Information-theory approach to the variable-mass harmonic oscillator

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A careful analysis of the well-known Kanai-Caldirola Hamiltonian yields rather surprising results concerning its physical origin. Recourse to the maximum entropy principle sheds some light regarding the meaning of the equivalence between different Lagrangians and/or Hamiltonians.

I. INTRODUCTION

During the last forty years, there have been numerous attempts to quantify dissipative forces, as experimental evidence of the presence of dissipation in several microscopic processes has been repeatedly found.[1] In spite of the fact that several techniques have been used to achieve that purpose, a unanimously accepted approach for quantifying dissipative systems has not been found so far. Different methods have been proposed.[2,3] In the present work we intend to investigate, using the information-theory approach, some features of one of these methods, on which there is a great amount of literature (see Refs. 4 and 5 and references therein): the use of a time-dependent Hamiltonian.

Idealizing the system that undergoes dissipation as a harmonic oscillator, the kind of Hamiltonian that would describe it, first proposed by Kanai,[6] adopts the following form:

$$H(t)=\frac{p^2}{2m(t)}+\tfrac{1}{2}m(t)\omega_0^2q^2 , \tag{1.1}$$

where

$$m(t)=m_0e^{\alpha t}\quad(\alpha>0) . \tag{1.2}$$

Some peculiarities of this model have been either widely criticized or misunderstood.[1-3,7-16] The numerous attempts to apparently obtain the correct quantization of the damped harmonic oscillator, starting from the Hamiltonian

$$H=\frac{p^2}{2m_0}e^{-\alpha t}+\tfrac{1}{2}m_0e^{\alpha t}\omega_0^2q^2 , \tag{1.3}$$

are due to the fact that this Hamiltonian leads to the equation of motion

$$\ddot{q}+\alpha\dot{q}+\omega_0^2q=0 , \tag{1.4}$$

which is actually the equation of motion of a damped harmonic oscillator. Several difficulties emerge when one uses the Hamiltonian (1.3) or, equivalently, the Lagrangian

$$L=e^{\alpha t}(\tfrac{1}{2}m_0\dot{q}^2-\tfrac{1}{2}m_0\omega_0^2q^2) , \tag{1.5}$$

in trying to quantify the system under study:

(a) One of the first problems to be observed is related to the apparent violation of Heisenberg's uncertainty principle. On quantifying Eq. (1.3) by changing the dynamical variables q and p into the quantum operators $\hat{q}$ and $\hat{p}$, one obtains

$$\hat{H}=\frac{\hat{p}^2}{2m_0}e^{-\alpha t}+\tfrac{1}{2}m_0e^{\alpha t}\omega_0^2\hat{q}^2 , \tag{1.6}$$

with

$$[\hat{q},\hat{p}]=i\hbar . \tag{1.7}$$

The commutator given by Eq. (1.7) occurs due to the fact that q and p are conjugate canonical variables and that the Dirac quantization method is used. Working in the Schrödinger picture, one can use the generalized Ehrenfest theorem (Ref. 17) and, in terms of its matrix G, the mean values of the position and momentum operators $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ result,

$$\frac{d}{dt}\langle\hat{q}\rangle=e^{-\alpha t}\frac{\langle\hat{p}\rangle}{m_0} , \tag{1.8}$$

$$\frac{d}{dt}\langle\hat{p}\rangle=-m_0e^{\alpha t}\omega_0^2\langle\hat{q}\rangle . \tag{1.9}$$

From Eq. (1.8) one obtains

$$\langle\hat{p}\rangle=m_0e^{\alpha t}\frac{d}{dt}\langle\hat{q}\rangle , \tag{1.10}$$

instead of the mechanical momentum usually considered to be the pertinent one for the problem at hand, i.e.,

$$\langle\hat{p}_k\rangle=m_0\frac{d}{dt}\langle\hat{q}\rangle . \tag{1.11}$$

We should mention that this feature of the operator $\hat{p}$ appears also in the corresponding classical treatment.

Using the same procedure as in Eqs. (1.8) and (1.9), one obtains

$$\frac{d}{dt}\langle\hat{q}^2\rangle=\frac{e^{-\alpha t}}{m_0}\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle , \tag{1.12}$$

$$\frac{d}{dt}\langle\hat{p}^2\rangle=-m_0e^{\alpha t}\omega_0^2\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle , \tag{1.13}$$

and

$$\Delta\hat{q}\,\Delta\hat{p}\geq\frac{\hbar}{2} . \tag{1.14}$$

On the other hand, for $\hat{p}_k$ one finds that

$$\Delta\hat{q}\,\Delta\hat{p}_k=\Delta\hat{q}\,\Delta(e^{-\alpha t}\hat{p})=e^{-\alpha t}\Delta\hat{q}\,\Delta\hat{p}\geq e^{-\alpha t}\frac{\hbar}{2} , \tag{1.15}$$

which implies

$$\Delta\hat{q}\,\Delta\hat{p}_k \xrightarrow[t\to\infty]{} 0 \ . \tag{1.16}$$

The result given by Eq. (1.16) was first interpreted as a violation of the uncertainty principle.[2,13,14,18] At a later time[9,19] a different interpretation of this situation was given in terms of a more general mathematical version of the uncertainty principle, namely,

$$\Delta\hat{A}\,\Delta\hat{B} \geq \tfrac{1}{2}|\langle[\hat{A},\hat{B}]\rangle| \ , \tag{1.17}$$

which indicates that a simultaneous accurate measurement is *not* possible for the operators $\hat{q}$ and $\hat{p}$ due to Eq. (1.7), but it *could* be possible for $\hat{q}$ and $\hat{p}_k$ since

$$[\hat{q},\hat{p}_k]=[\hat{q},e^{-\alpha t}\hat{p}]=e^{-\alpha t}[\hat{q},\hat{p}]=e^{-\alpha t}i\hbar \ . \tag{1.18}$$

When considering the measurement process at a quantum level, it was not clear whether the quantity measured was the mechanical momentum or the generalized momentum,[9] although this ambiguity vanishes when one carefully examines the real physical meaning of the Hamiltonian given by Eq. (1.3), as we will see in subsequent sections.

There were also some attempts to add a stochastic force to the Lagrangian [see Eq. (1.5)] or equivalently to the Hamiltonian [see Eq. (1.6)]. The effect of this stochastic force was to allow for an uncertainty principle to hold for $\hat{q}$ and $\hat{p}_k$.[20]

(b) Another problem that has been pointed out is that the Hamiltonian given by Eq. (1.6) is not the energy operator. This problem already appears at the classical level, as the energy should be defined according to[18]

$$E=e^{-\alpha t}H \ . \tag{1.19}$$

With the choice of Eq. (1.19), one obtains the "correct" rate of energy dissipation, namely,

$$\frac{dE}{dt}=-\alpha m_0\dot{q}^2 \ . \tag{1.20}$$

However, in the quantum approach, this energy steadily decreases (to zero). In a correct treatment, one would expect the decay to stop at the ground state of the oscillator. This is due to the fact that, notwithstanding the time dependence of this Hamiltonian, there is a minimum energy compatible with the uncertainty principle (see Ref. 3).

It is interesting to isolate the fact that makes the Hamiltonian given by Eq. (1.3) (or its quantum equivalent [Eq. (1.6)]) *not* equivalent to the energy operator. It can be demonstrated that Eq. (1.3) oscillates without dissipation.[10] In Sec. II, we shall see that this feature, far from being an obstacle, constitutes a signature of the actual nature of the system to be quantified.

(c) Finally, we mention a well-known alternative approach that consists in considering the Lagrangian given by Eq. (1.5), or equivalently the Hamiltonian given by Eq. (1.3), as corresponding to a harmonic oscillator with variable mass. According to this approach,[3] there is no contradiction, at a classical level, in considering that Eq. (1.5) or Eq. (1.3) represents a damped harmonic oscillator as well as a harmonic oscillator with variable mass. However, the problem appears at a quantum level, where the Hamiltonian given by Eq. (1.6) would represent just a harmonic oscillator with variable mass. The peculiar features that were observed in Eq. (1.6) when this Hamiltonian was misinterpreted as a harmonic oscillator subject to a damping force are to be reconsidered. In this sense, if Eq. (1.6) actually represents a harmonic oscillator with variable mass, the canonical momentum [see Eq. (1.10)] would be the usual mechanical momentum. No contradictions ensue concerning the validity of the uncertainty principle since the inequality given by Eq. (1.14) holds and refers to the position and the (usual) mechanical momentum of the particle. However, the problem of the definition of the energy still remains since, if one identifies the Hamiltonian with the energy, it oscillates without dissipation, and if, on the contrary, one identifies Eq. (1.19) with the energy, it steadily decreases towards zero.

It is the intention of the present work to shed some light upon the fact that the Lagrangian given by Eq. (1.5) [the Hamiltonian given by Eq. (1.3)] represents neither a damped harmonic oscillator nor a harmonic oscillator with variable mass, and that, consequently, the conclusions that can be drawn in this sense from the Hamiltonian given by Eq. (1.6) are incorrect. Besides, it should be noted that these consequences are not a result of the quantization procedure itself but of the special features of the problem at hand, even at the classical level, having to do with the method with which the Lagrangian given by Eq. (1.5) is obtained.

What we want to make conspicuous is the fact that the Kanai-Caldirola Hamiltonian does not describe the damped harmonic oscillator (Ref. 6) and that there does not exist any violation of the uncertainty principle,[2,3,9,12–14,18,20] or problems with the identification of the energy operator.[2,3,10,11,16,20] Moreover, we find that the reinterpretation of the Kanai-Caldirola Hamiltonian as corresponding to a variable-mass oscillator is wrong.[3]

It is important to emphasize the fact that this model, employed by many authors in the past, does not actually correspond to the problem to be solved. One should not try to apply it without carefully dealing with the problem of quantifying dissipation.

The paper is organized as follows: In Sec. II we analyze the problem from the Newtonian, Lagrangian, and Hamiltonian viewpoints. In Sec. III we show an alternative approach through information-theory techniques. Finally, in Sec. IV, the conclusions are drawn.

II. VARIABLE-MASS HARMONIC OSCILLATORS

A. Variable-mass harmonic-oscillator description in the Newtonian framework

According to Newton's mechanics, one describes a system consisting of a harmonic oscillator with variable mass by defining the momentum as

$$p(t)=m(t)\dot{q}(t) \ . \tag{2.1}$$

Recourse to Newton's second law of motion and to a

Hooke's-type force $-kq$, the particular choice of $m(t)=m_0e^{\alpha t}$, leads to

$$\ddot{q}+\alpha\dot{q}+\frac{k}{m_0}e^{-\alpha t}q=0 \ . \tag{2.2}$$

The solutions of Eq. (2.2) have been studied in Ref. 10. According to this reference, the solution would be

$$q(t)=\exp(-\alpha t/2)\left[AJ_1\left[\frac{2\omega_0}{\alpha}\exp(-\alpha t/2)\right]+BY_1\left[\frac{2\omega_0}{\alpha}\exp(-\alpha t/2)\right]\right] , \tag{2.3}$$

where J_1 and Y_1 are Bessel polynomials and A and B are arbitrary constants.

Thus, a harmonic oscillator with variable mass *does not* reproduce the equation of motion for a damped harmonic oscillator, which is given by $\ddot{q}+\alpha\dot{q}+\omega_0^2q=0$ [see Eq. (1.4)].

Indeed, one obtains Eq. (1.4) from Eq. (2.2), only if the elastic constant k is no longer a constant but $k=m(t)\omega_0^2$.

Thus, we must conclude that a system consisting of a variable-mass particle subject to a force

$$F=-m(t)\omega_0^2q \ , \tag{2.4}$$

with $m(t)=m_0e^{\alpha t}$, has Eq. (1.4) as its equation of motion.

The question that now naturally arises is whether such a system is in fact a damped harmonic oscillator. It is our intention now to discuss the problem within the Newtonian formalism, starting from the very definition of the energy.

Analysis of the energy for a variable-mass particle. Following a textbook procedure one can conclude that, if one is dealing with a variable-mass particle, the usual "definition" of the kinetic energy as $T=\frac{1}{2}m(t)\dot{q}(t)^2$ implies that its variation is not equal to the work done by the system, since it can be proved that (see Appendix I)

$$W=\tfrac{1}{2}m\dot{q}^2|_1^2+\tfrac{1}{2}\int_1^2\dot{m}\dot{q}^2dt \ . \tag{2.5}$$

Notice that the last term on the right-hand side (r.h.s.) of Eq. (2.5) is directly related to the variable-mass character of the system.

From the analysis of the behavior of the variation of the energy of a variable-mass system subject to forces like $-kq$, it is possible to establish that

(a) If $k=$const, by equating the work done by the system to Eq. (2.5), one obtains

$$\tfrac{1}{2}m(2)[\dot{q}(2)]^2+\tfrac{1}{2}k[q(2)]^2+\tfrac{1}{2}\int_1^2\dot{m}\dot{q}^2dt=\tfrac{1}{2}m(1)[\dot{q}(1)]^2+\tfrac{1}{2}k[q(1)]^2 \ . \tag{2.6}$$

Thus, although in this case F is a conservative force, there does not exist a quantity of the type of the usual mechanical energy, equal to the sum of a kinetic energy plus a potential energy, that should be conserved. This is due to the presence of the integral in either Eq. (2.5) or Eq. (2.6), which implies that a time-dependent mass makes the system a nonconservative one, though the forces acting on the particle are conservative. This feature may be related to the fact that a system with a time-dependent mass should be considered as an open system. Thus, if one wants to study the evolution of

$$\tfrac{1}{2}m(t)[\dot{q}(t)]^2+\tfrac{1}{2}k[q(t)]^2 \ ,$$

one finds that, with the particular choice $m(t)=m_0e^{\alpha t}$, the quantity $T+V$ decreases.

(b) If $k=m(t)\omega_0^2$, the force F can no longer be considered as a conservative one, and then (see Appendix B)

$$\tfrac{1}{2}m(2)[\dot{q}(2)]^2+\tfrac{1}{2}m(2)\omega_0^2[q(2)]^2+\tfrac{1}{2}\int_1^2(\dot{m}\dot{q}^2-\omega_0^2\dot{m}q^2)dt=\tfrac{1}{2}m(1)[\dot{q}(1)]^2+\tfrac{1}{2}m(1)\omega_0^2[q(1)]^2 \ . \tag{2.7}$$

Now, if one performs an analysis similar to the one made above, one can conclude that the evolution of the quantity

$$\tfrac{1}{2}m(t)[\dot{q}(t)]^2+\tfrac{1}{2}m(t)\omega_0^2[q(t)]^2$$

is just determined by the integral in Eq. (2.7). This quantity is not the usual energy of the system [in the light of Eqs. (2.5) and (2.7), it is not the counterpart of the so-called mechanical energy for either conservative or nonconservative systems *with constant mass*]. In choosing $m(t)=m_0e^{\alpha t}$, one finds that this integral oscillates, which in turn determines that

$$\tfrac{1}{2}m(t)[\dot{q}(t)]^2+\tfrac{1}{2}m(t)\omega_0^2[q(t)]^2$$

should oscillate. Thus, the oscillation of this quantity is not to be regarded as a disappointing feature, for this behavior is a direct consequence of the nature of the quantity under study, even at a classical level.

B. Variable-mass harmonic oscillator, Lagrangian framework

Within the Lagrangian framework, one is provided with a definite prescription in order to construct the Lagrangian L corresponding to a given system.[21]

For a time-dependent mass particle subject to a harmonic potential, recourse to the well-known Lagrange equations yields the equation of motion given by Eq. (2.2). If one chooses

$$V(q,t)=\tfrac{1}{2}m(t)\omega_0^2q^2 \ , \tag{2.8}$$

one obtains the equation of motion given by Eq. (1.4) via Kanai's Lagrangian [see Eq. (1.5)]. Finally, if one wants to describe a harmonic oscillator with dissipation, one should use the generalized Lagrange equations.[21] A nonconservative force expressed in terms of the Rayleigh dissipation function leads to

$$R=\tfrac{1}{2}\frac{\alpha}{m_0}\dot{q}^2 \ , \tag{2.9}$$

so that the Lagrange equation adopts the appearance:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{\partial R}{\partial \dot{q}} = 0 \ . \tag{2.10}$$

It is easily seen from Eq. (2.8), or Eqs. (2.9) and (2.10), that Lagrangians with different physical origins can yield the same equation of motion for the system. This feature, far from being a disappointing one, clarifies the actual meaning and origin of Kanai's Lagrangian. It was already seen in Sec. II A that different physical systems are described by the same equation of motion.

Consequently, it is not at all obvious that any Lagrangian that gives the equation of motion for the system one wants to describe should be the proper Lagrangian of the system. Furthermore, when such a Lagrangian is not constructed according to a very well-defined prescription (i.e., $L = T - V$), one may be actually dealing with a completely different physical system. In fact, many Lagrangians which yield the same equation of motion can be obtained by recourse to appropriate mathematical techniques such as the integrating factor method (see Ref. 22).

There are many examples of different physical systems (with different Lagrangians) that yield the same equation of motion [the case of a RLC circuit and a $L(t)C(t)$ circuit constitutes a good example (see Ref. 16)]. Of course, when obtaining these "equivalent" Lagrangians, one is to be surprised by the fact that *two* different physical systems can be described by the same equation of motion. It would be misleading to conclude that two or more different Lagrangians do represent the same physical system.

In the case of Kanai's Lagrangian, Ray[8] has found a physical system that is well described by it. Thus, the question that naturally arises is, if different Lagrangians which yield the same equations of motion really represent different systems, where can this difference be found? The answer is, in the *energy* of the system.

Let us conclude this analysis of the Lagrangian description of a system by studying changes of coordinates (keep in mind that one is dealing with one-dimensional problems here; extension to several dimensions is straightforward, provided all the coordinates are independent). If one transforms the coordinate q for a system into another coordinate Q by means of a point transformation,

$$Q = \exp(\alpha t/2) q \ , \tag{2.11}$$

Kanai's Lagrangian turns into

$$L_1 = L_1(Q,\dot{Q},t) = \tfrac{1}{2} m_0 \dot{Q}^2 - \tfrac{1}{2} m_0 \Omega^2 Q^2 - \tfrac{1}{2} m_0 \alpha Q \dot{Q} \ , \tag{2.12}$$

and by recourse to the Lagrange equation, one obtains the equation of motion for the Q coordinate,

$$\ddot{Q} + \Omega^2 Q = 0 \ , \tag{2.13}$$

where $\Omega = (\omega_0^2 - \alpha^2/4)^{1/2}$.

It can be easily seen that the time evolution of Q that arises from Eq. (2.13) would yield, via inversion of Eq. (2.11), the same evolution for q as the one obtained directly from Eq. (1.4). Then, though Lagrangians L and L_1 are different, they represent the same physical system for they are related through a point transformation. Nevertheless, one should recognize that q and *not* Q is the coordinate indicating the position of the particle. Q would give the actual position *only by inversion* of Eq. (2.11).

We can proceed further and realize[23] that the third term in the Lagrangian given by Eq. (2.12) can be expressed as the total time derivative of a function of the coordinate Q, i.e.,

$$-\tfrac{1}{2} m_0 \alpha Q \dot{Q} = \frac{dF(Q)}{dt} \ , \tag{2.14a}$$

where

$$F(Q) = -\tfrac{1}{4} m_0 \alpha Q^2 \ . \tag{2.14b}$$

Therefore the Lagrangian

$$L_2 = L_1 - \frac{dF(Q)}{dt} = L_2(Q,\dot{Q},t) = \tfrac{1}{2} m_0 \dot{Q}^2 - \tfrac{1}{2} m_0 \Omega^2 Q^2 \tag{2.15}$$

would yield the same equation of motion for Q [Eq. (2.13)] as the Lagrangian L_1. One may wonder whether both Lagrangians could represent the same physical system. It should be noticed that the Q coordinate would be *the same* for both Lagrangians. Answers to all the questions posed in this section are given in the next two sections.

C. Variable-mass harmonic oscillator, Hamiltonian framework

Within the Hamiltonian framework, the Hamiltonian H is the generator of the evolution of the system with time. However, it is the energy of the system only if (1) the Lagrangian of the system is the sum of functions, each homogeneous in the (generalized) velocities of degree 0, 1, and 2, respectively; (2) the equations defining the (generalized) coordinates do not depend explicitly on the time; and (3) the potential V does not depend on the (generalized) velocities. Anyway, it should be noticed that the energy one is referring to is defined by

$$E = T + V \ , \tag{2.16}$$

where

$$T = \tfrac{1}{2} m(t) \dot{q}^2 \quad \text{and} \quad V = V(q,t) \ .$$

If a canonical transformation (the analog of the point transformations for the Lagrangian description) is time independent, the new Hamiltonian H' is directly obtained by substituting coordinates in the old Hamiltonian H, while, if it is time dependent, the Hamiltonian transformation is[21]

$$H' = H + \frac{\partial F}{\partial t} \tag{2.17}$$

(F the generating function).

Now, we want to describe the situations studied with the Lagrangians given by Eqs. (1.5), (2.12), and (2.15). Starting from the Lagrangian

$$L = L(q,q,t) = e^{\alpha t}(\tfrac{1}{2} m_0 \dot{q}^2 - \tfrac{1}{2} m_0 \omega_0^2 q^2)$$

[Eq. (1.5)], one obtains

$$p=\frac{\partial L}{\partial \dot{q}}=m_0 e^{\alpha t}\dot{q}$$

[classical counterpart of Eq. (1.10)] and

$$H=H(q,p,t)=\frac{p^2}{2m_0}e^{-\alpha t}+\tfrac{1}{2}m_0 e^{\alpha t}\omega_0^2 q^2$$

[Eq. (1.3)]. This Hamiltonian is known as the Kanai-Caldirola Hamiltonian.[10]

If the Kanai's Lagrangian describes a system, it should be the system characterized by Eqs. (2.1), (2.4), and Newton's second law of motion. According to what was previously discussed, the Hamiltonian given by Eq. (1.3) must be the "energy" of the system $E=T+V$. Therefore, one should not try to define this energy as $E=e^{-\alpha t}H$ [Eq. (1.19)] because this definition would have no physical meaning.

If one starts from the Lagrangian

$$L_1=L_1(Q,\dot{Q},t)=\tfrac{1}{2}m_0\dot{Q}^2-\tfrac{1}{2}m_0\Omega^2Q^2-\tfrac{1}{2}m_0\alpha Q\dot{Q}$$

[Eq. (2.12)] that was obtained from Kanai's Lagrangian through the point transformation given by Eq. (2.11), one obtains

$$P=\frac{\partial L_1}{\partial \dot{Q}}=m_0\dot{Q}-\tfrac{1}{2}m_0\alpha Q \tag{2.18}$$

and

$$H_1=H_1(Q,P,t)=\frac{P^2}{2m_0}+\tfrac{1}{2}m_0\omega_0^2Q^2+\tfrac{1}{2}\alpha PQ\ . \tag{2.19}$$

Of course, for H_1 Hamilton's equations lead to $\ddot{Q}+\Omega^2Q=0$ [Eq. (2.13)].

As all point transformations are canonical, one may be interested in the canonical transformation leading from H to H_1, i.e.,

$$\begin{cases} Q=\exp(\alpha t/2)q\ , \\ P=\exp(-\alpha t/2)p\ . \end{cases} \tag{2.20}$$

Starting from the Lagrangian

$$L_2=L_2(Q,\dot{Q},t)=\tfrac{1}{2}m_0\dot{Q}^2-\tfrac{1}{2}m_0\Omega^2Q^2$$

[Eq. (2.15)], one obtains

$$P'=\frac{\partial L_2}{\partial \dot{Q}}=m_0\dot{Q}\ , \tag{2.21}$$

which is different from Eq. (2.18), and

$$H_2=H_2(Q,P',t)=\frac{P'^2}{2m_0}+\tfrac{1}{2}m_0\Omega^2Q^2\ . \tag{2.22}$$

From Hamilton's equations for the Hamiltonian $H_2(Q,P',t)$, one again obtains the equation of motion $\ddot{Q}+\Omega^2Q=0$ [Eq. (2.13)]. As well as with Lagrangians L_1 [Eq. (2.12)] and L_2 [Eq. (2.15)], one finds that two different Hamiltonians yield the same equation of motion for the Q coordinate.

If one starts from a Hamiltonian of the form of Eq. (2.19) and tries to construct a Hamiltonian of the form of Eq. (2.22) by means of a canonical transformation, one finds that[22]

$$H_1=H_1(Q,P,t)=\frac{P^2}{2m_0}+\tfrac{1}{2}m_0\omega_0^2Q^2+\tfrac{1}{2}\alpha PQ$$

[Eq. (2.19)] via

$$\begin{cases} \Pi=\dfrac{\Omega}{\omega_0}P\ , \\ R=\left[\dfrac{\alpha}{2m_0\Omega\omega_0}\right]P+\dfrac{\omega_0}{\Omega}Q \end{cases} \tag{2.23}$$

yields

$$H_2=H_2(R,\Pi)=\frac{\Pi^2}{2m_0}+\tfrac{1}{2}m_0\Omega^2R^2\ . \tag{2.24}$$

One thus obtains the equation of motion for the R coordinate,

$$\ddot{R}+\Omega^2R=0\ . \tag{2.25}$$

Therefore, it is obvious that, although Lagrangians L_1 [Eq. (2.12)] and L_2 [Eq. (2.15)] do yield the same equation of motion for the Q coordinate, one should not regard this coordinate as being identical for both Lagrangians. In spite of the fact that both Lagrangians give the same formal equation of motion for their respective coordinates (left unchanged on purpose when writing Lagrangian L_2 [Eq. (2.15)]), these coordinates are not identical but are connected through the transformation equations [Eq. (2.23)]. Thus, if Lagrangian L_1 [Eq. (2.12)] is the Lagrangian of the system one wants to describe, one should not consider dropping a term (that representing a total time derivative of a function of the coordinate and time) as just a simplification of the pertinent Lagrangian with no physical relevance. The Q coordinate of Lagrangian L_1 [Eq. (2.12)] cannot be regarded in the same light as the corresponding coordinate of Lagrangian L_2 [Eq. (2.15)], for that would lead to physical inconsistencies. For instance, if one is provided with proper initial conditions for Q and $\dot{Q}$ in L_1, those initial conditions should not be used straightforwardly for the coordinate and velocity in L_2, but should be transformed via Eq. (2.23).

Thus, when analyzing the physical meaning of the equivalent Lagrangians L_1 and L_2 within the Hamiltonian framework, one concludes that different Lagrangians yielding the same equation of motion should not be regarded as describing the same physical system.

We finish this section by mentioning that an alternative demonstration of the nondissipative character of the Kanai-Caldirola (KC) Hamiltonian can be obtained following, for example, Refs. 21, concerning phase-space area-preserving dynamical flows; in fact, the demonstration includes time-dependent Hamiltonians in general.

III. INFORMATION-THEORY APPROACH TO THE PROBLEM

Within Jaynes[24] information-theory context, the statistical operator $\hat{\rho}(t)$ that describes a system having a

Hamiltonian $\hat{H}$ is constructed according to a well-defined prescription.[17,25–27] Combining the normalization for $\hat{\rho}(t)$ (i.e., $\mathrm{Tr}\hat{\rho}(t)=1$) with the knowledge of the expectation values of N operators (i.e.,

$$\langle \hat{O}_j/\hat{\rho}\rangle = \mathrm{Tr}[\hat{\rho}(t)\hat{O}_j] = o_j, \quad 1 \le j \le N)$$

and with the maximum entropy condition for $S=-\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})$, one finds that solving the equation of evolution for

$$\hat{\rho}(t)=\exp\left[-\lambda_0-\sum_{j=1}^{N}\lambda_j(t)\hat{O}_j\right], \tag{3.1}$$

which is $i\hbar[\partial(\hat{\rho})/\partial t]=[\hat{H},(\hat{\rho})]$, is equivalent to finding the q (relevant) operators (with $N \le q$) that close a partial Lie algebra under commutation with the Hamiltonian $\hat{H}$:

$$[\hat{H},\hat{O}_j]=i\hbar\sum_{i=0}^{q} g_{ij}\hat{O}_i \ , \tag{3.2}$$

where the g_{ij} are the elements (c numbers) of a $q\times q$ matrix G (which may depend upon time if $\hat{H}$ is time dependent). Consequently, in order to build $\hat{\rho}(t)$, we need q observables $\hat{O}_i$.

As a result, one can directly obtain the evolution equations for the so-called Lagrange's multipliers λ_i's involved in Eq. (3.1) according to

$$\frac{d\lambda_i}{dt}=\sum_{l=0}^{q} g_{il}\lambda_l \ , \tag{3.3}$$

and the evolution equations for the expectation values of the $\hat{O}_i$ observables,

$$\frac{d\langle \hat{O}_i\rangle}{dt}=-\sum_{l=0}^{q} g_{li}\langle \hat{O}_l\rangle \ . \tag{3.4}$$

The whole dynamics of a system is contained in the G matrix. It is then interesting to see how canonical transformations affect it.

We first assume that the set of relevant operators one is dealing with constitutes the set of canonical coordinates of the system (which reflects the situation we are actually concerned with here). For the sake of simplicity, let us denote the set of (canonical) coordinate operators (which are constructed by directly associating the phase-space coordinates to quantum operators) by a column matrix $\hat{\eta}$, and by $\langle\hat{\eta}\rangle$ the column matrix representing the expectation values of those operators. Then, according to Eq. (3.4),

$$\frac{d\langle\hat{\eta}\rangle}{dt}=-\tilde{G}\langle\hat{\eta}\rangle \ , \tag{3.5}$$

where $\tilde{G}$ is the transpose of G.

A canonical transformation for the phase-space coordinates can be written in matrix notation as

$$\xi=M\eta \ , \tag{3.6}$$

where M is the invertible square matrix that characterizes the canonical transformation from the η coordinates to the ξ coordinates. Its elements are

$$M_{ij}=\frac{\partial\xi_i}{\partial\eta_j} \ . \tag{3.7}$$

Obviously, M will be time dependent if the canonical transformation is time dependent. The quantization is achieved by identifying the phase-space coordinates with their corresponding operators. One obtains

$$\langle\hat{\xi}\rangle=M\langle\hat{\eta}\rangle \ . \tag{3.8}$$

Differentiating Eq. (3.8) with respect to time, one gets

$$\frac{d\langle\hat{\xi}\rangle}{dt}=\left[\frac{dM}{dt}\right]\langle\hat{\eta}\rangle+M\left[\frac{d\langle\hat{\eta}\rangle}{dt}\right] \ . \tag{3.9}$$

From Eq. (3.8) one may write

$$\langle\hat{\eta}\rangle=M^{-1}\langle\hat{\xi}\rangle \tag{3.10}$$

and, replacing Eqs. (3.10) and (3.5) in Eq. (3.9), one obtains

$$\frac{d\langle\hat{\xi}\rangle}{dt}=\left[\left[\frac{dM}{dt}\right]M^{-1}-M\tilde{G}M^{-1}\right]\langle\hat{\xi}\rangle \ . \tag{3.11}$$

Comparing Eq. (3.11) with Eq. (3.5), it is seen that if one defines G' as the matrix obtained by closing the partial Lie algebra of the (relevant) operators denoted by $\hat{\xi}$ with the Hamiltonian $\hat{H}'=\hat{H}'(\hat{\xi},t)$ [that should be obtained through the transformation given by Eq. (2.17)], then

$$\frac{d\langle\hat{\xi}\rangle}{dt}=-\tilde{G}'\langle\hat{\xi}\rangle \tag{3.12}$$

and

$$\tilde{G}'=-\left[\frac{dM}{dt}\right]M^{-1}+M\tilde{G}M^{-1} \ . \tag{3.13}$$

Equation (3.13) gives the transformed matrix under a canonical transformation. Explicit knowledge of the new Hamiltonian $\hat{H}'(\hat{\xi},t)$ (to calculate the matrix elements of G') becomes unnecessary.

As the canonical transformations form a group, if one first applies a canonical transformation (described by M_1) and afterwards a second canonical transformation (described by M_2), this is equivalent to applying a single canonical transformation (described by $M_3=M_2M_1$). Indeed, it can be easily proved that this allows one to write

$$\tilde{G}'=-\left[\frac{dM_1}{dt}\right]M_1^{-1}+M_1\tilde{G}M_1^{-1} \ , \tag{3.13a}$$

$$\tilde{G}''=-\left[\frac{dM_2}{dt}\right]M_2^{-1}+M_2\tilde{G}'M_2^{-1} \ , \tag{3.13b}$$

or, directly,

$$\tilde{G}''=-\left[\frac{dM_2M_1}{dt}\right](M_2M_1)^{-1}+(M_2M_1)\tilde{G}(M_2M_1)^{-1} \ . \tag{3.13c}$$

Consequently, if two different matrices G and G' are

supposed to describe the same physical system, they must be related according to Eq. (3.13) so as to guarantee that the transformation of observables is canonical and therefore the evolution of the system is correctly described by Eq. (3.5) or Eq. (3.12). If two different Hamiltonians that *are not* connected by a canonical transformation *completely* describe a physical situation and one finds that they lead to different G matrices, one should conclude that these Hamiltonians actually pertain to *physical systems with different dynamics.*

We discuss now the Hamiltonians studied in Sec. (II C). If one starts from Kanai-Caldirola's Hamiltonian (in its quantized form)

$$\hat{H}=\hat{H}(\hat{q},\hat{p},t)=\frac{\hat{p}^2}{2m_0}e^{-\alpha t}+\tfrac{1}{2}m_0e^{\alpha t}\omega_0^2\hat{q}^2$$

[Eq. (1.6)], one easily ascertains that the set of observables $\{\hat{q},\hat{p}\}$ closes a semialgebra under commutation with the Hamiltonian $\hat{H}$. One finds

$$G=\begin{bmatrix} 0 & m_0e^{\alpha t}\omega_0^2 \\ -\dfrac{1}{m_0e^{\alpha t}} & 0 \end{bmatrix} . \tag{3.14}$$

We have already seen that, by means of the canonical transformation for the observables $\hat{q}$ and $\hat{p}$, one has

$$\begin{cases} \hat{Q}=\exp(\alpha t/2)\hat{q} \ , \\ \hat{P}=\exp(-\alpha t/2)\hat{p} \ . \end{cases} \tag{3.15}$$

We obtain the transformed Hamiltonian

$$\hat{H}_1=\hat{H}_1(\hat{Q},\hat{P},t)=\frac{\hat{P}^2}{2m_0}+\tfrac{1}{2}m_0\omega_0^2\hat{Q}^2+\tfrac{1}{4}\alpha(\hat{P}\hat{Q}+\hat{Q}\hat{P}) \ , \tag{3.16}$$

in which we use the symmetrization rule for the product PQ.

As expected, the set $\{\hat{Q},\hat{P}\}$ closes a semialgebra under commutation with $\hat{H}_1$, yielding

$$G'=\begin{bmatrix} -\dfrac{\alpha}{2} & m_0\omega_0^2 \\ -\dfrac{1}{m_0} & \dfrac{\alpha}{2} \end{bmatrix} . \tag{3.17}$$

The canonical transformation given by Eq. (3.15) of the observables $\hat{q}$ and $\hat{p}$ into $\hat{Q}$ and $\hat{P}$ can be described by the time-dependent transformation matrix

$$M_1=\begin{bmatrix} \exp(\alpha t/2) & 0 \\ 0 & exp(-\alpha t/2) \end{bmatrix} . \tag{3.18}$$

Thus as

$$M_1^{-1}=\begin{bmatrix} \exp(-\alpha t/2) & 0 \\ 0 & \exp(\alpha t/2) \end{bmatrix} \tag{3.19}$$

and

$$\frac{dM}{dt}=\begin{bmatrix} -\dfrac{\alpha}{2}\exp(-\alpha t/2) & 0 \\ 0 & \dfrac{\alpha}{2}\exp(\alpha t/2) \end{bmatrix} , \tag{3.20}$$

one can easily prove that the equality

$$\tilde{G}'=-\left[\frac{dM_1}{dt}\right]M_1^{-1}+M_1\tilde{G}M_1^{-1}$$

[Eq. (3.13a)] holds, yielding the G' matrix directly from the G and M matrices.

Consequently, if one starts from the Lagrangian

$$L_1=L_1(Q,\dot{Q},t)=\tfrac{1}{2}m_0\dot{Q}^2-\tfrac{1}{2}m_0\Omega^2Q^2-\tfrac{1}{2}m_0\alpha Q\dot{Q}$$

[Eq. (2.12)], which in turn yields the Hamiltonian given by Eq. (2.19), one can drop the last term of Eq. (2.12), as explained in Sec. II B, obtaining

$$L_2=L_2(Q,\dot{Q},t)=\tfrac{1}{2}m_0\dot{Q}^2-\tfrac{1}{2}m_0\Omega^2Q^2$$

[Eq. (2.15)]. If one carelessly assumes that the Lagrangian given by Eq. (2.15) describes the same physical system as the Lagrangian given by Eq. (2.12), then, in constructing the quantal Hamiltonian that corresponds to it, namely,

$$\hat{H}_2=\hat{H}_2(\hat{Q},\hat{P}',t)=\frac{\hat{P}'^2}{2m_0}+\tfrac{1}{2}m_0\Omega^2\hat{Q}^2 \ , \tag{3.21}$$

one would find out that the set $\{\hat{Q},\hat{P}'\}$ closes a partial Lie algebra under commutation with $\hat{H}_2$ yielding

$$G''=\begin{bmatrix} 0 & m_0\Omega^2 \\ -\dfrac{1}{m_0} & 0 \end{bmatrix} . \tag{3.22}$$

We deliberately wrote $\hat{P}'$ instead of $\hat{P}$ in Eq. (3.21), because they denote different operators, as the classical variables they come from are different, i.e., $P=m_0\dot{Q}-\tfrac{1}{2}m_0\alpha Q$ [Eq. (2.18)] and $P'=m_0\dot{Q}$ [Eq. (2.21)].

Apparently, if one does not consider what was discussed in Sec. II C, one may think that $\hat{Q}$ denotes the same operator in Eqs. (3.16) and (3.21). But, as G'' is different from G' and we claim to be describing the same physical system, there must be a canonical transformation that links both Hamiltonians $\hat{H}_1$ and $\hat{H}_2$. In applying Eq. (3.5), we should obtain the same equation of motion for the expectation value of the position operator, namely, $\langle\hat{Q}\rangle$:

$$\langle\ddot{\hat{Q}}\rangle+\Omega^2\langle\hat{Q}\rangle=0 \ . \tag{3.23}$$

As a consequence of this feature, and noticing the difference between G' and G'', we look now for a canonical transformation that leaves the position operator unchanged. This is of the form

$$M=\begin{bmatrix} 1 & 0 \\ x & y \end{bmatrix} . \tag{3.24}$$

The equation $\tilde{G}''=M\tilde{G}'M^{-1}$ [see Eq. (3.13b)] must

hold. Here we drop the term containing the time derivative of the matrix M, as the canonical transformation one is searching for is time independent.

It can be easily proved that there does not exist any matrix M of the form of Eq. (3.24) that fulfils Eq. (3.13d).

If one frees the four elements of the matrix M from any constraint, one easily finds that

$$M=\begin{pmatrix} \frac{\omega_0}{\Omega} & \frac{\alpha}{2m_0\Omega\omega_0} \\ 0 & \frac{\Omega}{\omega_0} \end{pmatrix} \tag{3.25}$$

connects G' and G'' according to Eq. (3.13b). It is then obvious that one is not dealing with the same operator $\hat{Q}$ in the Hamiltonians given by Eqs. (3.16) and (3.21). Within this context, although the Lagrangians given by Eqs. (2.12) and (2.15) lead to the same equation of motion for the coordinate operator, they do not describe the same physical system.

As a further remark, we can mention that the integrating function method (see, e.g., Refs. 22) can be applied in order to obtain different Lagrangians, and therefore different Hamiltonians, that yield the equation of motion (1.4). In fact, in these references, the possibility of obtaining not explicitly time-dependent Lagrangians (and Hamiltonians) is considered. But, in the light of our previous analysis, we can conclude that, if the dynamical variables position *and* momentum are to be the same as those involved in the KC Hamiltonian, the G matrices associated to these time-independent Hamiltonians will be different from (3.14), either in their elements or even in their dimensions. This fact enables us to assert that we would be dealing with different physical systems.

IV. CONCLUSIONS

Dissipative systems are interesting not only because one faces them quite often but also because they allow for a deeper understanding of some natural processes. As we have pointed out in the Introduction, the quantization of dissipation is still an open problem.

One of principal aims of this paper was the exhaustive analysis of the inspiring Kanai-Caldirola's Hamiltonian. After doing so, we applied the maximum entropy principle (MEP) procedure to it, with the aim of reaching unbiased results as far as the physical meaning of this Hamiltonian is concerned.

We can summarize our findings as follows:

(a) We conclude that the Kanai-Caldirola's Hamiltonian does not describe the system it was originally meant for, i.e., a damped harmonic oscillator of mass m_0. Instead, this Hamiltonian actually describes a particle of mass $m(t)=m_0e^{\alpha t}$ subject to a force $F=m_0e^{\alpha t}\omega_0^2q$, as pointed out in Sec. II (see Ref. 8). This conclusion can be easily drawn at the classical level. Many difficulties that are mentioned in literature as emerging from the quantization of this Hamiltonian are instead plain consequences of considering that the Kanai-Caldirola's Hamiltonian describes a physical system different from the one to which it actually refers. Therefore, we point out that all the modifications made to this Hamiltonian, in order to solve the above-mentioned difficulties, should be revised in the light of its real physical meaning. Besides, many and recent contributions to the literature that consider this Hamiltonian as corresponding to the damped harmonic oscillator should also be revised (see Refs. 4–6).

(b) We explicitly find that the Heisenberg's uncertainty principle is not violated in the case of the Kanai-Caldirola's Hamiltonian. In fact, previous interpretations indicating a violation of this principle arise from a confusion concerning the physical meaning of this Hamiltonian. After properly identifying the system that this Hamiltonian actually describes (see Sec. II), one finds that the mechanical momentum is just $p=m(t)\dot{q}$ and that the uncertainty principle holds for the coordinate q and for this momentum p. Then, it is obvious that one should not expect the uncertainty principle to hold for the coordinate q and the so-called "physical" momentum $p_k=m_0\dot{q}$. Besides, all the reinterpretations or modifications of this Hamiltonian[3,20] formulated on account of this apparent violation of the uncertainty principle prove to be meaningless.

(c) We also find that Kanai-Caldirola's Hamiltonian is the energy operator of the system. Therefore, recourse to spurious definitions of the energy operator [as $E=e^{-\alpha t}H$, Eq. (1.19)] becomes unnecessary, and, consequently, one avoids difficulties concerning the ground-state energy and the quantum energy levels. There is no problem in the quantization of the KC Hamiltonian, provided one always bears in mind the nature of the real physical system one is quantifying. Thus, the behavior of the quantized system agrees with what should be expected: well-defined energy levels and oscillating evolution of the expectation value of the Hamiltonian. These are just properties of the system to be quantized, and not problems of the quantization procedure. Since this system is not dissipative at all, we should not try to look for dissipation at the quantum level, or, what proves to be more misleading, to force dissipation at any rate. Anyway, it should be noted that the energy the KC Hamiltonian is referring to is to be analyzed in the light of the discussion in Sec. III, but keeping in mind that this is not a difficulty that arises when one quantifies the system.

(d) As a main conclusion of this paper, we believe that a great deal of confusion has been originated by having lost sight of the way the KC Hamiltonian can be obtained. One should carefully revise the use of "equivalent" Hamiltonians and Lagrangians. In this sense, we find a simple and straightforward method of analyzing the real equivalence of different Hamiltonians (and consequently Lagrangians) within the information theory (IT) context, without needing to get entangled with secondary aspects of the problem in hand. Above all, IT affords the possibility of obtaining unbiased results.

(e) Finally, we mention, at the end of Sec. II, that, following, for example, Refs. 21, a Hamiltonian which completely describes a physical system (even a time-dependent Hamiltonian) cannot yield any dissipation. Although the analysis that can be found in these references is made at the classical level (the phase-space

framework), the same conclusion can be drawn at the quantum level, since the fundamental assumption made is the validity of Hamilton's equations (opposite to the generalized Hamilton's equations that include forces that cannot be derived from a potential). Hamilton's equations can be proved to be directly connected to Schrödinger's equation, in the same way that Liouville's theorem is related to Ehrenfest's theorem.[17,25–27] The existence of a finite Lie algebra associated with this linear time-dependent Hamiltonian seems to be the corresponding aspect in the IT approach for making evident the nondissipative character of the system.

As a brief final remark, we can say that, owing to the oscillation in time of the KC Hamiltonian, it cannot be considered a dissipative Hamiltonian in the sense that it does not represent a decaying function. This oscillation, as was shown in Sec. II, is due to the open character of the system explicitly indicated in the time-dependent function $m(t)$, that in turn allows for an out-of-phase dynamical evolution for the decreasing position and increasing momentum.

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APPENDIX A

One wants to evaluate

$$W=\int_1^2 F\,dq\ , \tag{A1}$$

where [see Eq. (2.1)]

$$F=\frac{dp}{dt}=m(t)\ddot{q}(t)+\dot{m}(t)\dot{q}(t)\ . \tag{A2}$$

(From now on, we will omit writing in the different variables the explicit dependence on time.)

From (A1) and (A2) one can write

$$W=\int_1^2(m\ddot{q}+\dot{m}\dot{q})dq\ . \tag{A3}$$

As $dq=\dot{q}\,dt$, then

$$W=\int_1^2 m\ddot{q}\dot{q}\,dt+\int_1^2\dot{m}\dot{q}^2dt\ . \tag{A4}$$

On integrating by parts the first integral in the rhs of Eq. (A4), one obtains

$$\int_1^2 m\ddot{q}\dot{q}\,dt=\tfrac{1}{2}m\dot{q}^2|_1^2-\tfrac{1}{2}\int_1^2\dot{m}\dot{q}^2dt\ . \tag{A5}$$

Replacing (A5) in (A4),

$$W=\tfrac{1}{2}m\dot{q}^2|_1^2+\tfrac{1}{2}\int_1^2\dot{m}q^2dt\ . \tag{A6}$$

APPENDIX B

One wants to evaluate

$$W=\int_1^2 F\,dq\ , \tag{B1}$$

where [see Eq. (2.4)]

$$F=-m(t)\omega_0^2q(t)\ . \tag{B2}$$

(From now on, we will omit the explicit dependence on time.)

From (B1) and (B2) one can write

$$W=-\omega_0^2\int_1^2 mq\,dq\ . \tag{B3}$$

As $dq=\dot{q}\,dt$, then

$$W=-\omega_0^2\int_1^2 mq\dot{q}\,dt \tag{B4}$$

and, evaluating the integral in the r.h.s. of (B4) by parts, one obtains

$$W=\tfrac{1}{2}m(1)\omega_0^2q(1)^2-\tfrac{1}{2}m(2)\omega_0^2q(2)^2+\tfrac{1}{2}\omega_0^2\int_1^2\dot{m}q^2dt\ . \tag{B5}$$

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