PHYSICAL REVIEW A VOLUME 42, NUMBER 7 1 OCTOBER 1990

Continuum fields in quantum optics

K. J. Blow, Rodney Loudon,* and Simon J. D. Phoenix
British Telecom Research Laboratories, Martlesham Heath, Ipswich, IP5 7RE, United Kingdom

T. J. Shepherd
Royal Signals and Radar Establishment, St. Andrews Road, Malvern, Worcester, WR14 3PS, United Kingdom
(Received 3 May 1990)

We formulate the quantum theory of optical wave propagation without recourse to cavity quantization. This approach avoids the introduction of a box-related mode spacing and enables us to use a continuum frequency space description. We introduce a complete orthonormal set of operators that can describe states of finite energy. The set is countable and the operators have all the usual properties of the single-mode frequency operators. With use of these operators a generalization of the single-mode normal-ordering theorem is proved. We discuss the inclusion of material dispersion and pulse propagation in an optical fiber. Finally, we consider the process of photodetection in free space, concluding with a discussion of homodyne detection with both local oscillator and signal fields pulsed.

I. INTRODUCTION

Calculations in quantum optics have traditionally made use of a form of quantum electrodynamics in which the field is assumed to be confined within an optical cavity. The field is quantized in terms of a complete set of discrete eigenmodes of the cavity. When appropriate boundary conditions are applied at the cavity walls, the modes may have the form of a standing or running wave. The results of calculations are often independent of the size and shape of the assumed cavity, so that the discrete mode theory can sometimes be used to interpret experiments that are not contained within any real cavity.

There are, however, other experiments for which the assumption of some notional cavity produces quite spurious dependence of calculated quantities on the size or even the mere existence of the optical cavity. Thus, for example, in nonlinear optical processes such as self-phase modulation, the magnitudes of those effects by which the quantum theory differs from a classical calculation depend on the dimensions of the quantization cavity,[1] which has no real existence in the corresponding experiments. Again, in the theory of photodetection of an optical field in a closed cavity, it is necessary to take account of the depletion of the field caused by the destruction of photons in the photodetection process itself.[2-4] It is awkward to apply this theory to photodetection in free space and there has been some controversy over the reconciliation of photodetection theories in these two environments.[5-7]

Some optical experiments do, of course, employ a confined region of space, often in the form of a Fabry-Pérot cavity, and for these it is appropriate to use the discrete-mode formalism. However, the vast majority of optical experiments have no identifiable cavity, but rather the optical energy flows from sources through some kind of interaction region to a set of detectors. The positions of sources and detectors do not themselves define an optical cavity since they are part of a continuous unidirectional flow of optical energy with no significant reflection or recycling. The detection process does not deplete the photon number but rather acts as a sink that balances the optical energy production by the light sources. For systems of this nature, it is preferable to quantize the electromagnetic field in free space with a set of eigenmodes characterized by a continuous wave vector. The aims of this and a subsequent publication are to set up and apply a continuous-mode quantum theory of the electromagnetic field.

It is, of course, possible to formulate field theories in three-dimensional infinite space, as is done in standard relativistic quantum field theory. However, the arrangement of a typical optical experiment makes it feasible to introduce quite drastic simplifications while retaining a realistic model of the physical system. Most importantly, with a single light beam traveling in a straight line, perhaps in an optical fiber, when transverse effects are unimportant it is advantageous to take a quantization axis of infinite extent parallel to the beam direction and to retain a finite cross-sectional area A which is determined by the fiber mode or the geometry of the experiment. Also, since most optical experiments use a narrow-band source, in the sense that

$$B \ll \omega_0 , \tag{1.1}$$

where ω_0 is the central frequency of the bandwidth B, then further simplifications of the theoretical model can be made.

Previous authors[8-11] have presented quantum treatments of propagation in one-dimensional optical systems where the formulation has been in terms of the spatial evolution of temporal modes. Kennedy and wright[10] examined a quasi-one-dimensional treatment via the paraxial wave equation. Caves and Crouch[9] studied a one-dimensional parametric oscillator using a traveling wave formalism on a continuous frequency domain. In this paper we examine the properties of such a one-dimensional formalism. We derive the necessary operators and states which enable a complete description of transient phenomena.

II. FORMALISM

A. Continuous-mode operators

We begin by establishing the correspondence between the discrete modes of a one-dimensional cavity and the continuous modes in the absence of a cavity. This enables us to obtain expressions for the continuous-field operators and for important quantities in a propagation theory such as the Poynting vector and the photon flux. Fourier transforms can then be used to obtain these operators in time rather than frequency.

Consider an empty optical cavity of length L parallel to the z axis. For plane waves that propagate parallel to the axis, the eigenmodes are separated by wave vector

$$\Delta k=\frac{2\pi}{L} \tag{2.1}$$

and frequency

$$\Delta\omega=\frac{2\pi c}{L} \tag{2.2}$$

where periodic-boundary conditions have been used to provide running waves. Different modes of the cavity, labeled by i and j, have frequencies given by different integer multiples of the mode spacing (2.2). The field in the cavity is quantized by associating independent quantum harmonic oscillators with the different modes. Their creation and destruction operators satisfy the usual independent boson commutation relations

$$[\hat{a}_i,\hat{a}_j^\dagger]=\delta_{ij} \ . \tag{2.3}$$

The mode spectrum becomes continuous as $L\to\infty$ and $\Delta\omega\to 0$, and in the limit it is convenient to transform to continuous-mode operators according to

$$\hat{a}_i\to(\Delta\omega)^{1/2}\hat{a}(\omega) \ . \tag{2.4}$$

The Kronecker and Dirac δ functions are correspondingly related by

$$\delta_{ij}\to\Delta\omega\,\delta(\omega-\omega') \ , \tag{2.5}$$

and the commutation relation (2.3) is converted to the usual continuous-mode form

$$[\hat{a}(\omega),\hat{a}^\dagger(\omega')]=\delta(\omega-\omega') \ . \tag{2.6}$$

Sums over discrete quantities are converted to integrals over continuous frequency according to

$$\sum_i\to\frac{1}{\Delta\omega}\int d\omega \ . \tag{2.7}$$

The continuous-mode quantized electric and magnetic field operators are obtained from their discrete-mode counterparts by this procedure, with the respective results

$$\hat{E}^{+}(z,t)=i\int d\omega\left[\frac{\hbar\omega}{4\pi\varepsilon_0 cA}\right]^{1/2}\times\hat{a}(\omega)\exp\left[-i\omega\left[t-\frac{z}{c}\right]\right] , \tag{2.8}$$

and

$$\hat{B}^{+}(z,t)=i\int d\omega\left[\frac{\hbar\omega}{4\pi\varepsilon_0 c^3A}\right]^{1/2}\times\hat{a}(\omega)\exp\left[-i\omega\left[t-\frac{z}{c}\right]\right] , \tag{2.9}$$

where the field operators have been divided into their creation and destruction operator parts,

$$\hat{E}(z,t)=\hat{E}^{+}(z,t)+\hat{E}^{-}(z,t) \ , \quad \hat{B}(z,t)=\hat{B}^{+}(z,t)+\hat{B}^{-}(z,t) \ , \tag{2.10}$$

and the $-$ superscript operators are the Hermitian conjugates of the $+$ superscript operators given in (2.8) and (2.9). Only the parts of the field that correspond to propagation in the positive z direction are included in (2.8) and (2.9), and the ranges of integration are from 0 to ∞ (see the Appendix for details of the relation to standard three dimensional quantum field theory). The electric and magnetic fields are, of course, vectors with two independent polarization directions for $\hat{\mathbf{E}}$ in the xy plane and $\hat{\mathbf{B}}$ also lies in the plane at right angles to $\hat{\mathbf{E}}$. The electric and magnetic fields are generally taken to be oriented in the x and y directions, respectively, and the field operators will be written as scalars, as in the above equations.

The energy content of a propagating field is conveniently expressed by its intensity, rather than the energy density used for standing wave fields in cavities. We accordingly define a normally ordered Poynting vector operator

$$\hat{S}(z,t)=\frac{1}{\mu_0}[\hat{E}^{-}(z,t)\hat{B}^{+}(z,t)+\hat{B}^{-}(z,t)\hat{E}^{+}(z,t)] = \frac{\hbar}{2\pi A}\int d\omega\int d\omega'(\omega\omega')^{1/2}\hat{a}^\dagger(\omega)\hat{a}(\omega')\times\exp\left[i(\omega-\omega')\left[t-\frac{z}{c}\right]\right] \tag{2.11}$$

oriented parallel to the z axis. This somewhat complicated operator takes a very simple form upon integration over all time, to give the total energy flowing through a plane of constant z in the form

$$A\int_{-\infty}^{\infty}dt\,\hat{S}(z,t)=\int d\omega\,\hbar\omega\,\hat{a}^\dagger(\omega)\hat{a}(\omega) \ . \tag{2.12}$$

The physical interpretation of this result is that over the span of all time, all of the energy contained in the field must pass each point on the z axis. Similarly, at a given instant of time, the total-energy flow over the entire length of the z axis is

$$A\int_{-\infty}^{\infty}dz\,\hat{S}(z,t)=c\int d\omega\,\hbar\omega\,\hat{a}^\dagger(\omega)\hat{a}(\omega) \ , \tag{2.13}$$

with a similarly transparent physical interpretation.

The range of integration over ω in the above expressions strictly extends only from 0 to ∞, since the frequencies are defined to be positive. However, the range can be extended from $-\infty$ to ∞ without significant errors when the excitation bandwidths of the states of the radiation field to which the operators are applied satisfy the inequality (1.1). The need to satisfy (1.1) should be kept in mind when applying the theory that follows; for example, narrow bandwidth approximations cannot be made in formulating theories of the intrinsic lack of exact localization in the photodetection of individual photons.[12] If the range of integration *is* extended however, it is useful to define Fourier transformed operators

$$\hat{a}(t)=(2\pi)^{-1/2}\int d\omega\,\hat{a}(\omega)\exp(-i\omega t)\ . \tag{2.14}$$

Their commutation relation obtained with the use of (2.6) is

$$[\hat{a}(t),\hat{a}^{\dagger}(t')]=\delta(t-t')\ . \tag{2.15}$$

The narrow bandwidth approximation can be taken further by removing the square-root frequencies from the integrand of the Poynting operator (2.11). Then with ω_0 defined as for (1.1),

$$A\hat{S}(z,t)=\hbar\omega_0\hat{a}^{\dagger}\left[t-\frac{z}{c}\right]\hat{a}\left[t-\frac{z}{c}\right]\ . \tag{2.16}$$

The operator on the right-hand side of this equation clearly represents the flux of the light beam in units of photons per unit time and it is convenient to define a flux operator

$$\hat{f}(t)=\hat{a}^{\dagger}(t)\hat{a}(t)\ . \tag{2.17}$$

The mean flux is

$$f(t)\equiv\langle\hat{f}(t)\rangle =\frac{1}{2\pi}\int d\omega\int d\omega\langle\hat{a}^{\dagger}(\omega)\hat{a}(\omega')\rangle\exp[i(\omega-\omega')t]\ . \tag{2.18}$$

In the case of stationary light beams, the frequency correlation function can be written in the form

$$\langle\hat{a}^{\dagger}(\omega)\hat{a}(\omega')\rangle=2\pi h(\omega)h^{*}(\omega')\delta(\omega-\omega')\ , \tag{2.19}$$

and the mean flux is time independent with the value

$$f(t)=\int d\omega\, f(\omega)\equiv F\ , \tag{2.20}$$

where $f(\omega)=|h(\omega)|^2$ is the dimensionless mean flux per unit angular frequency bandwidth at angular frequency ω. In general, the total number of photons that pass each point on the z axis over the whole of time is represented by the dimensionless operator

$$\hat{N}=\int dt\,\hat{a}^{\dagger}(t)\hat{a}(t)=\int d\omega\,\hat{a}^{\dagger}(\omega)\hat{a}(\omega)\ . \tag{2.21}$$

Note that $N=\langle\hat{N}\rangle$ is infinite for any stationary light beam, as would be expected for a state with a nonzero flux for all time.

B. Noncontinuous basis functions

It will be convenient to express the operators $\hat{a}(\omega)$, defined on a one-dimensional continuous frequency space, in terms of a linear superposition of operators associated with an arbitrary noncontinuous set of basis functions,[11,13] which are not necessarily modes of the system. Thus, let $\phi_i(\omega)$ be a complete orthonormal set of functions on ω,

$$\int d\omega\,\phi_i(\omega)\phi_j^{*}(\omega)=\delta_{ij}\ , \tag{2.22}$$

$$\sum_i\phi_i^{*}(\omega)\phi_i(\omega')=\delta(\omega-\omega')\ , \tag{2.23}$$

where i and j label the members of the denumerably infinite set. For our purposes the set need only be complete for square integrable functions and can be chosen to take advantage of the features of any particular application of the theory.

Each function in the orthonormal set is assigned a destruction operator defined by

$$\hat{c}_i=\int d\omega\,\phi_i^{*}(\omega)\hat{a}(\omega)\ . \tag{2.24}$$

The inverse relation obtained with the use of (2.23) is

$$\hat{a}(\omega)=\sum_i\phi_i(\omega)\hat{c}_i \tag{2.25}$$

and the commutator obtained with the use of (2.6) and (2.22) is

$$[\hat{c}_i,\hat{c}_j^{\dagger}]=\delta_{ij}\ . \tag{2.26}$$

The operators defined by (2.24) therefore represent a set of independent bosons. It is likewise straightforward to obtain the commutators

$$[\hat{a}(\omega),(\hat{c}_i^{\dagger})^n]=n\phi_i(\omega)(\hat{c}_i^{\dagger})^{n-1}\ , \tag{2.27}$$

$$[(\hat{c}_i)^n,\hat{a}^{\dagger}(\omega)]=n\phi_i^{*}(\omega)(\hat{c}_i)^{n-1}\ , \tag{2.28}$$

between the continuous and noncontinuous operators.

The above relations can be converted to equivalent time-dependent forms in cases where the narrow bandwidth assumption is a valid approximation. Thus the Fourier transformed basis functions defined by

$$\phi_i(t)=(2\pi)^{-1/2}\int d\omega\,\phi_i(\omega)\exp(-i\omega t) \tag{2.29}$$

satisfy relations identical to (2.22) and (2.23) but with ω replaced by t. The destruction operator (2.24) then takes the form

$$\hat{c}_i=\int dt\,\phi_i^{*}(t)\hat{a}(t)\ , \tag{2.30}$$

with the inverse relation

$$\hat{a}(t)=\sum_i\phi_i(t)\hat{c}_i \tag{2.31}$$

where $\hat{a}(t)$ is defined by (2.14).

The flux operator (2.17) is expressed in terms of the noncontinuous operators according to

$$\hat{f}(t)=\sum_i\sum_j\phi_i^{*}(t)\phi_j(t)\hat{c}_i^{\dagger}\hat{c}_j\ , \tag{2.32}$$

and the total number operator (2.21) becomes

$$\hat{N}=\sum_i \hat{c}_i^\dagger \hat{c}_i \ . \tag{2.33}$$

The number states of the noncontinuous operators are defined in the usual way by operation on the vacuum state $|0\rangle$,

$$|n_i\rangle=(n_i!)^{-1/2}(\hat{c}_i^\dagger)^{n_i}|0\rangle, \quad n_i=0,1,2,\ldots \tag{2.34}$$

and they have the standard properties

$$\hat{c}_i|n_i\rangle=n_i^{1/2}|n_i-1\rangle \ , \tag{2.35}$$

$$\hat{c}_i^\dagger|n_i\rangle=(n_i+1)^{1/2}|n_i+1\rangle \ . \tag{2.36}$$

The $|n_i\rangle$ form a complete set of states on the space of the operator $\hat{c}_i$, and the product states

$$\prod_i |n_i\rangle \equiv |\{n_i\}\rangle \tag{2.37}$$

form a complete set of the one-dimensional continuous frequency space of ω.

A state in which the component associated with the operator $\hat{c}_i$ is in a number state and all the other components are in their vacuum states is denoted $|\{0\},n_i\rangle$. The properties of this state with respect to the continuous operators are

$$\hat{a}(\omega)|\{0\},n_i\rangle=\phi_i(\omega)n_i^{1/2}|\{0\},n_i-1\rangle \tag{2.38}$$

and

$$\hat{a}(t)|\{0\},n_i\rangle=\phi_i(t)n_i^{1/2}|\{0\},n_i-1\rangle \ . \tag{2.39}$$

It follows from (2.17) and (2.18) that the mean photon flux is

$$f(t)=|\phi_i(t)|^2 n_i \ . \tag{2.40}$$

The noncontinuous number state is *not* an eigenstate of the flux operator $\hat{f}(t)$ and the time-dependent flux accordingly has a nonzero variance. However, the number operator (2.33) has the eigenvalue equation

$$\hat{N}|\{0\},n_i\rangle=n_i|\{0\},n_i\rangle \tag{2.41}$$

and the total number of photons is, therefore, well defined, as would be expected. More generally, the state $|\{n_i\}\rangle$ in which all the noncontinuous components are in number states satisfies the eigenvalue equation

$$\hat{N}|\{n_i\}\rangle=\left[\sum_j n_j\right]|\{n_i\}\rangle \ . \tag{2.42}$$

III. SPECIFIC STATES OF THE FIELD

A. Coherent states

Continuum coherent states $|\{\alpha(\omega)\}\rangle$ are generated from the vacuum state $|0\rangle$ by a generalization of the usual displacement operator,

$$|\{\alpha(\omega)\}\rangle=\exp\left[\int d\omega[\alpha(\omega)\hat{a}^\dagger(\omega)-\alpha^*(\omega)\hat{a}(\omega)]\right]|0\rangle \ , \tag{3.1}$$

where $\alpha(\omega)$ is an arbitrary complex function of ω. It is not difficult to show with the use of the commutator (2.6) that this definition can be written in the equivalent form

$$|\{\alpha(\omega)\}\rangle=\exp\left[-\frac{1}{2}\int d\omega|\alpha(\omega)|^2 + \int d\omega\,\alpha(\omega)\hat{a}^\dagger(\omega)\right]|0\rangle \ , \tag{3.2}$$

where the vacuum state has the usual property

$$\hat{a}(\omega)|0\rangle=0 \ . \tag{3.3}$$

It can similarly be shown that the coherent state (3.1) is an eigenstate of the destruction operator,

$$\hat{a}(\omega)|\{\alpha(\omega)\}\rangle=\alpha(\omega)|\{\alpha(\omega)\}\rangle \ . \tag{3.4}$$

These relations can all be written equivalently in terms of time-dependent functions. Thus with

$$\alpha(t)=(2\pi)^{-1/2}\int d\omega\,\alpha(\omega)\exp(-i\omega t) \ , \tag{3.5}$$

the basic definition (3.1) can be written

$$|\{\alpha(\omega)\rangle \equiv |\{\alpha(t)\}\rangle =\exp\left[\int dt[\alpha(t)\hat{a}^\dagger(t)-\alpha^*(t)\hat{a}(t)]\right]|0\rangle \ , \tag{3.6}$$

and the eigenvalue equation is

$$\hat{a}(t)|\{\alpha(t)\}\rangle=\alpha(t)|\{\alpha(t)\}\rangle \ . \tag{3.7}$$

The photon flux (2.18) associated with the continuum coherent state is

$$f(t)=\langle\{\alpha(t)\}|\hat{f}(t)|\{\alpha(t)\}\rangle=|\alpha(t)|^2 \tag{3.8}$$

and the mean total number of photons from (2.21) is

$$N=\langle\hat{N}\rangle=\int dt\,|\alpha(t)|^2=\int d\omega|\alpha(\omega)|^2 \ . \tag{3.9}$$

It is sometimes the case that an experiment is performed with the light beam from a single-mode laser whose bandwidth is much smaller than that of any other components of the optical system. Such a "single-mode" coherent light beam can be represented by the above theory with complex functions

$$\alpha(\omega)=(2\pi F)^{1/2}\exp(i\theta)\delta(\omega-\omega_0) \ , \tag{3.10}$$

$$\alpha(t)=F^{1/2}\exp(-i\omega_0 t+i\theta) \ , \tag{3.11}$$

where F is the time-independent mean flux of photons per unit time, θ is its phase, and ω_0 is its frequency. The state represented by (3.10) has stationary statistics and is a cw light beam so that, as mentioned previously, the total number of photons is, of course, infinite.

The general coherent state (3.1) or (3.6) is readily expressed in terms of the noncontinuous set by means of (2.25). The exponent in the displacement operator from (3.1) thus takes the form

$$\int d\omega[\alpha(\omega)\hat{a}^\dagger(\omega)-\alpha^*(\omega)\hat{a}(\omega)]=\sum_i(\gamma_i\hat{c}_i^\dagger-\gamma_i^*\hat{c}_i) \ , \tag{3.12}$$

where

$$\gamma_i = \int d\omega\, \alpha(\omega)\phi_i^*(\omega) . \tag{3.13}$$

In view of the independence of different noncontinuous operators expressed by (2.26), the general coherent state (3.1) or (3.6) becomes

$$|\{\alpha(\omega)\}\rangle \equiv |\{\alpha(t)\}\rangle = \prod_i \exp(\gamma_i \hat{c}_i^\dagger - \gamma_i^* \hat{c}_i)|0\rangle$$
$$= \prod_i |\gamma_i\rangle \equiv |\{\gamma_i\}\rangle , \tag{3.14}$$

where the $|\gamma_i\rangle$ are coherent states associated with the operators $\hat{c}_i$. These have all the usual properties of Glauber coherent states, including the eigenvalue relation

$$\hat{c}_i|\gamma_i\rangle = \gamma_i|\gamma_i\rangle . \tag{3.15}$$

The eigenvalue $\alpha(\omega)$ of the continuum operator $\hat{a}(\omega)$ is expressed in terms of the individual noncontinuous contributions in accordance with the inverse of (3.13),

$$\alpha(\omega) = \sum_i \phi_i(\omega)\gamma_i , \tag{3.16}$$

and the mean total number from (2.33) or (3.9) breaks down into individual contributions as

$$N = \sum_i |\gamma_i|^2 . \tag{3.17}$$

The variance of the total photon number is

$$(\Delta N)^2 = N . \tag{3.18}$$

B. Number states

Define an operator

$$\hat{A}(\xi) = \int d\omega\, \xi^*(\omega)\hat{a}(\omega) , \tag{3.19}$$

where $\xi(\omega)$ is an arbitrary normalized complex function with

$$\int d\omega |\xi(\omega)|^2 = 1 . \tag{3.20}$$

The operator so defined has a commutator

$$[\hat{A}(\xi), \hat{A}^\dagger(\xi)] = 1 \tag{3.21}$$

and it can be used to construct number states in the usual way,

$$|n(\xi)\rangle = (n!)^{-1/2}[\hat{A}^\dagger(\xi)]^n|0\rangle . \tag{3.22}$$

This generalized number state can be expressed as a superposition of the number states $|n_i\rangle$ of the noncontinuous basis set discussed in Sec. II B. Thus with the use of (2.25),

$$\hat{A}(\xi) = \sum_i \xi_i^* \hat{c}_i \tag{3.23}$$

where

$$\xi_i = \int d\omega\, \xi(\omega)\phi_i^*(\omega) . \tag{3.24}$$

Indeed, it is, in general, possible for an arbitrary normalizable function $\xi(\omega)$ to construct a complete set $\phi_i(\omega)$ of orthonormal basis functions of which $\xi(\omega)$ is a member. This can be seen by considering the functions $\omega^n\xi(\omega)$ and using a Gram-Schmidt orthogonalization procedure to construct the set.[14] For example, if $\xi(\omega)$ is a Gaussian pulse the orthonormal set is the set of harmonic oscillator wave functions. The number states $|n(\xi)\rangle$ then become equivalent to one of the sets of number states $|n_i\rangle$ associated with the appropriate noncontinuous basis function, and the various results given in Sec. II B apply.

C. Noise

White noise is represented by a correlation function (2.19) in which the mean flux $f(\omega)$ per unit angular frequency bandwidth is a constant, f_0 say, so that

$$\langle \hat{a}^\dagger(\omega)\hat{a}(\omega')\rangle = 2\pi f_0 \delta(\omega-\omega') \tag{3.25}$$

and the corresponding time-dependent correlation function is

$$\langle \hat{a}^\dagger(t)\hat{a}(t')\rangle = 2\pi f_0 \delta(t-t') . \tag{3.26}$$

The mean flux $f(t)$ and the mean total number N are both infinite. If the noise is restricted to an angular frequency bandwidth B, the time-independent mean flux is

$$F = Bf_0 . \tag{3.27}$$

A Lorentzian noise spectrum is represented by a correlation function (2.19) in which

$$f(\omega) = \frac{\gamma/\pi}{(\omega_0-\omega)^2+\gamma^2} F . \tag{3.28}$$

The time-dependent correlation function in this case is

$$\langle \hat{a}^\dagger(t)\hat{a}(t')\rangle = F \exp[i\omega_0(t-t') - \gamma|t-t'|] . \tag{3.29}$$

The total number N is again infinite.

D. Squeezed states

Continuum squeezed vacuum states $|\{\eta(\omega)\}\rangle$ can be generated from the vacuum by a generalization of the usual discrete multimode squeezing operator,

$$|\{\eta(\omega)\}\rangle = \hat{S}(\{\eta(\omega)\})|0\rangle$$
$$= \exp\left[\int d\omega[\eta^*(\omega)\hat{a}(\omega)\hat{a}(2\Omega-\omega) - \eta(\omega)\hat{a}^\dagger(\omega)\hat{a}^\dagger(2\Omega-\omega)]\right]|0\rangle , \tag{3.30}$$

where $\eta(\omega)$ is an arbitrary dimensionless complex function of ω. The state given by (3.30) represents a field in which pairs of continuum modes are correlated about some central frequency Ω. The continuum creation and destruction operators transform under the continuum squeezing operator as follows:

$$\hat{a}(\omega)\rightarrow\cosh[r(\omega)]\hat{a}(\omega)-e^{i\chi(\omega)}\sinh[r(\omega)]\hat{a}^{\dagger}(2\Omega-\omega)\ ,$$
$$\hat{a}(2\Omega-\omega)\rightarrow\cosh[r(\omega)]\hat{a}(2\Omega-\omega)-e^{i\chi(\omega)}\sinh[r(\omega)]\hat{a}^{\dagger}(\omega)\ , \tag{3.31}$$

where we have written the squeezing parameter $\eta(\omega)$ as

$$\eta(\omega)=r(\omega)\exp[i\chi(\omega)]\ . \tag{3.32}$$

The frequency correlation function (2.19) can now be easily calculated with the help of these transformations and we find

$$\langle\hat{a}^{\dagger}(\omega)\hat{a}(\omega')\rangle=\exp\{i[\chi(\omega')-\chi(\omega)]\}\times\sinh[r(\omega)]\sinh[r(\omega')]\delta(\omega-\omega') \tag{3.33}$$

with the time-independent mean flux

$$F=\frac{1}{2\pi}\int d\omega\,\sinh^2[r(\omega)]\ . \tag{3.34}$$

The state represented by (3.30) is a stationary light beam.

The transformation of the continuum squeezing operator given in (3.30) to the noncontinuous basis set proceeds in a similar fashion to that already described in Sec. III A for the transformation of continuum coherent states. Using (2.25) we find that the continuum squeezing operator $\hat{\mathcal{S}}(\{\eta(\omega)\})$ can be written in the noncontinuous basis as

$$\hat{\mathcal{S}}(\{\eta(\omega)\})=\exp\left[\sum_i\sum_j[\Gamma^*_{ij}(\Omega)\hat{c}_i\hat{c}_j-\Gamma_{ij}(\Omega)\hat{c}^{\dagger}_i\hat{c}^{\dagger}_j]\right] \tag{3.35}$$

where we have written

$$\Gamma^*_{ij}(\Omega)=\int d\omega\,\eta^*(\omega)\phi_i(\omega)\phi_j(2\Omega-\omega)\ . \tag{3.36}$$

The matrix Γ with elements given by (3.36) now contains the information concerning the pairwise correlation of modes implicit in the form of the continuum squeezing operator. Consider a unitary transformation labeled $\mathcal{T}$ which converts the matrix Γ to a diagonal matrix K. The elements of K are, of course, the eigenvalues of the matrix Γ. Applying this unitary transformation to the squeezing operator (3.35) we find that $\hat{S}$ transforms according to

$$\hat{\mathcal{S}}(\{\eta(\omega)\})\rightarrow\prod_i\exp[K^*_{ii}(\Omega)\hat{d}_i\hat{d}_i-K_{ii}(\Omega)\hat{d}^{\dagger}_i\hat{d}^{\dagger}_i]\ , \tag{3.37}$$

where we have defined the new vector $\hat{\mathbf{d}}=\mathcal{T}\hat{\mathbf{c}}$. It should be noted that the noncontinuous basis functions are also transformed under this procedure. This transformation shows that, in principle, the continuous-mode squeezing operator can be written as a product of independent noncontinuous-mode squeezing operators. The individual squeezing operators in (3.37) act on the noncontinuous operators according to the prescription

$$\hat{d}_i\rightarrow\mu_i\hat{d}_i-\nu_i\hat{d}^{\dagger}_i \tag{3.38}$$

where we have written the squeezing parameters as

$$\mu_i=\cosh[|K_{ii}(\Omega)|]\ ,$$
$$\nu_i=\exp\{i\arg[K_{ii}(\Omega)]\}\sinh[|K_{ii}(\Omega)|]\ . \tag{3.39}$$

The pairwise correlations are retained upon transformation to the noncontinuous basis; the matrix elements $K_{ii}(\Omega)$ containing the relevant information.

The continuum squeezing operator considered above correlates field modes around a central frequency Ω. In practice there will exist a distribution of such frequencies so that each mode is correlated with infinitely many modes. The squeezing operator in this case can be generalized to

$$\hat{\mathcal{S}}(\{\eta(\omega)\})=\exp\left[\int\int d\omega\,d\omega'[\eta^*(\omega,\omega')\hat{a}(\omega)\hat{a}(\omega')-\eta(\omega,\omega')\hat{a}^{\dagger}(\omega)\hat{a}^{\dagger}(\omega')]\right] \tag{3.40}$$

This operator will, in general, produce a nonstationary field from the vacuum and an example of such a field state is considered in Sec. VI C where homodyne detection with pulsed fields is discussed.

IV. NORMAL-ORDERING THEOREM

It is often necessary to evaluate expectation values of operators of the form $\exp(\hat{O})$ where

$$\hat{O}=\int d\omega\,g(\omega)\hat{a}^{\dagger}(\omega)\hat{a}(\omega) \tag{4.1}$$

and $g(\omega)$ is an arbitrary function. It is straightforward with the use of the basic commutator (2.6) to prove the relation

$$\hat{a}(\omega)\hat{O}^{\,n}=[\hat{O}+g(\omega)]^n\hat{a}(\omega)\ , \tag{4.2}$$

and it follows that

$$\hat{a}(\omega)\exp(\hat{O})=\exp[\hat{O}+g(\omega)]\hat{a}(\omega)\ . \tag{4.3}$$

Equivalent relations hold for the corresponding time-dependent operators.

It is useful to have a normally ordered form of the operator $\exp(\hat{O})$ and we here prove the relation

$$\exp\left[\int d\omega\,g(\omega)\hat{a}^{\dagger}(\omega)\hat{a}(\omega)\right]=\,:\exp\left[\int d\omega(e^{g(\omega)}-1)\hat{a}^{\dagger}(\omega)\hat{a}(\omega)\right]: \tag{4.4}$$

where the colons denote normal ordering. The discrete mode version of this normal-ordering theorem is, of course, well known,[15] and we adopt a similar method of proof in the continuous-mode case, by showing that arbitrary coherent state matrix elements of the two sides of (4.4) are identical. Consider

$$\langle\{\alpha(\omega)\}|\exp(\hat{O})|\{\alpha(\omega)\}\rangle=\langle\{\gamma_i\}|\exp(\hat{O})|\{\gamma_i\}\rangle\ , \tag{4.5}$$

where the conversion to noncontinuous coherent states is made in accordance with (3.14). The operator $\hat{O}$ is also converted to noncontinuous form with the use of (2.25),

$$\hat{O}=\sum_{i,j}g_{ij}\hat{c}^{\dagger}_i\hat{c}_j\ , \tag{4.6}$$

where

$$g_{ij}=\int d\omega\, g(\omega)\phi_i^*(\omega)\phi_j(\omega) \ . \tag{4.7}$$

Let $\mathcal{U}$ be the unitary transformation that converts g to a diagonal matrix G, and define the new vectors

$$\hat{\mathbf{d}}=\mathcal{U}\hat{\mathbf{c}} \text{ and } \boldsymbol{\delta}=\mathcal{U}\boldsymbol{\gamma} \ . \tag{4.8}$$

The matrix element (4.5) thus becomes

$$\prod_i \langle \delta_i | \exp[G_{ii}\hat{d}_i^\dagger \hat{d}_i] | \delta_i \rangle = \prod_i \langle \delta_i | :\exp(e^{G_{ii}}-1)\hat{d}_i^\dagger \hat{d}_i: | \delta_i \rangle \ , \tag{4.9}$$

where the standard discrete normal-ordering theorem[15] can be used since the unitary transformation ensures that the $\hat{d}_i$ operators satisfy independent boson commutation relations and the $|\delta_i\rangle$ are coherent states with respect to these operators. Use of the inverse unitary transformation together with the orthogonality property (2.23) and the definition (4.7) now converts (4.9) to the coherent state $|\{\alpha(\omega)\}\rangle$ matrix element of the right-hand side of (4.4). The validity of the normal-ordering theorem (4.4) is thus established.

V. MATERIAL DISPERSION

A. Field quantization in a dielectric

Consider the fields in a lossless dielectric material with real dielectric function $\epsilon(\omega)$ and refractive index $n(\omega)$ related by

$$\epsilon(\omega)=n(\omega)^2 \ . \tag{5.1}$$

It is a consequence of general causality requirements[16] that a dielectric function whose value differs from unity for some range of frequencies cannot be real for all frequencies. The assumption made here is that the imaginary part of $\epsilon(\omega)$ is negligible over the narrow band of frequencies to which the present theory is restricted. The optical excitations within this bandwidth have frequency-dependent phase velocity

$$v_F(\omega)=\frac{\omega}{k}=\frac{c}{n(\omega)} \tag{5.2}$$

and group velocity $v_G(\omega)$ defined by

$$\frac{1}{v_G(\omega)}=\frac{\partial k}{\partial\omega}=\frac{\partial[\omega n(\omega)/c]}{\partial\omega} \ . \tag{5.3}$$

The expressions (2.8) and (2.9) for the electric and magnetic field operators need to be modified in the presence of a dielectric, where the expression for the energy density involves the displacement field D, and the relation between B and E involves the refractive index.[16] The Poynting vector $\hat{S}(z,t)$ defined in (2.11) satisfies an equation of continuity

$$\frac{\partial}{\partial z}\hat{S}(z,t)=-\frac{\partial}{\partial t}\hat{U}(z,t) \ , \tag{5.4}$$

where

$$\frac{\partial \hat{U}}{\partial t}=\hat{E}^-\frac{\partial \hat{D}^+}{\partial t}+\frac{\partial \hat{D}^-}{\partial t}\hat{E}^+ +\mu_0^{-1}\hat{B}^-\frac{\partial \hat{B}^+}{\partial t} +\mu_0^{-1}\frac{\partial \hat{B}^-}{\partial t}\hat{B}^+ \ . \tag{5.5}$$

The normalization of the field operators is fixed by insisting that the normally ordered energy density operator $\hat{U}(z,t)$ should satisfy the requirement

$$A\int_{-\infty}^{\infty} dz\, \hat{U}(z,t)=\int d\omega\, \hbar\omega\, \hat{a}^\dagger(\omega)\hat{a}(\omega) \ . \tag{5.6}$$

The continuous-mode quantized field operators then take the forms

$$\hat{E}^+(z,t)=i\int d\omega \left[\frac{\hbar\omega}{4\pi\varepsilon_0 c A n(\omega)}\right]^{1/2} \times\hat{a}(\omega)\exp\left[-i\omega\left[t-\frac{n(\omega)z}{c}\right]\right] \tag{5.7}$$

and

$$\hat{B}^+(z,t)=i\int d\omega\left[\frac{\hbar\omega\, n(\omega)}{4\pi\varepsilon_0 c^3 A}\right]^{1/2} \times\hat{a}(\omega)\exp\left[-i\omega\left[t-\frac{n(\omega)z}{c}\right]\right] , \tag{5.8}$$

where the total operators are still given by (2.10), the polarization directions are as described after this equation, and only the positive-z propagation parts ($\omega\geq 0$) are included (see the Appendix for further details). The expression (5.7) and (5.8) differ from quantized field results recently derived.[9]

The Poynting vector operator defined in the first line of (2.11) now takes the form

$$\hat{S}(z,t)=\frac{\hbar}{4\pi A}\int d\omega\int d\omega' \left[\frac{\omega\omega'}{n(\omega)n(\omega')}\right]^{1/2}[n(\omega)+n(\omega')]\hat{a}^\dagger(\omega)\hat{a}(\omega')\exp\left[i(\omega-\omega')t-i[\omega n(\omega)-\omega' n(\omega')]\frac{z}{c}\right] . \tag{5.9}$$

The Poynting vector operator is substantially more complicated than in a vacuum, but it has appealing properties upon integration. Thus the total energy flowing through a plane of constant z over all time is

$$A\int_{-\infty}^{\infty} dt\hat{S}(z,t)=\int d\omega\, \hbar\omega\, \hat{a}^\dagger(\omega)\hat{a}(\omega) \ . \tag{5.10}$$

This is unchanged from (2.12) and it retains the physical significance that all of the energy contained in the field must, in the fullness of time, pass each point on the z axis. Similarly, the total-energy flow over the entire length of the z axis at a given instant of time is

$$A\int_{-\infty}^{\infty} dz\, \hat{S}(z,t)=\int d\omega\, \hbar\omega\, v_G(\omega)\hat{a}^\dagger(\omega)\hat{a}(\omega) \ . \tag{5.11}$$

The contribution of each frequency component to the flow is now correctly weighted by the appropriate group velocity, instead of the uniform velocity c that appears in the free-space result (2.13).

B. Pulse propagation

In the preceding section expressions (5.7) and (5.8) describe the propagation in a linear dispersive medium of the electric and magnetic field operators, respectively. In this section we adopt an alternative approach where we start with the partial differential equation obtained by expanding the propagation constant to second order in frequency. The two treatments are equivalent to this order of approximation when the narrow-bandwidth assumption is taken. Representing (to within a constant factor) the slowly varying field envelope by the operator $\hat{a}(z,t)$ and assuming a narrow bandwidth then it is easily shown that the equation of motion for the field envelope in a frame moving with the group velocity can be written as

$$i\frac{\partial}{\partial z}\hat{a}(z,t)+\frac{k''}{2}\frac{\partial^2}{\partial t^2}\hat{a}(z,t)=0 \ , \tag{5.12}$$

where k'' is the second derivative, with respect to frequency, of the propagation constant, evaluated at the central frequency. The solution can be obtained by Fourier transformation and we find that

$$\hat{a}(z,\omega)=\hat{a}(0,\omega)\exp\left[-\frac{i}{2}k''\omega^2 z\right] \ . \tag{5.13}$$

Let us assume, initially, that our input pulse is a continuum coherent state with a temporal intensity profile given by $|\alpha(t)|^2$. In frequency space the propagated field state is

$$|\{\psi(z,\omega)\}\rangle=\exp\left[\int d\omega[\alpha(\omega)\hat{a}^\dagger(z,\omega) -\alpha^*(\omega)\hat{a}(z,\omega)]\right]|0\rangle \ . \tag{5.14}$$

We therefore find, using (5.13) in the above expression, that the propagated field state remains a coherent state with a z-dependent amplitude given by

$$\alpha(z,\omega)=\alpha(\omega)\exp\left[\frac{i}{2}k''\omega^2 z\right] \ . \tag{5.15}$$

The spectral intensity profile $|\alpha(z,\omega)|^2$ is unchanged on propagation, but the temporal intensity profile broadens. As expected, the propagation of a continuum coherent state in a linear dispersive medium reproduces the classical result.

The solution for the propagation of a continuum squeezed vacuum state, as given by (3.30), shows that the propagated field state remains a squeezed state but with a z-dependent squeezing parameter given by

$$\eta(z,\omega)=\eta(\omega)\exp\{ik''z[\Omega^2+(\Omega-\omega)^2]\} \ . \tag{5.16}$$

If we now specialize to the case of a *discrete* two-mode squeezed vacuum state we find the dispersion does not affect the magnitude of the squeezing but rotates the error contour in the phase plane through an angle θ given by

$$\theta=k''(\Omega-\omega)^2 z/2 \ . \tag{5.17}$$

The treatment of this problem in the noncontinuous basis proceeds from the replacement

$$\hat{a}(z,t)=\sum_j \phi_j(z,t)\hat{c}_j \ . \tag{5.18}$$

The advantage of this treatment is that the functional dependence on z and t is contained in the c-number noncontinuous basis functions rather than the operators as is (5.12), for example. The propagation equation (5.12) now yields the following c-number differential equation for the noncontinuous basis functions,

$$i\frac{\partial}{\partial z}\phi_j(z,t)+\frac{k''}{2}\frac{\partial^2}{\partial t^2}\phi_j(z,t)=0 \ . \tag{5.19}$$

The solution is obtained, as before, by Fourier transformation and the frequency space solution is given by

$$\phi_j(z,\omega)=\phi_j(0,\omega)\exp\left[-\frac{i}{2}k''\omega^2 z\right] \ . \tag{5.20}$$

A state in which the component associated with the operator $\hat{c}_m$ is in a coherent state and all the other components are in the vacuum is denoted by $|\{0\},\gamma_m\rangle$. A classical input with a *normalized* temporal intensity profile of $|\phi_m(z,t)|^2$ can be modeled by such a state. Using the relation

$$\langle\gamma_m,\{0\}|\hat{c}_j^\dagger\hat{c}_k|\gamma_m,\{0\}\rangle=|\gamma_m|^2\delta_{mj}\delta_{km} \tag{5.21}$$

we find that the mean photon flux $f(z,t)$ is given by

$$f(z,t)=\langle\hat{a}^\dagger(z,t)\hat{a}(z,t)\rangle=|\phi_m(z,t)|^2|\gamma_m|^2 \ . \tag{5.22}$$

The temporal intensity profile is broadened in accordance with (5.20). Note that here the coherent state amplitude γ_m determines the peak intensity of the pulse

VI. PHOTODETECTION THEORY

A. Direct detection

Consider an arrangement in which an optical signal falls directly on to an infinitely fast photodetector (in practice, this requires a detector response time much shorter than the characteristic fluctuation time of the optical signal). The photocurrent operator is proportional to Poynting's vector, which can be taken in the form (2.16) for a narrow bandwidth excitation. The results of sets of measurements in which the photocurrent is integrated for periods T can be predicted by the use of an operator

$$\hat{M}=\int_\tau^{\tau+T}dt\,\hat{a}^\dagger(t)\hat{a}(t)=\int_\tau^{\tau+T}dt\,\hat{f}(t) \ . \tag{6.1}$$

Here τ is the start time of the measurements, the detector is placed at $z=0$, the entire cross section A of the light beam is assumed to be detected, and the Poynting operator (2.16) has been divided by $\hbar\omega_0$ so that $\hat{M}$ represents the number of photons that arrive at the detector during the integration time. It is easily shown with the use of (2.15) that

$$\hat{M}^2 = :\hat{M}^2: + \hat{M} , \tag{6.2}$$

where the colons again denote normal ordering.

In practice detectors respond with a quantum efficiency of less than unity and there may be some mode mismatch between the detector and the system area A. These effects can be modeled by introducing an effective quantum efficiency η, where $0 \le \eta \le 1$ is the probability that a photon is indeed detected and thus registered as a photocount. It can be inserted in the above theory by means of the replacement[17]

$$\hat{a}(t) \to \eta^{1/2}\hat{a}(t) + i(1-\eta)^{1/2}\hat{v}(t) , \tag{6.3}$$

where $\hat{v}(t)$ represents a vacuum-state mode that is introduced to preserve the commutator (2.15). Equivalently, the effective quantum efficiency enters as a factor of η that must be attached to $\hat{M}$ in expectation values expressed in normally ordered form. Thus

$$\hat{M} \to \eta\hat{M} \text{ and } \hat{M}^2 \to \eta^2 :\hat{M}^2: + \eta\hat{M} . \tag{6.4}$$

The mean and the variance of the photocount obtained by this procedure are, respectively,

$$\langle m \rangle = \eta \langle \hat{M} \rangle \tag{6.5}$$

and

$$(\Delta m)^2 = \eta^2 \langle (\Delta\hat{M})^2 \rangle + \eta(1-\eta)\langle \hat{M} \rangle , \tag{6.6}$$

where the angular brackets on the right denote quantum-mechanical expectation values with respect to the continuous-mode excitation.

The statistical properties of a field are conveniently characterized by Mandel's Q parameter

$$Q = [\langle (\Delta\hat{M})^2 \rangle - \langle \hat{M} \rangle] / \langle \hat{M} \rangle , \tag{6.7}$$

which has the value 0 for a Poisson distribution. The analogous detected parameter is

$$Q_D = [(\Delta m)^2 - \langle m \rangle] / \langle m \rangle = \eta Q . \tag{6.8}$$

These parameters are readily evaluated for the various field excitations treated in Sec. III. Thus for the continuum coherent state defined in (3.6), use of the property (3.7) gives

$$\langle m \rangle = \eta \int_\tau^{\tau+T} dt |\alpha(t)|^2 \tag{6.9}$$

and

$$Q_D = 0 \tag{6.10}$$

corresponding to the usual Poisson statistics of coherent light. Similarly for the continuum number state defined by (3.22), use of the property analogous to (2.38) gives

$$\langle m \rangle = \eta n(\xi) \int_\tau^{\tau+T} dt |\xi(t)|^2 \tag{6.11}$$

and

$$Q_D = -\eta \int_\tau^{\tau+T} dt |\xi(t)|^2 \tag{6.12}$$

corresponding to sub-Poissonian statistics. Finally, for the chaotic light described by (3.29),

$$\langle m \rangle = \eta F T . \tag{6.13}$$

The photocount variance is evaluated with use of the factorization property for chaotic light,

$$\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t')\hat{a}(t)\hat{a}(t') \rangle = \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle \langle \hat{a}^\dagger(t')\hat{a}(t') \rangle + \langle \hat{a}^\dagger(t)\hat{a}(t') \rangle \langle \hat{a}^\dagger(t')\hat{a}(t) \rangle , \tag{6.14}$$

and the resulting Q parameter is

$$Q_D = \eta F[\exp(-2\gamma T) - 1 + 2\gamma T]/2\gamma^2 T , \tag{6.15}$$

corresponding to super-Poissonian statistics.

B. Homodyne detection

Consider a balanced homodyne detector in which the light beam of interest is superposed on a local oscillator by combining them at a 50-50 beam splitter. The measured quantity is the difference in the photocurrents of two detectors placed in the output arms of the beam splitter, and it can be represented by the operator[18]

$$\hat{O} = i \int_\tau^{\tau+T} dt [\hat{a}^\dagger(t)\hat{a}_L(t) - \hat{a}_L^\dagger(t)\hat{a}(t)] , \tag{6.16}$$

where $\hat{a}_L^\dagger$ and $\hat{a}_L$ are the continuum creation and destruction operators of the local oscillator field and $\hat{a}^\dagger$ and $\hat{a}$ correspondingly for the signal field. The relation analogous to (6.2) is

$$\hat{O}^2 = :\hat{O}^2: + \int_\tau^{\tau+T} dt [\hat{a}_L^\dagger(t)\hat{a}_L(t) + \hat{a}^\dagger(t)\hat{a}(t)] , \tag{6.17}$$

where the commutator (2.15) is satisfied by both the independent local oscillator and signal operators.

The effective quantum efficiencies, assumed to be the same for the two detectors are introduced by transformations analogous to (6.3), and they lead to the replacements

$$\hat{O} \to \eta\hat{O} \tag{6.18}$$

and

$$\hat{O}^2 \to \eta^2 :\hat{O}^2: + \eta \int_\tau^{\tau+T} dt [\hat{a}_L^\dagger(t)\hat{a}_L(t) + \hat{a}^\dagger(t)\hat{a}(t)] . \tag{6.19}$$

The mean and the variance of the difference photocount are, respectively,

$$\langle m \rangle = \eta \langle \hat{O} \rangle \tag{6.20}$$

and

$$(\Delta m)^2 = \eta^2 \langle (\Delta\hat{O})^2 \rangle + \eta(1-\eta) \int_\tau^{\tau+T} dt \langle \hat{a}_L^\dagger(t)\hat{a}_L(t) + \hat{a}^\dagger(t)\hat{a}(t) \rangle . \tag{6.21}$$

Note that the detection noise represented by the integral term involves the *sum* of the two beam fluxes despite the differencing in the detection process.

The above results apply for the arbitrary signal and local oscillator. The local oscillator is now assumed to be a single mode coherent light beam represented by a complex amplitude

$$\alpha_L(t)=(F_L)^{1/2}\exp(-i\omega_0 t+i\theta_L)\ , \tag{6.22}$$

similar to (3.11), with a flux F_L. No assumptions are made about the nature of the signal light, except that it incorporates a coherentlike amplitude variation described by

$$\langle \hat{a}(t)\rangle=[f_c(t)]^{1/2}\exp(-i\omega_0 t+i\theta)\ , \tag{6.23}$$

where the coherent component of the flux is much smaller than the local oscillator flux,

$$f_c(t) \ll F_L\ . \tag{6.24}$$

If only the terms of highest order in F_L are retained in the various expectation values, the measurement operator $\hat{O}$ from (6.16) can be written in the form

$$\hat{O}=(F_L)^{1/2}\int_\tau^{\tau+T} dt\,\hat{X}(\chi,t) \tag{6.25}$$

where

$$\hat{X}(\chi,t)=\hat{a}^\dagger(t)\exp(i\chi-i\omega_0 t) +\hat{a}(t)\exp(-i\chi+i\omega_0 t) \tag{6.26}$$

is the quadrature operator[19] of the signal beam for phase angle

$$\chi=\frac{\pi}{2}+\theta_L\ . \tag{6.27}$$

The mean and the variance of the difference photocount obtained with the use of (6.19)–(6.21) are, respectively,

$$\langle m\rangle=\eta(F_L)^{1/2}\int_\tau^{\tau+T} dt\langle\hat{X}(\chi,t)\rangle \tag{6.28}$$

and

$$(\Delta m)^2=\eta^2 F_L\int_\tau^{\tau+T} dt\int_\tau^{\tau+T} dt'\langle :\hat{X}(\chi,t),\hat{X}(\chi,t'):\rangle+\eta F_L T\ , \tag{6.29}$$

where the short-hand notation

$$\langle :\hat{A},\hat{B}:\rangle=\langle :\hat{A}\,\hat{B}:\rangle-\langle\hat{A}\rangle\langle\hat{B}\rangle \tag{6.30}$$

has been used and the inequality (6.24) has been assumed.

The integral contribution to the variance (6.29) vanishes for the continuum vacuum state and for a coherent state signal where (3.7) holds, and consequently

$$(\Delta m)^2=\eta F_L T\ . \tag{6.31}$$

The continuum state shows squeezing if its photocount variance is smaller than the vacuum state value for some phase angles χ. The squeezing criterion is, therefore,

$$\int_\tau^{\tau+T} dt\int_\tau^{\tau+T} dt'\langle :\hat{X}(\chi,t),\hat{X}(\chi,t'):\rangle<0 \tag{6.32}$$

with respect to homodyne detection with integration time T and local oscillator phase $\theta_L=\chi-\pi/2$.

C. Homodyne detection with pulsed fields

The results of the preceding section apply to homodyne detection with a stationary, single-mode, local oscillator field. For homodyne detection of pulsed signals it is advantageous to use a pulsed local oscillator. Accordingly, we now consider the homodyne detection of a pulsed signal, described by the noncontinuous basis function $\phi_0(t)$, with a pulsed local oscillator described by $\phi_L(t)$, where the local oscillator field is in a continuum coherent state as described in Sec. III A. The signal field is assumed to be described by a set of noncontinuous operators $\hat{d}_i$ at the output of a nonlinear system and the signal field at the input to the system is described by a similar set of operators $\hat{c}_i$. The action of the nonlinear system is defined to be such as to squeeze one member of the input signal operators and to leave the others unchanged so that we have

$$\begin{aligned}\hat{d}_0&=\mu\hat{c}_0+\nu\hat{c}_0^\dagger\ ,\\ \hat{d}_i&=\hat{c}_i,\quad i>0\ .\end{aligned} \tag{6.33}$$

The initial state is defined by

$$|\{\alpha_L\},0\rangle\equiv|\{\alpha_L\}\rangle|0\rangle \tag{6.34}$$

which would give a squeezed vacuum state after the transformation (6.33). The input coherent local oscillator field can be expressed as

$$\alpha_L(t)=(N_L)^{1/2}\exp(i\theta_L)\phi_L(t)\ , \tag{6.35}$$

where ϕ_L is a normalized function which gives the complex local oscillator pulse shape, N_L is the mean total number of photons in the pulse, and θ_L is the externally controlled local oscillator phase. We also define the complex overlap integral of the signal and local oscillator pulses

$$\Gamma\exp(i\theta_\Gamma)=\int_\tau^{\tau+T} dt\,\phi_0^*(t)\phi_L(t)\ . \tag{6.36}$$

The calculation of the variance of the measured signal using (6.17) and (6.33)–(6.36) proceeds by expanding the signal continuum operators in the discrete basis. The problem for the signal field reduces to calculating expectation values of operators quadratic in $\hat{d}_i$. Since the input field is a vacuum state only the diagonal parts of these expectation values contribute and these are easily calculated using (6.35), (2.35), and (2.36). We then obtain the following expression for the variance of the measured signal:

$$\langle \hat{O}^2 \rangle = \int_{\tau}^{\tau+T} dt [N_L |\phi_L(t)|^2 + |\nu|^2 |\phi_0(t)|^2] + N_L \Gamma^2 \{ |\mu|^2 + |\nu|^2 - \mu\nu \exp[2i(\theta_L - \theta_\Gamma)] - \mu^* \nu^* \exp[-2i(\theta_L - \theta_\Gamma)] - 1 \} . \quad (6.37)$$

Note that we could have included a term in (6.37) to represent the response function of the detector but this can easily be included by using an effective local oscillator field. We can see that the phase θ_Γ of the overlap integral simply redefines the relative phase between signal and local oscillator at which maximum squeezing will occur. The usual squeezing criterion, in the limit of large N_L,

$$|\mu|^2 + |\nu|^2 - \mu\nu - \mu^* \nu^* < 1 \quad (6.38)$$

is retained but the magnitude of the effect is reduced by a factor Γ^2 due to the overlap integral (6.36). We can also conclude that optimum detected squeezing will occur when Γ takes its maximum value which occurs when

$$\phi_L(t) \propto \phi_0(t) \quad (6.39)$$

within the detection window and zero outside. Remembering the property that any complete orthonormal set can be constructed from an arbitrary normalizable function, we can see that (6.39) establishes the general result that the optimum condition for squeezing is that the signal and local oscillator pulses have the same shape and coherent phase structure within the detection window. The constant of proportionality is unity when the detection window encompasses the signal pulse. This is independent of the generation process and similar conclusions have been drawn for pulsed squeezing in a parametic oscillator.[20]

Finally we consider the squeezing spectrum for a pulsed signal which can be obtained from an operator similar to (6.16)

$$\hat{O}(\omega,\tau) = i \int_{\tau}^{\tau+T} dt \exp(i\omega t) \times [\hat{a}^\dagger(t)\hat{a}_L(t) - \hat{a}_L^\dagger(t)\hat{a}(t)] , \quad (6.40)$$

and the result is

$$\langle \hat{O}(\omega)\hat{O}(-\omega) \rangle = \int_{\tau}^{\tau+T} dt [N_L |\phi_L(t)|^2 + |\nu|^2 |\phi_0(t)|^2] + N_L \Gamma^2(\omega) \{ |\mu|^2 + |\nu|^2 - \mu\nu \exp[i(2\theta_L - \psi_+ - \psi_-)] - \mu^* \nu^* \exp[-i(2\theta_L - \psi_+ - \psi_-)] - 1 \} , \quad (6.41)$$

where the frequency-dependent overlap integral is defined as

$$\Gamma(\omega)\exp[i\psi_\pm(\omega)] = \int_{\tau}^{\tau+T} dt \exp(\pm i\omega t) \phi_0^*(t)\phi_L(t) . \quad (6.42)$$

The squeezing spectrum is thus proportional to the modulus squared of the overlap integral which gives a simple interpretation in two limits, we will also assume that the optimum local oscillator pulse (6.39) has been used. First, when the signal is entirely within the range of detection the squeezing spectrum is given by the Fourier transform of the signal intensity. Second, when the integration time is small compared with the signal duration the overlap integral becomes

$$\Gamma(\omega)\exp[i\psi_+(\omega)] = |\phi_0(\tau)|^2 \int_{\tau}^{\tau+T} dt \exp(i\omega t) \quad (6.43)$$

which is the simple adiabatic turning on and turning off of the source. Thus, (6.41) contains the squeezing spectrum from the adiabatic regime where a quasi-cw signal is observed to the averaging regime where the properties of a pulse are observed as a whole.

VII. CONCLUSIONS

The formalism presented in this paper is suited to the treatment of quantum optical systems where the propagation effectively takes place in one dimension, as in an optical fiber, with no restrictions on the optical length of the one-dimensional axis. The notional optical cavity traditionally used in the theories of such systems is thus dispensed with, and the allowed optical wave vectors form a continuum. The traditional field energy is replaced by Poynting's vector, which remains well defined for stationary field excitations whose total-energy content must be infinite.

Stationary field excitations, such as Lorentzian noise or the squeezed vacuum state, are conveniently handled by the continuum formalism of Sec. II A, but pulsed or transient excitations are more readily treated in terms of the noncontinuous basis functions of Sec. II B. Thus, for example, the amplitude spectrum of a pulse of a number state or coherent state light can generally be taken as one member of a complete set of noncontinuous basis functions, with much reduction in the complexity of subsequent calculations. The discrete nature of the noncontinuous basis function also simplifies the derivations of certain general properties of the infinite one-dimensional field theory, as illustrated by the proof of the normal-ordering theorem in Sec. IV.

The usefulness of the formalism has been illustrated by application to some basic problems in quantum optics. Thus in Sec. VI it has been shown that photodetection theory is more straightforwardly treated in the continuum formalism, with none of the difficulties sometimes encountered in theories that are quantized in a cavity of finite length. The homodyne detection of pulsed signals is also conveniently handled in terms of the noncontinuous basis functions. The calculation presented in Sec.

VI C demonstrates the manner in which the quantum-optical operator properties of the signal-local oscillator superposition are separated from the c-number pulse-shape overlap integral. In a subsequent publication[1] we shall apply the methods developed here to the theory of quantum effects in self-phase modulation, where spurious length dependences can appear in calculations that employ quantization in a finite cavity.

Finally, it should be emphasized that the continuum approach applies to the majority of optical experiments, where the overall flow of electromagnetic energy through the apparatus is not subject to longitudinal discrete-mode structure, and that the removal of the notional cavity is accomplished without any increase in the mathematical complexity of the formalism.

ACKNOWLEDGMENTS

We thank Matthew Collett for helpful discussions and important insights on the formalisms of continuum coherent states and number states.

APPENDIX

We show in this appendix how the quantized field expressions (2.8) and (2.9) are related to the standard expressions of quantum field theory. The vector potential operator in the Coulomb gauge derived by Bjorken and Drell[21] can be written

$$\hat{\mathbf{A}}(\mathbf{r},t)=\hat{\mathbf{A}}^{+}(\mathbf{r},t)+\hat{\mathbf{A}}^{-}(\mathbf{r},t) \tag{A1}$$

where

$$\hat{\mathbf{A}}^{+}(\mathbf{r},t)=\int d\mathbf{k}\left[\frac{\hbar}{16\pi^3\varepsilon_0 c|\mathbf{k}|}\right]^{1/2}\times\sum_{\lambda=1,2}\boldsymbol{\varepsilon}(\mathbf{k},\lambda)\hat{a}(\mathbf{k},\lambda)\times\exp(-ic|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{r}) \tag{A2}$$

and $\hat{\mathbf{A}}^{-}(\mathbf{r},t)$ is the Hermitian conjugate of this expression. Here λ denotes the two transverse polarizations, with

$$\boldsymbol{\varepsilon}(\mathbf{k},\lambda)\cdot\mathbf{k}=0\ , \tag{A3}$$

the unit polarization vectors satisfy

$$\begin{aligned}&\boldsymbol{\varepsilon}(\mathbf{k},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{k},\lambda')=\delta_{\lambda\lambda'}\ ,\\&\boldsymbol{\varepsilon}(\mathbf{k},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{k},\lambda')=(-1)^{\lambda}\delta_{\lambda\lambda'}\ ,\end{aligned} \tag{A4}$$

and the creation-destruction operator commutator is

$$[\hat{a}(\mathbf{k},\lambda),\hat{a}^{\dagger}(\mathbf{k}',\lambda')]=\delta_{\lambda\lambda'}\delta^3(\mathbf{k}-\mathbf{k}')\ . \tag{A5}$$

The fields considered in the present paper are defined in a spatial region of infinite extent parallel to the z axis but of finite cross-sectional area $\mathcal{A}$ in the xy plane. The x and y wave-vector components are thus restricted to discrete values and the three-dimensional integral in (A2) is converted according to

$$\int d\mathbf{k}\to\frac{(2\pi)^2}{\mathcal{A}}\sum_{k_x,k_y}\int dk_z\ . \tag{A6}$$

Only field excitations with $k_x=k_y=0$ are used in the paper. The summation in (A6) can, therefore, be removed, and putting $k_x=k$, the other required conversions are

$$\delta^3(\mathbf{k}-\mathbf{k}')\to\frac{\mathcal{A}}{(2\pi)^2}\delta(k-k')\ , \tag{A7}$$

$$\hat{a}(\mathbf{k},\lambda)\to\frac{\mathcal{A}^{1/2}}{2\pi}\hat{a}(k,\lambda)\ . \tag{A8}$$

The field operator (A2) thus becomes

$$\hat{\mathbf{A}}^{+}(z,t)=\int_{-\infty}^{\infty}dk\left[\frac{\hbar}{4\pi\varepsilon_0 c|k|\mathcal{A}}\right]^{1/2}\times\sum_{\lambda=1,2}\boldsymbol{\varepsilon}(k,\lambda)\hat{a}(k,\lambda)\times\exp[-ic|k|t+ikz]\ , \tag{A9}$$

and the commutator (A5) becomes

$$[\hat{a}(k,\lambda),\hat{a}^{\dagger}(k',\lambda')]=\delta_{\lambda\lambda'}\delta(k-k')\ . \tag{A10}$$

The field operators (2.8) and (2.9), with $\omega=c|k|$, are obtained from (A9) in accordance with

$$\hat{E}^{+}=-\frac{\partial\hat{A}^{+}}{\partial t},\quad \hat{B}^{+}=\frac{\partial\hat{A}^{+}}{\partial z}\ , \tag{A11}$$

where in (2.8) and (2.9) we have chosen the polarization vector to be parallel to the x axis and we have restricted the propagation to the positive z direction.

The above expressions are valid in free space. In the presence of a dielectric with frequency-dependent refractive index $n(\omega)$, the vector potential operator (A2) is replaced by

$$\hat{\mathbf{A}}^{+}(\mathbf{r},t)=\int d\mathbf{k}\left[\frac{\hbar v_G(\omega)}{16\pi^3\varepsilon_0 c\omega n(\omega)}\right]^{1/2}\times\sum_{\lambda=1,2}\boldsymbol{\varepsilon}(\mathbf{k},\lambda)\hat{a}(\mathbf{k},\lambda)\times\exp(-i\omega t+i\mathbf{k}\cdot\mathbf{r})\ , \tag{A12}$$

where the group velocity is defined in (5.3) and

$$\omega=c|\mathbf{k}|/n(\omega)\ . \tag{A13}$$

This expression is consistent with field operators for dispersive dielectric media derived previously.[22,23] The conversion to one dimension proceeds in accordance with (A6)–(A8), and the resulting replacement for (A9) is

$$\hat{A}^{+}(z,t)=\int_{-\infty}^{\infty}dk\left[\frac{\hbar v_G(\omega)}{4\pi\varepsilon_0 c\omega n(\omega)\mathcal{A}}\right]^{1/2}\times\sum_{\lambda=1,2}\boldsymbol{\varepsilon}(k,\lambda)\hat{a}(k,\lambda)\times\exp(-i\omega t+ikz)\ . \tag{A14}$$

When the wave-vector variable is converted to frequency in accordance with (5.2) so that

$$dk\to d\omega/v_G(\omega),\quad \hat{a}(k,\lambda)\to v_G^{1/2}(\omega)\hat{a}(\omega) \tag{A15}$$

and the polarization is again taken to be parallel to the x axis, the fields (5.7) and (5.8) now follow from (A11).

*Permanent address: Physics Department, Essex University, Colchester, CO4 3SQ, United Kingdom.

[1]K. J. Blow, R. Loudon, and S. J. D. Phoenix (unpublished).
[2]B. R. Mollow, Phys. Rev. **168**, 1896 (1968).
[3]M. O. Scully and W. E. Lamb, Phys. Rev. **179**, 368 (1969).
[4]T. J. Shepherd, Opt. Acta **28**, 567 (1981).
[5]M. D. Srinivas and E. B. Davies, Opt. Acta **29**, 981 (1981).
[6]L. Mandel, Opt. Acta **28**, 1447 (1981).
[7]M. D. Srinivas and E. B. Davies, Opt. Acta **29**, 235 (1982).
[8]I. Abram, Phys. Rev. A **35**, 4661 (1987).
[9]C. M. Caves and D. D. Crouch, J. Opt. Soc. Am. B **4**, 1553 (1987).
[10]T. A. B. Kennedy and E. M. Wright, Phys. Rev. A **38**, 212 (1988).
[11]N. Imoto (unpublished).
[12]See, for example, E. R. Pike and S. Sarkar, Phys. Rev. A **35**, 926 (1987), and earlier references therein.
[13]P. Kumar, O. Aytür, and J. Huang, Phys. Rev. Lett. **64**, 1015 (1990).
[14]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).
[15]W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
[16]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1977).
[17]H. P. Yuen and J. H. Shapiro, IEEE Trans. Inf. Theory **TI-26**, 78 (1980).
[18]M. J. Collett, R. Loudon, and C. W. Gardiner, J. Mod. Opt. **34**, 881 (1987).
[19]R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987).
[20]B. Yurke, P. Grangier, R. E. Slusher, and M. J. Potasek, Phys. Rev. A **35**, 3586 (1987).
[21]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).
[22]R. Loudon, Proc. Phys. Soc. **82**, 393 (1963).
[23]P. D. Drummond (unpublished).