PHYSICAL REVIEW A VOLUME 42, NUMBER 6 15 SEPTEMBER 1990

Transition to turbulence in a discrete Ginzburg-Landau model

Tomas Bohr and Anders W. Pedersen
The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark

Mogens H. Jensen
Nordita, Blegdamsvej 17, DK-2100 Copenhagen, Denmark
(Received 16 November 1989)

We present a numerical study of the onset of turbulence in a discretized version of the complex Ginzburg-Landau equation. The transition point is determined by computing Lyapunov exponents, which show a first-order transition at a parameter value α_1 below the linear stability threshold for the uniform state. On further decreasing the parameter, the finite-time Lyapunov exponent remains positive only up to a characteristic transient time, after which the vortices get entangled and the asymptotic Lyapunov exponents become zero. The finite-time exponent goes to zero at $\alpha_c < \alpha_1$ as a power law.

The transition to turbulence in spatially extended dynamical systems is currently a field of very active research with many important open questions. The classical field of hydrodynamical turbulence is complemented by a wide range of other phenomena showing complex motion, e.g., in chemical reactions, liquid crystals, and surface waves as well as simplified models in the form of coupled map lattices or cellular automata. Chemical reactions like the Belousov-Zhabotinsky reaction provide beautiful examples. In a stirred reactor this system can undergo a Hopf bifurcation and thus go into a temporally periodic spatially uniform state.[1] In the absence of stirring, inhomogeneities appear which affect the local periods and thus tend to dephase the different parts of the system. In two dimensions (shallow dish) one observes target patterns, spiral waves, or *vortices* and the dynamics can be quite complicated.[2,3]

In the present work we determine the transition to turbulence in a two-dimensional coupled map lattice introduced in Ref. 4 (from now on referred to as I) which closely resembles the complex Ginzburg-Landau partial differential equation (PDE) and which should, therefore, be a good model for the dynamics of an extended system close to a Hopf bifurcation. The transition point is found by computing the largest Lyapunov exponent of the system.

The complex Ginzburg-Landau equation[5–8] is derived by assuming that the "order parameter" A, which is a complex field giving the amplitude and phase of the lowest temporal Fourier mode, is small and slowly varying in space and time. It then takes the form

$$\dot{A} = \mu A - (1+i\alpha)|A|^2 A + (1+i\beta)\nabla^2 A \ , \tag{1}$$

where μ, α, and β are real numbers. The parameter μ is the usual Landau coefficient: Negative μ implies a quiescent state ($A=0$), whereas positive μ gives nonzero values to the order parameter. In fact there is a homogeneous solution $A=\sqrt{\mu}e^{-i\alpha\mu t}$ for positive μ, and it is seen that the frequency of this periodic state is $\omega=\alpha\mu$.

The linear stability of the homogeneous rotating state can be investigated by standard techniques.[5] As long as $1+\alpha\beta>0$, it is stable, but when $1+\alpha\beta<0$, long-wavelength modes ($|\mathbf{k}|<k_c \propto |1+\alpha\beta|^{1/2}$) will be exponentially enhanced. In this analysis only small fluctuations around the uniform state are taken into account. The periodicity of the phase variable leads, however, to the possibility of topological defects in the form of vortices as in the planar XY model or superfluid helium films.[9] At the center of the vortex the phase is singular, but, by letting the modulus vanish, A itself remains well defined. When vortices are present the phase field is not single-valued: The total variation of phase on traversing a loop enclosing one or more vortices is an integer multiple of 2π.

Our choice of model was motivated by several different considerations. First of all, especially in the coupled map version to be described below, it is a computationally very simple system which has an interesting transition to turbulence. It is also quite closely related to experiments, the main problem being that our model is *driven* uniformly so one has to find good ways of pumping chemicals into the reactor homogeneously. The simple topology gives hope of understanding the "mechanism" underlying the transition in terms of interacting vortices, maybe to some extent analogous to the "coherent structures" of hydrodynamical turbulence. Further, the system allows interesting comparison between equilibrium and nonequilibrium dynamics: by setting $\alpha=\beta=0$ and adding noise, we get a dynamical description of the Kosterlitz-Thouless transition in, e.g., superfluid helium films.

The coupled map lattice is described in detail in I and we shall only briefly recapitulate. It consists of two parts: a *local* map $A'=F(A)$ representing the two first terms of (1) and a *nonlocal* part representing the complex heat equation which results from omitting the local terms. The properties of the local map F are very simple. In contrast to most of the literature on coupled map systems they are completely nonchaotic. Without the diffusion term (1) can be written as $\dot{r}=\mu r - r^3$ and $\dot{\phi}=-\alpha r^2$. The general structure of this can be easily reproduced by maps $r_{n+1}=f(r_n)$ and $\phi_{n+1}=\phi_n-\tau\alpha r_n^2$, where the map f has an *unstable* fixed point in 0 and a *stable* one in $r=\sqrt{\mu}$. Specifically, one can choose f as the exact integral $r(t+\tau)^2=\mu r(t)^2/[\lambda\mu+(1-\lambda)r(t)^2]$ of the radial equation with $\lambda=e^{-2\mu\tau}$.

The heat equation has the solution $A(t+\tau_0)=\exp[\tau_0(1+i\beta)\nabla^2]A(t)$. As nonlocal map we thus take $\tilde{A}=[1+(\tau_0/M)(1+i\beta)\Delta]^M A$, where Δ is a discrete Laplacian[10] and M is an integer that determines the range of the effective interaction. The limit $M\to\infty$ reproduces the exponential above (except that Δ and ∇^2 are not precisely the same). We take M to be around 5, large enough to ensure that short-wavelength instabilities do not occur.

The full map lattice can now be written $A_{n+1}(\mathbf{r})=F(\tilde{A}_n(\mathbf{r}))$. For our simulations we have used periodic boundary conditions on $L\times L$ lattices with L between 25 and 200. One can again ask for linear stability of the homogeneously rotating state $A_n=\sqrt{\mu}e^{-i\alpha\mu\tau n}$ and the resulting criterion (replacing $1+\alpha\beta>0$) is $1+2[\mu/(1-s)]\tau\alpha\beta>0$ where $s=f'(\sqrt{\mu})=\exp(-2\mu\tau)$, and it is seen that the new stability criterion approaches the old one as $\tau\to 0$. In the following we shall vary α keeping the other parameters fixed. Our standard configuration has been $\mu=0.2$, $\tau=1$, $\tau_0=0.2$, and $\beta=-1$ and here the uniform state is linearly stable for $\alpha<\alpha_2=0.82$. Most earlier work in this field[8,11] has focused on the region close above the linear stability threshold. Here the homogeneous state is unstable and vortex-antivortex pairs nucleate. Our approach (as described in I) has been to look at the states generated by *random* initial conditions (which then would contain a large number of vortices) and ask when such states would become turbulent. The transition point found in this way actually lies in the *linearly stable* region.

We determine whether states are turbulent by computing the largest Lyapunov exponent. The standard method is to follow the growth of a randomly chosen tangent vector iterated by acting on it with the Jacobian matrix of the map. In Fig. 1 the logarithm of the length of the iterated vector versus the discrete time n is plotted for a 50×50 lattice. The Lyapunov exponent is the average slope of the curve, which, in Fig. 1, is clearly positive—around 0.02 averaged over 7000 iterates. The value of α is here 0.74, which is below the linear stability threshold. At lower values of α something interesting is seen, as illustrated in the inset: We get a positive slope only for a finite time T (around 2000 in the figure) and insofar as one can extract a well-defined slope from times less than T, it is possible to assign a positive "finite-time Lyapunov exponent" λ_f. Figure 2 shows the variation of T with the parameter α (fixing other parameters and system size). First of all the plot shows a large scatter: T depends strongly on the choice of random initial configuration! However, the data show a sharp increase in T around $\alpha=\alpha_1\approx 0.75$ consistent with a divergence of T. The true, asymptotic Lyapunov exponents are only nonzero above α_1 and the exponent jumps there from 0 to some positive value (around 0.02). To test this, it is crucial to know what happens when the size of the system is varied and see if we can extrapolate sensibly to the "thermodynamical limit" $L\to\infty$. Figure 2(b) shows a plot of transient time T as function of size L and again we observe a large scatter from different initial conditions. On top of the scatter there appears to be an overall increase with L. The increase seems very slow for L larger than around 100, but our data are not sufficient to determine whether the curve actually continues to increase slowly, maybe even logarithmically, or saturates.

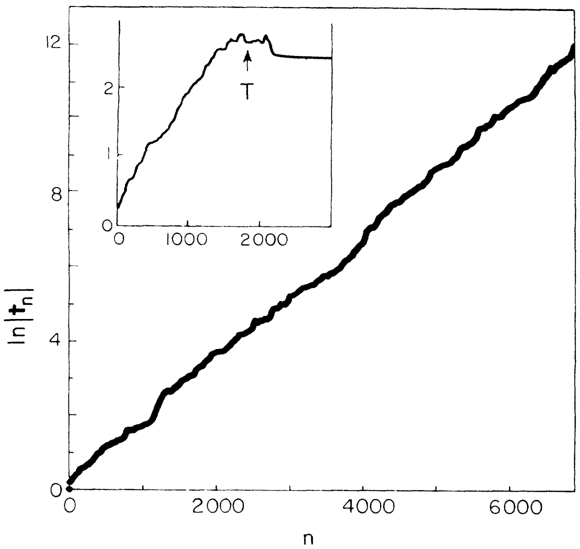

FIG. 1. The logarithm of the length of an iterated tangent vector $|\mathbf{t}_n|$ vs discrete time n for $\alpha=0.74$. The inset has $\alpha=0.71$ and shows that the turbulent state is transient. It only has a positive Lyapunov exponent up to $n=T$.

We have also measured the Lyapunov exponents in a different, more "direct," way. A given configuration of our system is slightly perturbed by rotating the A field at a single site by a small angle (typically 10^{-4}). We then follow the evolution of both the perturbed (A_n') and the unperturbed (A_n) system and monitor their "distance" at equal times. As a measure of distance we have chosen simply

$$d_n=\left[\sum_{\mathbf{r}}|A_n'(\mathbf{r})-A_n(\mathbf{r})|^2\right]^{1/2} \tag{2}$$

and the Lyapunov exponent emerges as the slope of $\ln d_n$ vs n. The results obtained in this way were always compatible with the results obtained from the growth of tangent vectors (as in Fig. 1). Figure 3 shows the results of such computations. The squares represent values of λ_f on a hexagonal lattice obtained by iterating a tangent vector, whereas the triangles are obtained on a square lattice using Eq. (2). As seen on the figure there seems to be a well-defined point $\alpha=\alpha_c\approx 0.5$, where λ_f becomes zero and the variation can be fitted fairly well as

$$\lambda_f=(\alpha-\alpha_c)^\nu , \tag{3}$$

with $\nu\approx 0.5$. Clearly, the closer we get to α_c the harder it is to find a good estimate of λ_f; thus the considerable error bars. In contrast to T, the Lyapunov exponents, λ_f

of the transients show no systematic variation with L.

In the turbulent state, whether transient or not, the number of vortices fluctuates strongly due to creation and annihilation of vortex pairs. The statistical properties, such as the mean vortex density or λ_f, are reasonably independent of the initial condition as opposed to T itself. The final state ($n \gg T$) in the transient turbulent regime $\alpha_c < \alpha < \alpha_1$ contains typically a few *entangled* vortices. In the entangled state, as described in I, the centers of the vortices are stuck and although they continue to send out waves the Lyapunov exponent is zero. Opposite vortices basically attract (for $\alpha=\beta=0$ this is the logarithmic attraction responsible for the Kosterlitz-Thouless transition[9]), but the spiral arms (which are not present for $\alpha=\beta=0$) can screen each other, strongly giving rise to entanglement. The time scale T seems to be determined by the event that one vortex (or maybe two) starts outgrowing the others. Within a few hundred time steps it can then eat up most of the others, leaving a final state of a few entangled vortices. And the occurrence of this event depends on very particular conditions related to small details in the state and is thus strongly dependent on the initial condition. In a larger system more vortices are competing and it may not be unreasonable that the turbulent state can persist for a longer time. A recent analytical calculation[12] of the equations of motion for an interacting vortex-antivortex pair leads to a sequence of bound states that might describe entangled configurations.

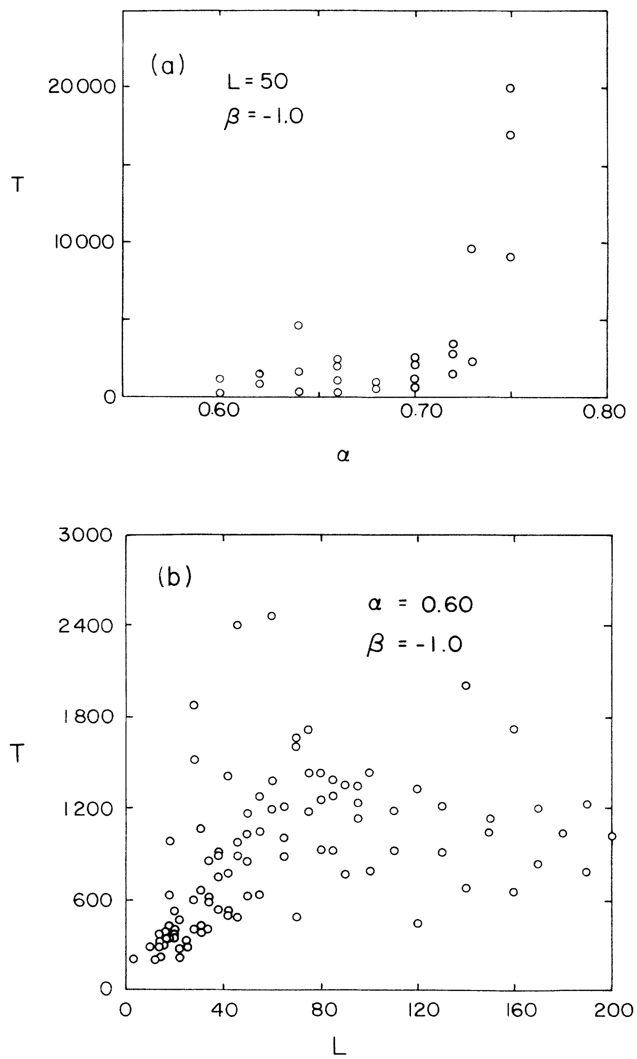

FIG. 2. (a) The transient time T vs the parameter α with $L=50$ and $\beta=-1.0$. Note the divergence around $\alpha\approx 0.75$. (b) The transient time T vs the system size L. α is fixed at 0.60. Each point corresponds to a particular random initial state and the large scatter indicates a strong variation with initial condition.

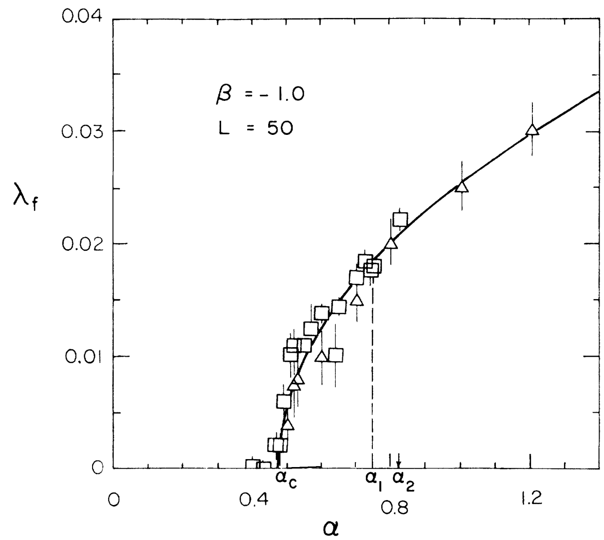

FIG. 3. The Lyapunov exponent as a function of the parameter α. The squares were obtained by iterations of a unit tangent vector (as in Fig. 1) on a hexagonal lattice. The triangles were obtained by measuring the distance [Eq. (2)] between two nearby states on a quadratic lattice. For α less than $\alpha_1 \approx 0.75$ the turbulent state is transient and the Lyapunov exponent is only defined for a finite time T. The true Lyapunov exponent is zero below α_1 as shown by the dotted line. The curve is a power-law fit as Eq. (3) with $\nu=0.50$ and $\alpha_c=0.48$. The threshold of linear instability for the uniform state is at $\alpha_2=0.82$.

If, as discussed earlier, $T(L)\to\infty$ for $L\to\infty$ in the whole parameter interval $[\alpha_c,\alpha_1]$, the transient turbulence will become genuinely turbulent. In that case the infinite system will have a *continuous* transition probably at a value close to the α_c that we already determined. We would then expect the true Lyapunov exponent to scale as (3) close to the transition. One might ask whether it is possible to give an analytic estimate of the parameter value at which the transition to turbulence takes place. Far away from the core the spirals basically generate plane waves and it has therefore been conjectured[13] that the transition point should be determined by the "sideband" instability of the periodic state with that particular wavelength. We have checked this for the parameters of Fig. 3 and also for $\beta=0$ and find that the transition point α_c to transient turbulence coincides with the corresponding linear instability threshold within our numerical accuracy.[14] One should note that the existence of a transition to turbulence for $\beta=0$ rules out any correspondence with the linear instability of the *uniform state,* whereas the sideband instability predicts $\alpha_c=1.2$, which roughly corresponds to the numerical simulations.

We would like to thank C. Bennett, E. Bodenschatz, P. Coullett, G. Grinstein, F. Hynne, L. Kramer, J. Lega, E. Meron, A. Newell, I. Procaccia, D. Rand, and P. Graae Sørensen for helpful discussions. A.W.P. is grateful to The Carlsberg Foundation for support.

---

[1]J.-C. Roux, R. H. Simoyi, and H. L. Swinney, Physica **8D**, 257 (1983).

[2]A. N. Zaikin and A. M. Zhabotinsky, Nature **225**, 535 (1970).

[3]A. T. Winfree, *The Geometry of Biological Time* (Springer, Berlin, 1980).

[4]T. Bohr, M. H. Jensen, A. W. Pedersen, and D. Rand, in *New Trends in Nonlinear Dynamics and Pattern Forming Phenomena,* edited by P. Coullet and P. Huerre (Plenum, New York, 1990).

[5]A. C. Newell and J. A. Whitehead, J. Fluid Mech. **38**, 279 (1969); A. C. Newell, in *Lectures in Applied Mathematics,* (American Mathematical Society, Providence, 1974), Vol. 15.

[6]Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, 1980).

[7]H. R. Brand, P. S. Lomdahl, and A. C. Newell, Physica **23D**, 345 (1986).

[8]P. Coullet, L. Gil, and J. Lega, Phys. Rev. Lett. **62**, 1619 (1989).

[9]See, e.g., J. M. Kosterlitz, in *Nonlinear Phenomena at Phase Transitions and Instabilities,* edited by T. Riste (Plenum, New York, 1982), p. 397; or D. R. Nelson, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, New York, 1983), Vol. 7, p. 1.

[10]On the lattice we approximate the Laplacian by an average ΔA over neighbors. Thus on a two-dimensional hexagonal lattice $(x,y)=(i+j/2,\sqrt{3}/2j)$ with $i,j=1,\ldots,N$ we take $\Delta A(i,j)=\frac{2}{3}\sum_{i',j'}A(i',j')-A(i,j)$ where the sum is over the six nearest neighbors (i',j').

[11]A. V. Gaponov-Grekhov and M. I. Rabinovich, Usp. Fiz. Nauk **152**, 159 (1987) [Sov. Phys.—Usp. **30**, 433 (1987)]. A. V. Gaponov-Grekhov, A. S. Lomov, G. V. Osipov, and M. I. Rabinovich (unpublished).

[12]C. Elphick and E. Meron (unpublished).

[13]A. Newell (private correspondence).

[14]T. Bohr, A. W. Pedersen, M. H. Jensen, and D. A. Rand, in *Proceedings of the Conference on Nonlinear Evolution of Spatio-Temporal Structures,* City, 1990, edited by F. Busse and L. Kramer (Plenum, New York, in press). See also related work in L. Kramer, E. Bodenschatz, W. Pesch, A. Weber, and W. Zimmermann, in *New Trends in Nonlinear Dynamics and Pattern Forming Phenomena,* (Ref. 4); E. Bodenschatz, Ph.D. thesis, University of Bayreuth (1989); A. Weber, M. Sc. thesis, University of Bayreuth (1989).