

Heat transport in high-Rayleigh-number convection

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The heat flux (Nusselt number) as a function of Rayleigh number, $Nu \approx 0.3 Ra^{2/7}$, is deduced from the presence of a mean flow and the nesting of the thermal boundary layer within the viscous one. The numerical coefficients are obtained from those known empirically for turbulent boundary layers. The consistency of our assumptions as a function of Prandtl number limits this regime to $(10^7-10^8)Pr^{5/3} \lesssim Ra \lesssim (10^{13}-10^{15})Pr^4$. The Bolgiano-Obukhov $k^{-7/5}$ spectrum for the temperature fluctuations is inconsistent with a simple scaling treatment of the equations.

Recent experiments^{1,2} on convection in helium of unparalleled precision and parameter range have revealed a new scaling regime with exponents distinct from those obtained from the hypothesis of a marginally stable thermal boundary layer by Malkus and Howard.^{3,4} Convectively driven turbulence is sufficiently complex that dimensional reasoning need not lead to a unique answer and, on the other hand, the same conclusion can follow from a variety of assumptions. Thus even though the new Rayleigh number exponents have already been derived,¹ we obtain a more detailed picture of the flow and additional testable predictions by making the hypothesis that the heat transport is controlled by a thermal boundary layer itself created by the shear flow near the walls. Our analysis is valid when the entire temperature drop occurs within the viscous sublayer of the turbulent boundary layer. The latter is well characterized experimentally,^{5,6} and the desired ratio of viscous to thermal lengths can easily be achieved by adjusting the Prandtl number; so to observe the crossover point becomes an interesting experiment. For the spectra, we find that a buoyancy-dominated regime with exponents different from Kolmogorov's $k^{-5/3}$ cannot be consistently obtained within a simple scaling analysis for a meaningful range of scales.

The asymptotic properties of the Boussinesq equations are best extracted by nondimensionalizing with the thermal diffusivity and cell height, viz.,

$$[\partial_t \mathbf{v} + \mathbf{v} \cdot (\nabla \mathbf{v}) - \nabla p] / Pr = \nabla^2 \mathbf{v} + Ra \Theta \hat{z}, \tag{1a}$$

$$\partial_t \Theta + \mathbf{v} \cdot \nabla \Theta = \nabla^2 \Theta, \tag{1b}$$

where Pr is the Prandtl number, Ra the Rayleigh number, and the scaled temperature Θ equals $\pm \frac{1}{2}$ on the bottom and top plates. In these units, the Reynolds number Re is just the large-scale velocity divided by Pr, or $\langle v^2 \rangle_V \sim (Re Pr)^2$. We denote averages over a plane $z = \text{const}$ or the entire volume as $\langle \rangle_{A,V}$, respectively, and we assume them to be stationary; if not, an additional time average can be done.

Several exact relations then follow. By averaging the heat equation and assuming insulating lateral boundaries we find that the vertical heat flux or Nusselt number Nu, is independent of z and is related to the thermal dissipa-

tion

$$Nu \equiv \langle v_z \Theta \rangle_A - \partial_z \langle \Theta \rangle_A = \langle (\nabla \Theta)^2 \rangle_V, \tag{2}$$

where the second equality comes from the $\Theta \partial_t \Theta$ equation. Because the flow is entirely buoyancy driven, averaging the equation for the total energy ($v^2/2 - z \Theta Ra Pr$) yields

$$\langle (\nabla v)^2 \rangle = (Nu - 1) Ra. \tag{3}$$

It should be emphasized that the physical velocity and temperature boundary conditions are used and nothing is assumed about a mean flow that may be present in spite of the averaging.

The logic of our argument below is to relate the heat flux to the rate of shear in the viscous boundary layer, i.e., $Nu(Re)$, and then use standard turbulent boundary layer theory in conjunction with the dissipation equation (3) to yield $Re(Ra)$ and $Nu(Ra)$.

An important aspect of the high-Ra experiments of Refs. 1 and 2 was a persistent mean flow, which sufficiently close to the horizontal plates can be approximated by a linear profile $u_x = z/\tau$ with $0 \leq z \ll 1$ and $u_z \sim O(zu_x)$.^{5,6} If the entire temperature drop occurs within this range, which will always be true for large enough Pr, (1b) becomes

$$(z/\tau) \partial_x \Theta = \partial_z^2 \Theta \tag{4a}$$

or $\Theta = \frac{1}{2} - 0.269 I(z/(\tau x)^{1/3})$, with I the integral of $\exp(-\xi^3/9)$ and $I(0) = 0$. As a practical matter, we define the thermal boundary thickness z_{th} to be when $\Theta \sim 0.02-0.05$ or when the argument of I is $2.5-2$ for $x \sim 1$. Then computing the Nusselt number and assuming an aspect ratio 1 cell,

$$Nu \sim 1.0/z_{th} \sim \tau^{-1/3}. \tag{4b}$$

Observations also indicate that the flow in the cell is turbulent so we find τ by matching to the empirically verified scaling relations for a turbulent boundary layer. The mean velocity is given by⁷

$$\langle v_A \rangle = u_* [2.5 \ln(z/z_*) + 6.0] \tag{5a}$$

with $z_* = Pr/u_*$, and where the characteristic velocity u_*

is related to the large-scale Reynolds number by

$$u_* = \text{Pr Re} / [2.5 \ln(\text{Re}) + 6.0]. \quad (5b)$$

The logarithmic velocity profile matches onto the viscous-buffer sublayer, $0 < z < z_v$,

$$\langle v \rangle_A = u_* z / z_* \quad (5c)$$

at $z \approx z_v \sim (7-12)z_*$.

Since we already have $\text{Nu}(\text{Re})$, to find the Rayleigh number dependence we will exploit (3) by inserting the kinetic-energy dissipation estimated as in turbulent shear flow in pipes or channels⁸

$$\text{Pr} \langle (\nabla v)^2 \rangle_V \sim 100 u_*^3. \quad (6)$$

The numerical coefficient in (6) is somewhat arbitrarily taken to be marginally larger than the contribution for the viscous layer (5c) alone, $\sim z_v u_*^3 / z_*$, times 6 walls. We combine (3) and (6) to obtain $\text{Nu Ra} \sim 100 u_*^3 / \text{Pr}$ and then eliminate u_* using (4b), $\text{Nu} \sim (u_*^2 / \text{Pr})^{1/3}$, to yield

$$\text{Nu} \sim 0.27 \text{Pr}^{-1/7} \text{Ra}^{2/7}, \quad (7a)$$

$$\text{Re} \sim 0.14 \text{Pr}^{-5/7} \text{Ra}^{3/7} [2.5 \ln(\text{Re}) + 6.0]. \quad (7b)$$

The leading Ra dependence in (7a) and (7b) was proposed in Ref. 1 on the basis of purely dimensional arguments that did not take specific account of the mean flow.⁹ While experiment seems to rule out logarithms in Nu as in (7a), the fit,² $\text{Re} \sim 0.31 \text{Ra}^{0.485}$ involves an exponent larger than $\frac{3}{7}$ which is plausibly accounted for by the logarithms in (7b). Our numerical prefactors are also reasonable (N.B. $\text{Nu} = 0.25 \text{Ra}^{2/7}$ from Ref. 1). The container shape enters the energy balance equation as a numerical factor which has a nonzero limit for large aspect ratios L , whereas $\text{Nu}(\text{Re}) \sim L^{-1/3}$. For $L \gg 1$, (7a) and (7b) acquire factors of $L^{-3/7}$, $L^{-1/7}$, respectively.

Note that a correction has to be made following (5a) to the experimental Reynolds number since it is defined using the velocity close to the wall rather than the maximum velocity which is assumed to occur at $z \sim 1$ in (5a)-(5c). In the numerical estimates below we use the experimental values of $\text{Nu}(\text{Ra})$ and $\text{Re}(\text{Ra})$.

The upper limit to the validity of (7) follows from the condition that the thermal boundary layer equals the viscous one, namely, $z_{\text{th}} \approx \text{Nu}^{-1} = (7-12)z_*$. A solution must occur with increasing Ra since z_* decreases more rapidly than z_{th} at fixed Pr . The lower limit of validity is just the condition that $\text{Re} \gtrsim 2 \times 10^3$ to create sufficient turbulence. We therefore have

$$5 \times 10^7 \text{Pr}^{5/3} \lesssim \text{Ra} \lesssim (10^{13} - 5 \times 10^{14}) \text{Pr}^4. \quad (8)$$

The quoted spread in the upper limit is large because $z_{\text{th}}/z_* \sim \text{Ra}^{1/7}$ and came solely from the ambiguity in defining the viscous layer. It should be further increased by the uncertainty in $z_{\text{th}}\text{Nu}$.

Within the "logarithmic region" (5c), the mean temperature follows a similar law,⁶ $\langle \Theta \rangle_A \sim (\text{Nu}/u_*) \times \ln(z/z_*)$, suggesting a bulk temperature scale of $\text{Nu} \ln(u_*)/u_* \sim \text{Ra}^{-1/7} \ln(\text{Ra})$. This accords well with an estimate based on factoring the expression $\langle v_z \Theta \rangle_A$ for Nu and assuming $\langle v^2 \rangle^{1/2} \sim \text{Re Pr}$ as done in Ref. 1. Therefore, the temperature drop across each boundary is

$\frac{1}{2} - O(\text{Ra}^{-1/7})$, verifying an earlier assumption made in deriving (4b).

Since our thermal boundary is thicker than the Malkus-Howard limit,³ its stability must be considered. As first noted in Ref. 1, shear should suppress rolls perpendicular to itself but we disagree with the estimates given there. Specifically, one should perturb around solutions to (4a) not $\Theta = -z$. By scaling the boundary layer equations with τ , one finds an effective Rayleigh number $\text{Ra}_\perp = \tau^{5/3} \text{Ra} \sim 10^5 \text{Pr}^{5/7} \text{Ra}^{-3/7} \ll 10^3$ where we set $\text{Re} \sim 0.5 \times 10^5$ in the \ln terms of (5a) and took $\text{Re}(\text{Ra})$ from (7). If the mean flow were absolutely steady, which is probably untrue, then Gorter-like vortices could form parallel to the shear when $\text{Ra}_\parallel \sim z_{\text{th}}^3 \text{Ra} \sim \text{Ra}^{1/7}$ exceeds 10^3 .

The reader may note that no specific reference to thermal plumes was made in deriving (7a) and (7b) and none is necessary.¹⁰ The heat flux was calculated from the diffusive term very close to the wall where buoyancy is immaterial, and we never had to inquire how the temperature is transported in the bulk. Buoyancy clearly drives the mean flow but should not alter Eqs. (4)-(6) as we discuss in more detail below.

For buoyancy-driven turbulence it is instructive to explicitly evaluate the constants in the Kolmogorov $k^{-5/3}$ spectrum for the velocity and the analogous spectra for the temperature, assumed passive.^{5,6} One will then find consistency in that the $\text{Ra}\Theta$ term in (1a) equals $(\mathbf{v} \cdot \nabla \mathbf{v})/\text{Pr}$ only on the largest scales. Specifically if typical isotropic fluctuations over a scale r are denoted by δ , then

$$\begin{aligned} \delta v(r) &\sim \varepsilon_K^{1/3} r^{1/3}, \\ \delta \Theta(r) &\sim \varepsilon_\Theta^{1/2} \varepsilon_K^{-1/6} r^{1/3}, \end{aligned} \quad (9)$$

where the kinetic-energy dissipation $\varepsilon_K \sim u_*^3$ in our units and the temperature dissipation ε_Θ in the interior is bounded according to (2) by Nu . For $r \sim 1$, (9) predicts $\delta \Theta \lesssim (\text{Nu}/\text{Re Pr})^{1/2}$ which is marginally larger than the $\delta \Theta$ correlated with v_z which we estimated above as $\text{Nu}/\text{Re Pr}$. We prefer to use $\text{Nu}/\text{Re Pr}$ for $\delta \Theta(r \sim 1)$ which we can reconcile with (2) by the assumption that $\langle (\nabla \Theta)^2 \rangle_V$ is mostly accounted for by the thermal boundary layer. We, therefore, readjust ε_Θ downward to $\text{Nu}^2/\text{Re Pr}$. Finally, we have $r^{-1}(\delta v)^2/(\text{Ra Pr} \delta \Theta) \sim r^{-2/3} \gtrsim 1$ justifying our neglect of buoyancy on all but the largest scale. [Using (2) for ε_Θ would add a factor of $\text{Ra}^{1/14}$ to the left-hand side of this inequality which combined perhaps with logarithms and numerical factors could open up a small range of scales where buoyancy is dominant.]

Motivated by experiment,² an alternative scaling for $\delta v, \delta \Theta$ has been proposed^{11,12} that we will now show is inconsistent with the Boussinesq equations within the confines of a theory that allows only a single velocity and temperature scale. Imagine balancing (1a) and (1b) in such a way that the buoyancy is always important and Θ cascades,

$$\delta v^2(r)/r \sim \delta \Theta(r) \text{Ra Pr}, \quad (10a)$$

$$\delta v(r) \delta \Theta^2(r)/r \sim \varepsilon_\Theta, \quad (10b)$$

which imply¹³

$$\delta v \sim (\epsilon_\Theta \text{Ra}^2 \text{Pr}^2)^{1/5} r^{3/5}, \tag{11a}$$

$$\delta \Theta \sim [\epsilon_\Theta^2 / (\text{RaPr})]^{1/5} r^{1/5}, \tag{11b}$$

provided r is larger than a diffusive cutoff,

$$r_b \sim (\text{Ra}^2 \text{Pr}^2 \epsilon_\Theta)^{-1/8}, \tag{11c}$$

where Θ variance is destroyed by diffusion at a rate of ϵ_Θ . The kinetic-energy dissipation from (11a) is just $\langle (\nabla v)^2 \rangle_V \sim \text{RaPr} \epsilon_\Theta^{1/2}$ which is inconsistent with (3) (for reasonable Pr), hence (10) forces the velocity to fall off too rapidly to dissipate sufficient energy in the bulk.

If one assumes (10a) is violated for $r \lesssim r_b$ and $\delta v \sim r^{1/3}$ so as to allow for sufficient dissipation, then the inconsistency of (1) and (11a) can be demonstrated directly for $r \gtrsim r_b$ by following the von Karman-Howarth analysis.⁶ Using only homogeneity and stationarity one finds from (1),

$$\begin{aligned} \langle \mathbf{v} \cdot \nabla v_i(\mathbf{r}) v_i(0) \rangle_V \\ = \text{RaPr} \langle v_z(\mathbf{r}) \Theta(0) \rangle_V + \text{Pr} \langle \nabla^2 v_i(\mathbf{r}) v_i(0) \rangle_V. \end{aligned} \tag{12}$$

Isotropy implies that the left-hand side can be expressed as the r derivative of $\langle \{\mathbf{f} \cdot [\mathbf{v}(\mathbf{r}) - \mathbf{v}(0)]\}^3 \rangle$ and thus $\delta v^3(r)/r$ within our scaling theory. The v_z - Θ correlation on the right is just Nu for $r=0$ and only deviates from Nu for $r \sim O(1)$ since under all estimates the heat is carried predominantly by the largest scales. Where the last term balances the second, defines the dissipation scale, and thus for larger r

$$\delta v^3(r) \sim \text{RaPr} \text{Nu} r, \tag{13}$$

which agrees with (9) and not (11a). Actually, to achieve homogeneity we should only average (12) in the center of the cell, but the lateral boundary layers carry a heat flux $\sim u_* z_* \ll \text{Nu}$. If the mean flow is included, it mixes with v in (12) only as a strain rate which is small compared with $\delta v/r$.

One could imagine an anisotropic scaling with $\delta v_z \sim r^\alpha f(z/r_\perp^a)$, $a < 1$, $\perp = x, y$, and similar equations for δv_\perp and $\delta \Theta$. One can satisfy (10b) and (13) with z replacing r , and incompressibility, but (12) has to be resolved into \perp and z components. A pressure term then appears which forces isotropy. Kolmogorov scaling predicts a correction exponent $E_k \sim \epsilon_k^{2/3} k^{-5/3} [1 + O(S/\epsilon_k^{1/3} k^{2/3})]$ due to a large scale strain S but no new leading exponents.

Experiments^{2,11} have revealed a $\omega^{-7/5}$ regime in the scalar spectrum [i.e., (11b) if $k \sim \omega$], beginning at the largest scales and extending a factor of ~ 30 in scale size irrespective of Ra. Although advection by the large scales is always the dominant frequency for a given k , in the absence of a large mean flow the two are not proportional

since the integral of the energy spectrum up to k serves as the effective advection velocity. The actual wave-number spectrum will, therefore, be steeper than the measured frequency spectrum.

Another possible aspect of these experiments and a long outstanding violation of Kolmogorov scaling for a passive scalar is the nonzero skewness seen in shear flows right down to dissipative scales (Fig. 25, Ref. 14). Interestingly enough, the longitudinal structure functions for the temperature (Fig. 18, Ref. 14) exhibits a limited regime of scaling with a -1.5 exponent followed by something steeper not unlike those of Refs. 2 and 11. Presumably both the skewness and discrepancy with (9) are due to the ejection of large temperature gradients from the boundary layer by the turbulent bursts. Clearly buoyancy can only enhance the ejection of scalar variance, but once outside of the boundary layers, the spectrum in Refs. 2 and 11 seems insensitive to the distance from the walls. Therefore, while we reject the arguments leading to (11b) the alternatives need to be quantified.

It remains a significant open problem to show directly from (1a) and (1b) that a mean flow exists.¹⁵ An analogy might be drawn with the $\text{Pr} \rightarrow \infty$ limit, where $z_{\text{th}} \ll$ a viscous length as here, and there is a single velocity mode.⁹ One could also envisage calculating a laminar large scale flow from the Boussinesq equations with a turbulent eddy damping $\sim z \text{Re}$ and a virtual bulk temperature drop $\sim \text{Ra}^{-1/7}$. The effective Rayleigh number is $\text{Ra}_T \sim \text{Ra} \text{Ra}^{-1/7} / \text{Re}^2 \sim O(1)$ suggesting that a single mode might be consistent. The turbulent "medium" could also be likened to convection between insulating plates where the heat flux but not the temperature is prescribed and convection occurs at onset in a single large cell.

When the boundary layers cross around the maximum Ra in (8), the bulk temperature drop becomes $1 - O(\ln \text{Re})^{-1}$ by virtue of the mixing length expressions⁶ $\langle \Theta \rangle_A \sim \frac{1}{2} - (\text{Nu}/u_*) \ln(z/z_*)$ and $z_{\text{th}}/z_* \sim O(1)$ written with respect to the bottom plate. Even if the mean flow disappears, large eddies will generate a shear and estimating $\text{Nu}(\text{Re}) \sim u_* / |\ln z_*|$ from $\langle \Theta \rangle_A(z \sim 1) \sim 0$ and using (3), one finds⁴ $\text{Nu} \sim \text{Ra}^{1/2} \text{Pr}^{1/2} / (\ln \text{Re})^3$. Although we have assumed a passive scalar, buoyancy should not decrease Nu, and there is a rigorous numerical upper bound¹⁶ on $\text{Nu}/\text{Ra}^{1/2}$ suggesting that buoyancy acts like a generic large scale force.

Finally, note that the strong Pr dependence in (8) should facilitate experimental verification of our predictions. In particular the Rayleigh number range of the $\frac{2}{5}$ regime will be much reduced in mercury.

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- ¹B. Castaing *et al.*, *J. Fluid Mech.* **204**, 1 (1989).
- ²M. Sano, A. Z. Wu, and A. Libchaber, *Phys. Rev. A* **40**, 6421 (1989).
- ³W. V. R. Malkus, *Proc. R. Soc. London A* **225**, 185 (1954); L. N. Howard, *J. Fluid Mech.* **17**, 405 (1963).
- ⁴R. H. Kraichnan, *Phys. Fluids* **5**, 1374 (1962).
- ⁵H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (MIT, Cambridge, MA, 1972).
- ⁶L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, London, 1959).
- ⁷H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (Ref. 5), pp. 157–161 and Fig. 5.6.
- ⁸See Ref. 5, p. 153 and J. Hinze, *Turbulence* (McGraw-Hill, New York, 1975), pp. 646–651. The dissipation one would have estimated from Kolmogorov theory, $(\text{PrRe})^3$, is absent in these shear flows. For convection, the experiments in Refs. 1 and 2 imply directly that $\langle(\nabla v)^2\rangle/(\text{Re}^3\text{Pr}^2)$ decreases with increasing Ra.
- ⁹It is useful to note that when both the temperature and velocity exhibit Blasius scaling, $\text{Nu} \sim \text{Re}^{1/2}$, the kinetic-energy dissipation is $(\text{RePr})^2\text{Re}^{1/2}$, and (3) implies $\text{Nu} \sim \text{Ra}^{1/4}\text{Pr}^{1/2}$. When $\text{Pr} \gg 1$ and $\text{Re} \lesssim O(1)$, the Nu follows from (4b) with $\tau^{-1} = \text{RePr}$ and $\langle(\nabla v)^2\rangle \sim \langle v^2\rangle \sim (\text{RePr})^2$ so $\text{Nu} \sim \text{Ra}^{1/5}$. This last formula agrees with G. O. Roberts, *Geophys. Astrophys. Fluid Dyn.* **12**, 235 (1979).
- ¹⁰To see that the buoyancy has negligible effect on the logarithmic velocity profile, recompute the Reynolds stress from the horizontally averaged vorticity equation and observe that in addition to the constant term $u\bar{z}$ there is a contribution $\text{Ra} \int \delta \int \delta' \Delta \Theta(z'')$, where $\Delta \Theta$ is the temperature difference on the two side walls. Since $\Delta \Theta \leq 1$, buoyancy will only matter for $z \gtrsim \text{Ra}^{-1/4}$, so we must match (5a) to Re at this point rather than at $z \sim 1$ which is immaterial.
- ¹¹X. Z. Wu, L. Kadanoff, A. Libchaber, and M. Sano (unpublished).
- ¹²I. Procaccia and R. Zeitak, *Phys. Rev. Lett.* **62**, 2128 (1989).
- ¹³R. Bolgiano, *J. Geophys. Res.* **64**, 2226 (1959); A. M. Obukhov, *Dokl. Akad. Nauk SSSR* **121**, 1246 (1959) [*Sov. Phys. Dokl.* **3**, 61 (1959)].
- ¹⁴P. Mestayer, *J. Fluid Mech.* **125**, 475 (1982).
- ¹⁵R. Krishnamurti and L. Howard, *Proc. Nat. Acad. Sci.* **78**, 1981 (1981); *J. Fluid Mech.* **170**, 385 (1986).
- ¹⁶L. Howard, *Annu. Rev. Fluid Mech.* **4**, 473 (1972).