\exp-q|x-x'|\left[1+\frac{\eta N}{x'}\right]J_N(x')\ . \qquad (2.12)$$

We use known asymptotic forms for $J_N(x')$ when N and x' are both large and positive.[8] For $x'>N$

$$J_N(x')\approx\left[\frac{2}{\pi}\right]^{1/2}(x'^2-N^2)^{-1/4}$$
$$\times\cos\left[(x'^2-N^2)^{1/2}-N\tan^{-1}\left[\frac{(x'^2-N^2)^{1/2}}{N}-\frac{\pi}{4}\right]\right]\ , \qquad (2.13)$$

and for $x'<N$

$$J_N(x')\approx(2\pi)^{-1/2}(N^2-x'^2)^{-1/4}$$
$$\times\exp\left[(N^2-x'^2)^{1/2}-\tfrac{1}{2}\ln\left[\frac{N+(N^2-x'^2)^{1/2}}{N-(N^2-x'^2)^{1/2}}\right]\right]\ . \qquad (2.14)$$

These are both weakly singular as $x'\to N$ so we introduce a factor which will ensure that

$$\lim_{x'\to N}J_N(x')=J_N(N)\to\frac{2^{1/3}}{3^{2/3}\Gamma(\frac{2}{3})}N^{-1/3}$$
$$=C_1\left[\frac{2}{N}\right]^{1/3}+O(N^{-2/3})\ . \qquad (2.15)$$

It will prove to be of no importance here. We simply replace the singular denominators by

$$(x'^2-N^2)^{-1/4}\to[x'^2-N^2(1-\beta_1)]^{-1/4}\ ,$$
$$\beta_1=(\pi N)^{-2}[J_N(N)]^{-4} \qquad (2.16)$$

in (2.13) and

$$(N^2-x'^2)^{-1/4}\to[N^2(1+\beta_2)-x'^2]^{-1/4}\ ,$$
$$\beta_2=(2\pi N)^{-2}[J_N(N)]^{-4} \qquad (2.17)$$

in (2.14).

The integral in (2.12) is divided into three regions $0\to x$, $x\to N$, and $N\to\infty$. In the first two regions (2.13) and (2.16) are used. Then (2.14) and (2.17) are used in the final one. In the first region ($I_1^{(1)}$) we let $x'=N\eta'$ and use $x=N\eta$ to obtain

$$I_1^{(1)}=\frac{e^{-Nq\eta}}{2\eta^2 q}\left[\frac{N}{2\pi}\right]^{1/2}\int_0^\eta\frac{d\eta'(1+\eta/\eta')}{(1-\eta'^2+\beta_2)^{1/4}}e^{NF(\eta')}\ , \qquad (2.18)$$

where

$$f(\eta')=q\eta'+(1-\eta'^2)^{1/2}-\tfrac{1}{2}\ln\left[\frac{1+(1-\eta'^2)^{1/2}}{1-(1-\eta'^2)^{1/2}}\right]. \tag{2.19}$$

The large N in the exponent in the integrand ensures that the main contribution comes from the region of maximum $f(\eta')$ which is at $\eta'=\eta$, where

$$f(\eta)=2(1-\eta^2)^{1/2}-\tfrac{1}{2}\ln\left[\frac{1+(1-\eta^2)^{1/2}}{1-(1-\eta^2)^{1/2}}\right]. \tag{2.20}$$

The rest of the integrand is evaluated at $\eta'=\eta$ and the integration is easily performed with the result

$$I_1^{(1)}=(8\pi N)^{-1/2}(1-\eta^2)^{-1}(1-\eta^2+\beta_2)^{-1/4}\times e^{-N\epsilon(\eta)}(1-e^{-2N\eta q}), \tag{2.21}$$

where

$$\epsilon(\eta)=q\eta-f(\eta)=\tfrac{1}{2}\ln\left[\frac{1+(1-\eta^2)^{1/2}}{1-(1-\eta^2)^{1/2}}\right]-(1-\eta^2)^{1/2}. \tag{2.22}$$

The largest contribution to (2.21) comes from the smallest $\epsilon(\eta)$ because of the large factor N in the first exponent. $\epsilon(\eta)$ is monotonically decreasing, so the largest contribution comes from the region near $\eta=1$. The last form of (2.7) shows that η maximizes at $\sin\theta=1$ and $N=2N_m=2(U_P+|W_0|)/\omega$, where

$$\eta_{\max}=\left[1+\frac{|W_0|}{U_P}\right]^{-1/2}\approx 1-\frac{|W_0|}{2U_P}\approx 1-\frac{|W_0|}{2\omega N_m}. \tag{2.23}$$

From (2.9) we see that this is slightly less than 1. The exponent in the last factor of (2.21) is then of the order $2N(1-\eta^2)^{1/2}\approx 2(2N_m|W_0/\omega|)^{1/2}\approx 2\times 10^3$ so the exponential can be dropped. The factor $(1-\eta^2+\beta_2)^{-1/4}$ can be evaluated at this value of η, but

$$\frac{(1-\eta^2)_{\min}}{\beta_2}\approx(4\pi C_1^2)^2\frac{|W_0|}{\omega}N_m^{-1/3}\gg 1, \tag{2.24}$$

so β_2 can be dropped yielding

$$I_1^{(1)}\approx(8\pi N)^{-1/2}(1-\eta^2)^{-5/4}e^{-N\epsilon(\eta)}. \tag{2.25}$$

The contribution of the second region of the integral in (2.12) can be written in a similar way to

$$I_1^{(2)}=(2\eta^2 q)^{-1}e^{N\eta q}\left[\frac{N}{2\pi}\right]^{1/2}\times\int_\eta^1\frac{d\eta'(1+\eta/\eta')}{(1-\eta'^2+\beta_2)^{1/4}}e^{-Ng(\eta')}, \tag{2.26}$$

where

$$g(\eta')=q\eta'-(1-\eta'^2)^{1/2}+\tfrac{1}{2}\ln\left[\frac{1+(1-\eta'^2)^{1/2}}{1-(1-\eta'^2)^{1/2}}\right]. \tag{2.27}$$

The dominant region of the η' integral is near the lower limit $\eta'\approx\eta$ where g is minimum. It is stationary at that point, so the Taylor series yields

$$g(\eta')\approx g(\eta)+\tfrac{1}{2}(\eta-\eta')^2 g'',\quad g(\eta)=\tfrac{1}{2}\ln\left[\frac{1+(1-\eta^2)^{1/2}}{1-(1-\eta^2)^{1/2}}\right],\quad g''=\eta^{-2}(1-\eta^2)^{-1/2}. \tag{2.28}$$

When this is substituted into the integrand of (2.26) and the slowly varying terms are evaluated at $\eta'=\eta$, the remaining integration can be performed with the result

$$I_1^{(2)}=\tfrac{1}{2}(1-\eta^2)^{-1/2}e^{-N\epsilon(\eta)}\times\operatorname{erf}\left[\left[\frac{N}{2}\right]^{1/2}\frac{(1-\eta^2)^{3/4}}{\eta(1+\eta)}\right], \tag{2.29}$$

where erf is the error function.[8]

The third region of the integral of (2.12) is evaluated with the use of (2.13) and (2.16) and the substitution $x'=N\eta'$. It can then be written

$$I_1^{(3)}=\frac{e^{Nqm}}{\eta^2 q}\left[\frac{N}{2\pi}\right]^{1/2}\int_1^\infty\frac{d\eta'(1+\eta/\eta')}{(\eta'^2-1+\beta_1)^{1/4}}\times e^{-Nq\eta'}e^{iN\Phi(\eta')}, \tag{2.30}$$

where

$$\Phi(\eta')=(\eta'^2-1)^{1/2}-\tan^{-1}(\eta'^2-1)^{1/2}. \tag{2.31}$$

The point $\eta'=1$ is a stationary phase as well as the dominant region of contribution due to the factor $e^{-Nq\eta'}$. We therefore substitute $\eta'=1+t/qN$ and expand about $t=0$ where possible. The result is

$$I_1^{(3)}=\pi^{-1/2}e^{-Nq(1-\eta)}\left[\frac{2}{Nq^3}\right]^{1/4}\times\operatorname{Re}e^{-i\pi/4}\int_0^\infty\frac{dt}{(t+\beta_1 qN/2)^{1/4}}e^{-t+i\alpha t^{3/2}} \tag{2.32}$$

where $\alpha=3^{-1}(2/q)^{3/2}N^{-1/2}$. The parameters in the integrand are neither large nor small, so it is left in this form.

We can compare the three contributions to I_1 by focusing on the exponents $Nq(1-n)$ in the last and $N\epsilon(\eta)$, (2.22), in the first two. These are both large but the first is always larger so the contribution $I_1^{(3)}$ can be neglected. The remaining terms are

$$I_1\approx\tfrac{1}{2}(1-\eta^2)^{-1/2}e^{-N\epsilon(\eta)}\times\left\{(2\pi N)^{-1/2}(1-\eta^2)^{-3/4}+\operatorname{erf}\left[\left[\frac{N}{2}\right]^{1/2}\frac{(1-\eta^2)^{3/4}}{\eta(1+\eta)}\right]\right\}. \tag{2.33}$$

The contribution of the two terms can be compared in the dominant region $N\approx 2N_m$, $\eta\approx 1-|W_0|/2\omega N_m$. The argument of the error function is about

$\frac{1}{2}(|W_0|)/\omega^{3/4}N_m^{-1/4}\approx 1.88$, at which point erf$\approx 0.99$. The first term is $(4\pi)^{-1/2}N_m^{1/4}(|W_0|/\omega)^{-3/4}\approx 0.075$. So the second term of (2.31) is the dominant one.

$$I_1 \approx \tfrac{1}{2}(1-\eta^2)^{-1/2}e^{-N\epsilon(\eta)} . \qquad (2.34)$$

We now turn to a comparison with the KFR theory for the same experiment. The only difference from T_N^A, (2.1), is that the vector **Q**, (2.2), is replaced by **q** in the KFR theory. Then in (2.4) the factor $(N-|\Delta W/\omega|)^{-1}I_1 \approx N^{-1}I_1$ is replaced by $(N-U_P/\omega)^{-1}I_0$ in the KFR theory. We then use $I_0=J_N(x)$ and (2.14) to get instead of $N^{-1}I_1$ the factor

$$(N-U_P)^{-1}(2\pi N)^{-1/2}(1-\eta^2)^{-1/4}e^{-N\epsilon(\eta)} \qquad (2.35)$$

for the KFR theory. Notice that the exponential factor here and in the adiabatic theory is the same, but the multiplicative factors are different. If we form the ratio of the T matrices we obtain

$$T^A/T^{\rm KFR} \approx \left[\frac{\pi N}{2}\right]^{1/2}\left[1-\frac{U_P}{N\omega}\right](1-\eta^2)^{-1/4}$$
$$\approx \frac{\pi^{1/2}}{2}N_m^{3/4}\left[\frac{|W_0|}{\omega}\right]^{-1/4} \approx 170 ,$$

where (2.23) has been used. Then the ionization rate from the adiabatic theory is a factor of $(170)^2\approx 3\times 10^4$ times larger than the KFR theory, which improves the comparison with numerical experiments that have been done.[9]

The total ionization rate is obtained from

$$W=\sum_N \int \frac{d^2q}{(2\pi)^3}\delta\left[\frac{q^2-q^2(N)}{2m}\right]|T_N^A|^2 , \qquad (2.36)$$

where the sum runs from the integer greater than N_m to infinity and $q(N)$ is given by (2.3). The sum can be converted to an integral with negligible error. Then the azimuthal and the radial integrations can be simply performed. We drop the first term of (2.32), replace the error function by unity, and make the substitution $N=N_m u$ to obtain

$$W=\frac{2\omega}{\pi N_m}\left[\frac{|W_0|}{\omega}\right]^3\int_0^{\pi}\sin\vartheta\, d\vartheta\int_1^{\infty}\frac{du}{u^2}a_0 q(N_m u)\frac{e^{-2N_m u\epsilon(\eta)}}{(1-\eta^2)} , \qquad (2.37)$$

where $a_0 q(N_m u)=[(\omega N_m/|W_0|)(u-1)]^{1/2}$ is expanded about its point of maximum contribution since $N_m \gg 1$ and the remaining part of the integrand can be evaluated at that point. We define $s=\sin\theta$ and

$$\mathcal{E}(u,s)=2u\epsilon(\eta)\approx\frac{2u}{3}(1-\eta^2)^{3/2}+O((1-\eta^2)^{5/2}) , \qquad (2.38)$$

the last step coming from the fact that $(1-\eta^2)$ will be small. More explicitly,

$$\mathcal{E}(u,s)=\frac{2}{3u^2}\left[u^2-\frac{4U_P s^2}{\omega N_m}(u-1)\right]^{3/2} . \qquad (2.39)$$

The derivatives, evaluated at $u=2$, $s=1$, are

$$\mathcal{E}_u=\tfrac{2}{3}\delta^{3/2},\quad \mathcal{E}_s=-4\delta^{1/2},\quad \mathcal{E}_{us}=-6\delta^{1/2} ,$$
$$\mathcal{E}_{ss}=4\delta^{-1/2},\quad \mathcal{E}_{uu}=\delta^{1/2},\quad \mathcal{E}(2,1)=\tfrac{8}{3}\delta^{3/2} , \qquad (2.40)$$

where $\delta=|W_0|/\omega N_m$ and terms of relative order δ have been dropped. The factor in the exponent (2.37) can then be written as

$$\mathcal{E}(u,s)=\left[\mathcal{E}(2,1)-\frac{\mathcal{E}_u^2}{2\mathcal{E}_{uu}}-\frac{1}{2}\frac{(\mathcal{E}_s-\mathcal{E}_u\mathcal{E}_{us}/\mathcal{E}_{uu})^2}{(\mathcal{E}_{ss}-\mathcal{E}_{us}^2/\mathcal{E}_{uu})}\right]$$
$$+\tfrac{1}{2}\mathcal{E}_{uu}\{u-2+\mathcal{E}_{uu}^{-1}[\mathcal{E}_{us}(s-1)+\mathcal{E}_u]\}^2+\tfrac{1}{2}\left[\mathcal{E}_{ss}-\frac{\mathcal{E}_{us}^2}{\mathcal{E}_{uu}}\right]\left[s-1+\frac{\mathcal{E}_s-\mathcal{E}_u\mathcal{E}_{us}/\mathcal{E}_{uu}}{\mathcal{E}_{ss}-\mathcal{E}_{us}^2/\mathcal{E}_{uu}}\right]^2+\cdots . \qquad (2.41)$$

This form indicates that the point of maximum contribution is not $N=N_m u=2N_m$ but is slightly shifted by an amount that depends upon $s=\sin\theta$.

The u integration in (2.35) can then be done, and in the limit $N_m\to\infty$ it is an integral over a Gaussian in the domain $-\infty$ to $+\infty$, for $s>\frac{5}{6}$. For $s<\frac{5}{6}$ the domain shrinks to zero, which says that the angular distribution is effectively confined to the region

$$\tfrac{5}{6}\le\sin\vartheta\le 1 . \qquad (2.42)$$

The u integration is then simply performed. The s integral, with this constraint is, of the form

$$2\int_{5/6}^{1}\frac{s\,ds}{(1-s^2)^{1/2}}\exp[-\tfrac{1}{2}N_m\mathcal{E}_{ss}(s-1)^2]$$
$$\approx(2N_m\mathcal{E}_{ss})^{-1/4}\Gamma(\tfrac{1}{4}) . \qquad (2.43)$$

Assembling all this we arrive at

$$W=\frac{\omega}{\pi^{1/2}}\frac{\Gamma(\frac{1}{4})}{2^{5/4}}\left[\frac{|W_0|}{\omega}\right]^{11/8}N_m^{5/8}$$
$$\times\exp-\left[\frac{2}{3}N_m^{-1/2}\left[\frac{|W_0|}{\omega}\right]^{3/2}\right], \qquad (2.44)$$

where the term in the set of large parentheses in (2.41) has been expanded in powers of δ and for simplicity only the first term $\frac{2}{3}\delta^{3/2}$ is retained. This is somewhat suspect since it is then multiplied by the large N_m and exponentiated.

The central result of this section is that the new form of the T matrix, introduced to remedy the formal difficulties of the KFR theory, can give much greater ionization rates in some cases. The experiment that motivated this calculation provided only relative ionization rates, and since the dominant exponents for I_1, and in the KFR theory, are identical, the electron energy spectra in the two theories are essentially the same. Therefore no experimental differentiation between the theories is now available. There exists a numerical model calculation,[9] which has been used to test the KFR theory, which shows it to be many orders of magntiude too small. The adiabatic theory then seems to be a correction in the right direction. We hope to expand upon this in the future.

Two subsidiary results of this calculation are not new. The first is that the energy distribution of the emerging electrons peaks at about $N=2N_m$. It is implicit in previous calculations,[1–3,7] but it is difficult for us to interpret in a manner other than that described above. The second is the peaking of the angular distribution about the plane of polarization. This is simply the statement that the electron emerges preferentially in the direction of the electric field (averaged over a period). The more quantitative statement, Eq. (2.42), is, we believe, new.

III. INITIAL *P* STATE AND MORE GENERAL *S* STATES

The simplest hydrogenlike bound P state has the form

$$u_{P\mu}(\mathbf{r})=(\pi a^5)^{-1/2}\mathbf{r}_\mu e^{-r/a}, \qquad (3.1)$$

where the binding energy is

$$|W_0|=(2ma^2)^{-1}, \qquad (3.2)$$

and we retain the estimates given in (2.9). For an unpolarized initial state, the squared T matrix must be averaged over the initial magnetic sublevels. It is simpler to do this first and then perform the spatial integral of (2.1). We obtain

$$|\bar{T}_N^A|^2=\frac{1}{3}|\sum_\mu|T_N^A(p\mu)|^2=\frac{64\pi e^2}{3a^5}\int\int\frac{d\beta\, d\beta'}{2\pi}\exp i[N(\beta-\beta')-x(\sin\beta-\sin\beta')]\frac{\mathbf{Q}(\beta)\cdot\mathbf{Q}(\beta')}{[Q^2(\beta)+a^{-2}]^2[Q^2(\beta')+a^2]^2} \qquad (3.3)$$

where

$$\mathbf{Q}(\beta)\cdot\mathbf{Q}(\beta')=q^2-\sqrt{2}m\omega\alpha_0 q_\perp(\cos\beta+\cos\beta')+\tfrac{1}{2}(m\omega\alpha_0)^2\cos(\beta-\beta'). \qquad (3.4)$$

We can use

$$\int\frac{d\beta}{2\pi}\frac{\cos\beta}{(1-\eta\cos\beta)^2}e^{iN(\beta-x\sin\beta)}=\eta^{-1}(I_2-I_1), \qquad (3.5)$$

where I_n is defined in (2.5), and by integration by parts

$$\int\frac{d\beta}{2\pi}\frac{\sin\beta}{(1-\eta\cos\beta)^2}e^{i(N\beta-x\sin\beta)}=i\eta^{-1}\left[N-\frac{x}{\eta}\right]I_1+i\frac{x}{\eta^2}J_N(x)=i\frac{x}{\eta^2}J_N(x) \qquad (3.6)$$

(we have used $N-x/\eta=0$ in the limit $\Delta W\to 0$). Then (3.3) can be rewritten as

$$|\bar{T}_N^A|^2=\frac{64\pi}{3}e^4a\left[\frac{|W_0|}{\omega}\right]^3N^{-4}\left[\left[N-N_m+\frac{U_P}{\omega\eta^2}\right]I_2^2-2I_1I_2\left[N\eta+\frac{U_P}{\omega\eta^2}\right]+\frac{U_P}{\omega\eta^2}I_1^2+\frac{U_PN^2}{\omega\eta^2}J_N^2(x)\right], \qquad (3.7)$$

where (2.6) and (2.7) have been used. Then we use $U_P/\omega\approx N_m$, $\eta\approx 1$, and anticipate that $I_2\gg I_1$ to obtain

$$|\bar{T}_N^A|^2\approx\frac{64\pi}{3}e^4a\left[\frac{|W_0|}{\omega}\right]^3N^{-4}[NI_2^2+N_mN^2J_N^2(x)]. \qquad (3.8)$$

The definitions (2.5) of I_n immediately yield

$$I_2=\frac{\partial}{\partial\eta}(\eta I_1), \qquad (3.9)$$

when x and η are treated as independent parameters. We use (2.12) for I_1 and (2.10a) let $x'=N\eta'$ to obtain

$$I_2=\frac{N}{2(1-\eta^2)^{1/2}}\int_0^\infty d\eta' J_N(N\eta')e^{-Nq|\eta-\eta'|}\left[\frac{\eta}{1-\eta^2}(1+\eta/\eta')+\eta'^{-1}+\frac{N}{\eta^2(1-\eta^2)^{1/2}}|\eta-\eta'|(1+\eta/\eta')\right]. \tag{3.10}$$

This is evaluated by a technique similar to that used for I_1, (2.12). Again the dominant contributions come from the first two regions of the integrand, $0<\eta'<\eta$ and $\eta<\eta'<1$. An intermediate result is written as

$$I_2=(1-\eta^2)^{-1}I_1+\frac{N^{3/2}e^{-N\epsilon(\eta)}}{(2\pi)^{1/2}(1-\eta^2)^{5/4}}\times\left[\frac{1}{4\eta^2q^2}+\frac{1}{Ng''}(1-e^{-N/2g''(1-\eta)})\right], \tag{3.11}$$

where g'' is given in (2.28). The exponential containing g'' is set equal to zero, which is the same approximation as replacing the error function in (2.34) by unity. The second term in the last set of large parentheses (from region 2) dominates the first (from region 1), just as it did in the evaluation of I_1. Thus we obtain, using the limiting form of (2.34),

$$I_2=e^{-N\epsilon(\eta)}\left[(1-\eta^2)^{-3/2}+\left[\frac{N}{2\pi}\right]^{1/2}(1-\eta^2)^{3/4}\right], \tag{3.12}$$

where the two terms are similar in magnitude. This is substituted into the set of square brackets in (3.8) and (2.14) is used for J_N. The I_2^2 term is seen to dominate the J_N^2 term by a factor of $(1-\eta^2)^{-1}$, so the J_N^2 term is dropped. We obtain

$$|\overline{T}_N^A|^2=\frac{64\pi}{3}e^4a\left[\frac{|W_0|}{\omega}\right]^3\frac{e^{-2N\epsilon(\eta)}}{N^3(1-\eta^2)^3}\times\left[1+\left[\frac{N}{2\pi}\right]^{1/2}(1-\eta^2)^{3/4}\right]^2. \tag{3.13}$$

At this point it is possible to compare this to the analogous result of the KFR theory:

$$|\overline{T}_N^{\text{KFR}}|^2=\frac{64\pi}{3}e^4a\left[\frac{|W_0|}{\omega}\right]^3\frac{e^{-2N\epsilon(\eta)}}{(N-N_m)^3}\times[2\pi N(1-\eta^2)^{1/2}]^{-1}. \tag{3.14}$$

The ratio of the two, evaluated $N=2N_m$, which is the approximate peak of the exponential, is

$$\frac{|\overline{T}_N^A|^2}{|\overline{T}_N^{\text{KFR}}|^2}=\frac{\pi}{2}\frac{N_m}{(1-\eta^2)^{5/2}}\left[1+\left[\frac{N_m}{\pi}\right]^{1/2}(1-\eta^2)^{3/4}\right]^2. \tag{3.15}$$

If this is evaluated at the approximate point of maximum exponential, (2.23), then

$$\frac{|\overline{T}_N^A|^2}{|\overline{T}_N^{\text{KFR}}|^2}\approx 10^9. \tag{3.16}$$

The transition rate, evaluated in a fashion similar to that used for (2.44), is

$$W=\frac{2^{15/4}}{\beta\pi^{1/2}}\Gamma(\tfrac{1}{4})\omega\left[\frac{Ry}{W_0}\right]^2N_m^{11/8}\left[\frac{|W_0|}{\omega}\right]^{3/8}\times\left[1+\pi^{-1/2}N_m^{-1/4}\left[\frac{|W_0|}{\omega}\right]^{3/4}\right]^2\times\exp-\left[\tfrac{2}{3}N_m^{-1/2}\left[\frac{|W_0|}{\omega}\right]^{3/2}\right]. \tag{3.17}$$

Again the exponential factor of (3.17) is the same as for the KFR result but the prefactors are very different.

We now turn to more general S state given by a linear combination of Slater orbitals

$$u_s=\sum_{\mu\nu}A_{\mu\nu}r^\mu\exp-\frac{r}{a_\nu}. \tag{3.18}$$

The last integral in (2.1) then becomes

$$\int d^3r\,e^{-i\mathbf{Q}\cdot\mathbf{r}}V(r)u_s(r)=\sum_{\mu\nu}A_{\mu\nu}\left[W_0-\frac{Q^2}{2m}\right]\frac{4\pi}{Q}(\mu+1)!\text{Im}(a_\nu^{-1}-iQ)^{-\mu-2}, \tag{3.19}$$

where we have used the fact that $u(r)$ satisfies a single-particle Schrödinger equation containing the potential V. [Parenthetically, we note that the right side of (3.19) is unchanged when V is a nonlocal potential. In order to demonstrate this, one must return to the derivation of Ref. 4 and replace V with a nonlocal potential in its *gauge-invariant form.*] The dominant contributions in the integrals evaluated above come from large Q, so a power series of Q^{-1} in (3.19) yields

$$\sum_\nu\left[W_0-\frac{Q^2}{2m}\right]8\pi\left[\frac{A_{0\nu}}{a_\nu}-A_{1\nu}\right](Q^2+a_\nu^{-2})^{-2}+O(Q^{-6}).$$

The remaining integral over β produces $I_1^{(\nu)}$, where this is the I_1 integral (2.5) that results when a_0 is replaced by a_ν. It is not difficult to see that the only change is to replace η, (2.7) by η_ν such that $\delta_\nu \approx 1-\eta_\nu^2 \approx (|W_\nu|/\omega)1/N_m$ replaces δ, (2.38) where

$$|W_\nu| = (2ma_\nu^2)^{-1} .$$

The exponent in (2.37) is, however, unchanged. Then the resultant change in I_1 is to replace the factor $(1-\eta^2)^{-1/2}$ with $(1-\eta_\nu^2)^{-1/2}$, i.e., to multiply I_1 by (a_ν/a_0). If we assemble the steps in this argument, we see that the only change in the S-state T matrix is to replace I_1 in T^A by

$$I_1 \to \frac{a_0^{3/2}}{\sqrt{2N}} \sum_\nu \frac{(A_{0\nu} - 2A_{1\nu} a_\nu)}{(1-\eta^2)^{5/4}} e^{-N\epsilon(\eta)} .$$

This does not significantly alter the numerical conclusions reached in Sec. II.

ACKNOWLEDGMENTS

This research was supported by the National Science Foundation Grant No. PHY-8807781, and by The City College of the City University of New York, Faculty Research Award No. 665287.

[1]L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1945 (1964) [Sov. Phys.—JETP **20**, 1307 (1965)].
[2]F. H. M. Faisal, J. Phys. B **6**, L89 (1973).
[3]H. R. Reiss, Phys. Rev. A **22**, 1786 (1980).
[4]M. H. Mittleman, Phys. Rev. A **40**, 463 (1989).
[5]P. B. Corkum, N. H. Burnett, and F. Brunel, Phys. Rev. Lett. **62**, 1259 (1989).
[6]I. J. Bersons, J. Phys. B **8**, 3078 (1975).
[7]See, for example, M. H. Mittleman, *Introduction to the Theory of Laser-Atom Interaction* (Plenum, New York, 1982).
[8]Handbook of Mathematical Functions, Natl. Bur. Stand. Appl. Math. Ser. No. 55, edited by M. Abramowitz and I. A. Stegun (U.S. GPO, Washington, D.C., 1964).
[9]J. Javanainen (private communication).