Frequency shifts induced in laser pulses by plasma waves

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An analytical theory is developed that describes how a radiation pulse is modified as it propagates through a plasma with arbitrary temporal and spatial density variations. Expressions are derived for the shifts induced in both the frequency and the wave number of the radiation pulse. The possibility of upshifting the frequency of the laser pulse using plasma waves is analyzed. It is found that maximum shifts occur when the phase velocity of the plasma wave is equal to the group velocity of the laser pulse. Nonlinear analysis of laser propagation in underdense plasmas ($\omega_p^2/\omega^2 \ll 1$, where ω_p is the plasma frequency and ω is the laser frequency) shows that the frequency shift scales asymptotically as the square root of the propagation distance. Furthermore, it is shown that the laser power remains constant as the pulse propagates through the plasma.

I. INTRODUCTION

The propagation of electromagnetic radiation through plasmas is a problem of general interest with a wide variety of applications ranging from communications[1] to laser-driven particle accelerators.[2,3] For example, recent plasma simulation studies[4,5] suggest two possible methods by which the frequency of an electromagnetic (em) wave may be upshifted. In the first method, the plasma density through which the em wave is propagating is suddenly increased in time,[4] while the second method utilizes the interaction of a plasma-wave wake field (having a phase velocity near the speed of light)[2,3,6,7] with a short em pulse.[5] Phenomena such as these, which result from variations in the plasma density, may offer a way of tuning the radiation from a laser or, alternatively, they may describe the distortion of radio signals in the ionosphere.

The possibility of using an intense driving laser pulse to upshift the frequency of a trailing laser pulse within a plasma may be of practical significance. In such a scheme, an intense driving laser pulse of length $\simeq \lambda_p$ (where λ_p is the plasma wavelength) is propagated through a plasma generating large amplitude plasma wave wake fields.[2,3] A trailing, less intense laser pulse of length $<\lambda_p/2$ may be properly phased in the wake field such that its frequency is upshifted as the pulse propagates through the plasma. Frequency upshifts result when the trailing pulse is phased such that it "rides" the wake field in a region where the local wake-field density gradient is negative (see Fig. 1). Alternatively, a driving electron beam may be used to produce the large-amplitude plasma wake field.[6,7] This process of upshifting the frequency of a laser pulse by use of a plasma wake field was proposed and demonstrated in Ref. 5 through the use of particle simulation codes. To explain this phenomenon, Ref. 5 uses arguments based upon Lorentz transformations of the dispersion relation for an em wave in a plasma. The goal of the present work is to develop a rigorous analytic theory describing this process and to clearly delineate the underlying physics responsible for the frequency upshift.

In the following, an analytic theory is developed from first principles that describes how an em radiation pulse is affected by variations in the plasma density. Specifically, the one-dimensional (1D) wave equation is used to derive expressions for the shifts induced in the frequency and in the wave number by arbitrary plasma variations (both in space and in time). In Sec. II, a linear theory is presented which is valid for sufficiently small shifts in the frequency and wave number. It is shown that temporal plasma variations lead to shifts in the radiation frequency, whereas spatial plasma variations lead to shifts in the radiation wave number. For the case of a short radiation pulse interaction with a plasma wave with finite phase velocity and arbitrary amplitude, it is shown that shifts are induced in both the wave number and in the frequency. Maximum frequency shifts may be obtained when the phase velocity of the plasma wave is equal to the group velocity of the radiation pulse evaluated at the ambient plasma density. In Sec. III, a nonlinear theory is developed that is capable of describing large frequency shifts (and amplitude changes) induced in a laser pulse by a nonlinear wake field in an underdense plasma. It is found that the frequency shift asymptotically scales as the square root of the propagation distance. Furthermore, the amplitude of the vector potential of the laser changes in such a manner as to keep the laser power constant as the pulse propagates. This paper concludes with a discussion in Sec. IV.

II. LINEAR THEORY

The 1D wave equation for the normalized transverse vector potential $a = eA_\perp/(mc^2)$ of the radiation field is given by

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] a(z,t) = k_p^2(z,t)\, a(z,t) , \qquad (1)$$

where $k_p^2 \equiv k_{p0}^2 n(z,t)/[\gamma(z,t)n_0]$. Here, $\gamma(z,t)$ is the relativistic factor associated with the motion of the plasma electrons, $n(z,t)$ is the plasma electron density, and $k_{p0}^2=\omega_{p0}^2/c^2$, where ω_{p0} is the electron plasma frequency in the ambient density n_0. In deriving the above equation, conservation of canonical momentum was used, $p_\perp = eA_\perp/c$, which gives a transverse plasma current (in the fluid limit) of $J_\perp \equiv -enp_\perp/(m\gamma)$. Throughout the following it will be assumed that $a^2 \ll 1$ such that $k_p^2(z,t)$ is independent of $a(z,t)$, i.e., the effects of the radiation field on the plasma wave will be ignored.

To solve the above wave equation, it is helpful to write $a(z,t)=b(z,t)\exp(ik_0 z - i\omega_0 t)$, where $b(z,t)$ is the radiation envelope and where ω_0 and k_0 are the frequency and wave number of the radiation in the ambient plasma (in the absence of a plasma wave) that satisfy the dispersion relation $\omega_0^2=c^2k_0^2+c^2k_{p0}^2$. Furthermore, it is convenient to introduce a change of variables $\zeta=z-v_g t$ and $\tau=t$, where $v_g=c^2k_0/\omega_0$ is the group velocity of the radiation in the ambient plasma. The wave equation is then given by

$$\left[2i\frac{\omega_0}{c^2}\frac{\partial}{\partial\tau}+2\frac{v_g}{c^2}\frac{\partial^2}{\partial\zeta\partial\tau}+\frac{1}{\gamma_g^2}\frac{\partial^2}{\partial\zeta^2}-\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\right]b = \delta k_p^2(\zeta,\tau)b(\zeta,\tau) \ , \quad (2)$$

where $\delta k_p^2 \equiv k_p^2 - k_{p0}^2$ and $1/\gamma_g^2 = 1 - v_g^2/c^2$.

It will now be assumed that the radiation envelope $b(\zeta,\tau)$ is slowly varying compared to the radiation frequency ω_0, that is, $|\partial b/\partial\tau| \ll |\omega_0 b|$ and $|\partial b/\partial\zeta| \ll |\omega_0 b/c|$. Assuming this, the second-order derivatives in Eq. (2) may be neglected. Equation (2) may then be solved giving

$$b(\zeta,\tau)=|b_0(\zeta)|\exp\left[-\frac{ic^2}{2\omega_0}\int_0^\tau d\tau'\delta k_p^2(\zeta,\tau')\right] . \quad (3)$$

The condition $|\partial b/\partial\tau| \ll |\omega_0 b|$ implies $|c^2\delta k_p^2/(2\omega_0^2)| \ll 1$. For a plasma density perturbation δn, this condition implies $|\omega_{p0}^2\delta n/(2\omega_0^2 n_0)| \ll 1$. The condition $|\partial b/\partial\zeta| \ll |\omega_0 b/c|$ implies $|\int_0^\tau d\tau'(\partial\delta k_p^2/\partial\zeta)c^3/(2\omega_0^2)| \ll 1$ as well as $|(db_0/d\zeta)c/(b_0\omega_0)| \ll 1$. The first of these inequalities generally implies that the wave-number shift must be small compared to ω_0/c, while the second inequality indicates that the initial envelope $|b_0(\zeta)|$ must be slowly varying compared to ω_0/c.

The total phase $\Phi(\zeta,\tau)$ of the radiation field may be identified by writing $a(\zeta,\tau)=|a(\zeta,\tau)|\exp[i\Phi(\zeta,\tau)]$. It is then possible to examine the evolution of the frequency as well as the wave number of the radiation through the definitions $\omega(\zeta,\tau)\equiv -\partial\Phi/\partial t = -(\partial/\partial\tau - v_g\partial/\partial\zeta)\Phi$ and $k(\zeta,\tau)\equiv\partial\Phi/\partial z = \partial\Phi/\partial\zeta$. Using the above solution for $b(\zeta,\tau)$ gives

$$\omega(\zeta,\tau)=\omega_0+\frac{c^2}{2\omega_0}\delta k_p^2(\zeta,\tau)-\frac{v_g c^2}{2\omega_0}\int_0^\tau d\tau'\frac{\partial}{\partial\zeta}\delta k_p^2(\zeta,\tau') \ , \quad (4a)$$

$$k(\zeta,\tau)=k_0-\frac{c^2}{2\omega_0}\int_0^\tau d\tau'\frac{\partial}{\partial\zeta}\delta k_p^2(\zeta,\tau') \ . \quad (4b)$$

The above equations are valid for arbitrary variations $\delta k_p^2(\zeta,\tau)$. [Recall, $k_p^2=k_{p0}^2 n/(n_0\gamma)$ and the effects of γ become important for nonlinear relativistic plasma waves.[7,8]] Provided $b(\zeta,\tau)$ remains slowly varying compared to ω_0, the amplitude of the normalized vector potential $|a|$ does not change. That is, Eq. (3) indicates that *in the linear theory* $|a|=|b_0(\zeta)|$ and, hence, the initial envelop of the vector potential is simply convected forward at the group velocity v_g (amplitude changes in the vector potential are high-order effects).

To illustrate the above theory, consider a plasma density variation that is a function only of space. For example, consider a radiation pulse entering a plasma ($\omega_p^2/\omega_0^2 \ll 1$) from vacuum with a plasma density profile $\delta k_p^2=\delta k_p^2(z)$ for $z>0$ and equal to zero for $z<0$. Assume that at $t=0$ the radiation pulse extends from $-c\tau_L<z<0$, where $\omega_0\tau_L \gg 1$. Equations (4a) and (4b) indicate that, as the pulse propagates, the frequency and wave number evolve according to $\omega(z,t)=\omega_0$ and $k(z,t)=k_0-c^2\delta k_p^2(z)/(2v_g\omega_0)$. This is in agreement with the well-known result[1] that as radiation propagates into a plasma with spatial density variations the frequency remains constant, whereas the wave number changes such that the dispersion relation $\omega^2=c^2(k^2+k_p^2)$ remains satisfied.

On the other hand, consider a plasma density variation which is a function only of time. For example, consider a radiation pulse (of length $\tau_L \gg 1/\omega_0$) propagating through a long, uniform plasma column (where $\omega_p^2/\omega_0^2 \ll 1$) in which the density is temporally changing, $\delta k_p^2=\delta k_p^2(t)$. Equations (4a) and (4b) indicate that, as the pulse propagates, the frequency and wave number evolve according to $\omega(z,t)=\omega_0+c^2\delta k_p^2(t)/(2\omega_0)$ and $k(z,t)=k_0$. This is in agreement with the simulations of Ref. 4, which indicate that as radiation propagates through a plasma with temporal density variations the wave number remains constant, whereas the frequency changes such that the dispersion relation $\omega^2=c^2(k^2+k_p^2)$ remains satisfied. Alternatively, such a result may be intuitively deduced from standard eikonal theory.[9]

Equations (4a) and (4b) may also be used to examine the case in which a plasma wave (with phase velocity near c)[2,3,6,7] is used to upshift the frequency of a laser pulse, as suggested by the simulations of Ref. 5. (Here the plasma variation is a function both of time and space.) Assuming a plasma wave with a phase velocity v_p such that the plasma-wave quantities are functions of only $z-v_p t$ implies $\delta k_p^2(\zeta,\tau)=\delta k_p^2(\zeta-\Delta v\tau)$, where $\delta k_p^2(\zeta-\Delta v\tau)$ has the form of a periodic oscillation and $\Delta v=v_p-v_g$. Defining the shift in frequency and in wave number as $\Delta\omega=\omega(\zeta,\tau)-\omega(\zeta,0)$ and $\Delta k=k(\zeta,\tau)-k(\zeta,0)$, and using Eqs. (4a) and (4b), gives

$$\Delta\omega=\frac{c^2 v_p}{2\omega_0\Delta v}[\delta k_p^2(\zeta-\Delta v\tau)-\delta k_p^2(\zeta)] \ , \quad (5a)$$

$$\Delta k=\frac{c^2}{2\omega_0\Delta v}[\delta k_p^2(\zeta-\Delta v\tau)-\delta k_p^2(\zeta)] \ . \quad (5b)$$

The above equations indicate that the frequency shift will be maximum for the case $v_p=v_g$. For this case $\Delta\omega = -(v_g \tau c^2/2\omega_0) d\delta k_p^2/d\zeta$ and $\Delta k = -(c^2\tau/2\omega_0) d\delta k_p^2/d\zeta$. Assuming a plasma wave (with $v_p = v_g$) such that $\delta k_p^2(\zeta) = k_{p0}^2 \delta n(\zeta)/n_0$, where $\delta n(\zeta) = \delta n_0 \sin(k_{p0}\zeta)$ is the plasma-wave density perturbation, gives a frequency shift of

$$\frac{\Delta\omega}{\omega_0} = -\pi \frac{\omega_{p0}^2}{\omega_0^2} \frac{\delta n_0}{n_0} \frac{v_g \tau}{\lambda_{p0}} \cos(k_{p0}\zeta) , \tag{6}$$

where $\lambda_{p0} = 2\pi/k_{p0}$. Hence, a positive frequency shift $\Delta\omega > 0$ requires the laser pulse to be positioned properly in the phase of the plasma wave such that $d\delta n/d\zeta < 0$. This is illustrated in Fig. 1. Equation (6) shows that $\Delta\omega$ is a linear function of $v_g\tau$, the distance that the pulse has propagated through the plasma. In such a way $|\Delta\omega|$ will increase until it becomes sufficiently large so that the approximation that $b(\zeta,\tau)$ is slowly varying is not longer valid. For $b(\zeta,\tau)$ to be slowly varying requires $|c\Delta\omega/(v_g\omega_0)|^2 \ll 1$. This gives a limit on the propagation distance $c\tau < \lambda_{p0}\omega_0^2 n_0/(\pi\omega_{p0}^2 \delta n_0)$ for which Eq. (6) remains valid. A convenient method for producing such a plasma wave may be the laser wake field accelerator,[3] in which a driving laser pulse of frequency ω_0 is used to generate a plasma wave with $v_p = v_g$.

For the case $v_p \neq v_g$, $|\Delta\omega|$ is no longer a linearly increasing function of $c\tau$. In fact, Eq. (5a) indicates that $\Delta\omega$ will oscillate as a function of $c\tau$. Consider a plasma wave of the form $\delta k_p^2(\zeta,\tau) = k_{p0}^2 |\delta n/n_0| \cos[k_{p0}(\zeta - \Delta v \tau)]$. Assuming the laser pulse (with a pulse length $c\tau_L \ll \lambda_{p0}$) is initially centered about $\zeta = 0$, Eq. (5a) indicates that the maximum frequency shift is given by

$$|\Delta\omega_m| = |\omega_{p0}^2 v_p \delta n_0/(\omega_0 n_0 \Delta v)|$$

and occurs when $c\tau = |\lambda_{p0}/(2\Delta v)|$. This value of $c\tau$ is the linear detuning distance, i.e., the distance it takes for the laser pulse to phase slip a distance of $\lambda_{p0}/2$ with respect to the plasma wave. Furthermore, notice that $\Delta\omega_m$ may either be positive or negative, depending on the sign of Δv (for the present example, $\Delta\omega_m > 0$ for $v_p < v_g$). Again, the assumption that $b(\zeta,\tau)$ is slowly varying implies $|c\Delta\omega_m/(2v_p\omega_0)|^2 \ll 1$.

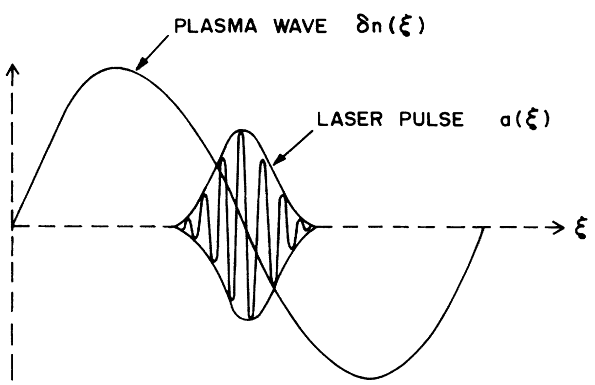

FIG. 1. Schematic of upshifting the laser pulse frequency by a plasma wave with $v_p = v_g$. Positive frequency shifts require the laser pulse to be centered about regions of the wave with a decreasing density.

Physically, the frequency shifts induced in a radiation pulse by a plasma wave may be understood as follows. A plasma wave gives rise to variations in the plasma parameter $k_p^2(\zeta,\tau)$. This leads to variations in the "local" phase velocity of the laser pulse. Heuristically, the "local" dispersion relation for the radiation field is given by $\omega^2(\zeta,\tau) = c^2k^2(\zeta,\tau) + c^2k_p^2(\zeta,\tau)$. For small $c\tau$, this gives $v_p^2(\zeta)/c^2 = 1 + k_{p0}^2 n(\zeta)/(k_0^2 n_0)$. For example, the local phase velocity near the front of the laser pulse ($\zeta = \zeta_+$) will be less than the local phase velocity near the back of the pulse ($\zeta = \zeta_-$) provided $n(\zeta_+) < n(\zeta_-)$. Hence, the individual phase peaks in the pulse $a(z,t) = |a| \exp(ikz - i\omega t)$ may move relative to one another (i.e., closer together for the present example). In such a way both the frequency and the wave number of the radiation pulse will change, as is given by Eqs. (4a) and (4b).

It should be pointed out that due to the local nature of the frequency shift (the dependence of $\Delta\omega$ on ζ), a laser pulse with a finite pulse length $c\tau_L$ will develop a spread in frequency shifts. That is, the frequency shift at the front of the pulse may be different from the frequency shift at the back of the pulse. For example, consider a plasma wave with $v_p = v_g$ of the form $\delta k_p^2 = k_{p0}^2(\delta n_0/n_0)\sin(k_{p0}\zeta)$ and a laser pulse centered about $\bar{\zeta} = \pi$ with $c\tau_L < \lambda_{p0}/2$ (see Fig. 1). The difference in the frequency shift at the center of the pulse with a point $\delta\zeta$ away is given by

$$|[\Delta\omega(\bar{\zeta}) - \Delta\omega(\bar{\zeta} + \delta\zeta)]/\Delta\omega(\bar{\zeta})| = 1 - \cos(\delta\zeta) .$$

This spread in frequency shift may be quite significant.

III. NONLINEAR THEORY

It is possible to calculate nonlinear frequency shifts ($\Delta\omega \geq \omega$) analytically in the limit $v_p \simeq v_g \simeq c$. As is discussed below, this is a valid approximation when $\omega_p^2/\omega^2 \ll 1$. Introducing the variables $\zeta = z - ct$ and $\tau = t$, it is possible to calculate the nonlinear source current for the wave equation by use of the quasistatic approximation.[8] The quasistatic approximation may be used to model the interaction of a short laser pulse with a plasma provided that the laser pulse evolution time τ_e is long compared to the transit time of the plasma electrons through the pulse. Typically, $\tau_e \sim (\omega/\omega_p)/\omega_p$, hence the quasistatic approximation is valid for laser pulse lengths $\tau_L \ll (\omega/\omega_p)/\omega_p$. Within the quasistatic approximation, cold fluid equations for the plasma electrons may be used to calculate the transverse current.[8] In 1D, it may be shown $J_\perp = -ena_\perp/\gamma = -en_0 a_\perp/(1+\phi)$, where $\phi = e\phi_{es}/mc^2$ is the normalized electrostatic potential of the plasma. Hence, the transverse wave equation is given by

$$\left[\frac{2}{c}\frac{\partial}{\partial\zeta} - \frac{1}{c^2}\frac{\partial}{\partial\tau} \right] \frac{\partial}{\partial\tau} a = k_{p0}^2 \frac{a}{(1+\phi)} \equiv k_p^2 a . \tag{7}$$

In applying the above equation to the frequency shifts induced in a short laser pulse $c\tau_L < \lambda_p/2$ by a large amplitude plasma wave, ϕ represents the electrostatic potential of the plasma wave (with $v_p \simeq c$), which is assumed to be known and independent of the vector potential of the short laser pulse.

To calculate the nonlinear frequency shifts induced by such a plasma wave, the laser pulse vector potential is written $a(\zeta,\tau)=a_0\exp[i\Phi(\zeta,\tau)]$, where a_0 is a constant. Assuming $|\partial\Phi/\partial\zeta| \gg |\partial\Phi/\partial(c\tau)|$ (which is shown below to be valid when $\omega_p^2/\omega^2 \ll 1$), the wave equation becomes

$$-\left[\frac{\partial\Phi}{\partial\zeta}\right]\left[\frac{\partial\Phi}{\partial\tau}\right]+i\frac{\partial^2\Phi}{\partial\zeta\partial\tau}=\frac{c}{2}k_p^2 . \quad (8)$$

The above equation describes the nonlinear evolution of the complex phase Φ of the radiation field. Furthermore, when $|\partial\Phi/\partial\zeta| \gg |\partial\Phi/\partial(c\tau)|$, the pulse frequency is given by $\omega(\zeta,\tau)\simeq c\partial\,\mathrm{Re}(\Phi)/\partial\zeta=ck(\zeta,\tau)$, where $\mathrm{Re}(\Phi)$ is the real part of Φ.

For the case in which the plasma gradient may be adequately approximated by its first-order Taylor expansion, $k_p^2\simeq k_{p0}^2+\zeta\partial(k_p^2)/\partial\zeta$, the above equation may be solved analytically. Assuming a solution of the form $\Phi(\zeta,\tau)=k(\tau)\zeta+\Theta(\tau)$, one finds

$$k(\tau)=k_0\left[1-\frac{c\tau}{L}\frac{k_{p0}^2}{k_0^2}\right]^{1/2} , \quad (9a)$$

$$\Theta(\tau)=L(k-k_0)+i\ln(k/k_0)+\Theta_0 , \quad (9b)$$

where $k_0=k(\tau=0)$, $\Theta_0=\Theta(\tau=0)$, and $\partial(k_p^2)/\partial\zeta\equiv k_{p0}^2/L$. Equation (9a) indicates that upshifting the frequency requires $L<0$. Furthermore, Eq. (9a) shows that asymptotically, for large $c\tau$, $\omega\simeq ck\sim\sqrt{c\tau}$. Recalling that $k_p^2\equiv k_{p0}^2(1+\phi)^{-1}$ and expanding about $\zeta=0$ (chosen such that $\phi=0$ at $\zeta=0$) gives $k_p^2\simeq k_{p0}^2\simeq k_{p0}^2(1-\zeta\partial\phi/\partial\zeta)$. Thus, $L^{-1}=-\partial\phi/\partial\zeta$. Assuming $\phi=\phi_0\sin k_p\zeta$ gives $L^{-1}=-k_p\phi_0$. In the limit $|\phi_0|\ll 1$, one has $\phi_0\simeq-\delta n_0/n_0$, where $\delta n_0/n_0$ is the normalized amplitude of the density perturbation $\delta n=\delta n_0\sin k_p\zeta$. This indicates that in order to have frequency upshifts, $L<0$, the pulse must be located in a region of increasing $\phi(\zeta)$, which corresponds to a region of decreasing $\delta n(\zeta)$.

Using the above expressions for $k(\tau)$ and $\Theta(\tau)$, the vector potential of the laser field may be written as

$$a(\zeta,\tau)=\frac{a_0k_0}{k(\tau)}\exp[ik\zeta+iL(k-k_0)+i\Theta_0] . \quad (10)$$

Notice that now the amplitude of the laser pulse evolves as a function of τ, in addition to the phase, such that the power of the laser pulse $P\sim|\partial a/\partial\zeta|^2$ is constant in τ. In order to upshift the laser frequency $\omega(\tau)\simeq ck(\tau)$, it is necessary for $L<0$, which implies that the amplitude of the vector potential $|a|$ decreases. Furthermore, notice that $|\partial\Phi/\partial(c\tau)|/|\partial\Phi/\partial\zeta|\simeq k_{p0}^2/2k^2$. Hence, validity of the above theory requires $k_{p0}^2/k^2\ll 1$. [Notice that by using the definition $\omega=(c\partial/\partial\zeta-\partial/\partial\tau)\Phi$ along with Eqs. (9a) and (9b), one finds $\omega\simeq ck(1+k_p^2/2k^2)$ and, hence, $\omega\simeq ck$ to order $k_p^2/k^2\ll 1$.] For the case $L>0$, which corresponds to decreasing the pulse frequency, Eq. (9a) becomes invalid when $k(\tau)$ decreases to the point where $k^2\sim k_{p0}^2$.

Although Eq. (10) indicates that the pulse power $P\sim|\partial a/\partial\zeta|^2$ is constant in τ, the total energy in the pulse is not constant. This is true because the pulse length will evolve in τ. Since the local group velocity of the radiation field is a function of ζ, $v_g(\zeta)/c\simeq 1-k_p^2(\zeta)/(2k^2)$, the local group velocity at the leading edge of the pulse will be different from the local group velocity at the trailing edge of the pulse, hence the pulse length will change. The evolution of the pulse length may be estimated as follows. Consider a laser pulse with a square profile of initial length $c\tau_{L0}=2\delta\zeta_0$, centered about $\zeta=0$, where $k_p(\zeta=0)=k_{p0}$. The difference between the group velocity at the leading edge and the group velocity at the trailing edge, $\Delta v_g=v_g(\delta\zeta_0)-v_g(-\delta\zeta_0)$, is $\Delta v_g/c\simeq\delta\zeta_0k_{p0}^2/(|L|k^2)$, where $L<0$ has been assumed. The pulse length is given by $c\tau_L=c\tau_{L0}+\int_0^\tau d\tau\Delta v_g$ and, using Eq. (9a), one finds

$$c\tau_L=c\tau_{L0}[1+\ln(k/k_0)] . \quad (11)$$

The total energy in the laser pulse may be estimated by $U\sim c\tau_L|\partial a/\partial\zeta|^2$. The change in the pulse energy $\Delta U=U(\tau)-U_0$ is then $\Delta U/U_0\simeq\ln(k/k_0)$ and, thus, the pulse energy increases as $k(\tau)$ increases. The analysis presented in Ref. 5 assumes that the number of photons $(\sim U/k)$ in the laser pulse remains constant. The above results indicate that $U/k\sim(k_0/k)\ln(k/k_0)$. Hence, it is only for small frequency shifts $k=k_0+\delta k$, where $|\delta k/k_0|\ll 1$, for which U/k may be assumed to be constant.

Equation (9a) indicates that asymptotically, for large $c\tau\gg|L|(k_0^2/k_{p0}^2)$, the frequency scales as $\omega\simeq\omega_{p0}(c\tau/|L|)^{1/2}$, where $c\tau$ represents the distance the pulse has propagated through the plasma. Notice that frequency shifts $\Delta\omega\simeq\omega_0$ require $c\tau\simeq|L|(k_0^2/k_{p0}^2)\simeq\lambda_p(k_0^2/k_{p0}^2)/(2\pi\phi_0)$, which can be quite a large propagation distance when $k_0^2/k_{p0}^2\gg 1$ and $\phi_0\ll 1$. In principle, Eq. (9a) indicates that there is no upper limit to how far the frequency may be upshifted (assuming that the plasma wave and laser pulse may be sustained over a sufficiently large distance within the plasma), however this is a result of assuming $v_p\simeq v_g\simeq c$. In practice, the phase velocity of the plasma wave v_p will not be equal to the group velocity of the laser pulse v_g. The fact that $v_p\neq v_g$ implies that the laser pulse will "phase slip" out of the region of the plasma wave for which $\partial\phi/\partial\zeta>0$ and thus will no longer be frequency upshifted. The detuning distance $c\tau_d$, defined to be the propagation distance required for the pulse center to phase slip a distance $\lambda_p/2$ with respect to the position on the plasma wave for which $\phi=0$, is given by $\lambda_p/2=|\int_0^{\tau_d}d\tau(v_p-v_g)|$. Assuming $v_p=c$ and $v_g/c\simeq 1-k_{p0}^2/(2k^2)$, where $k=k(\tau)$ is given by Eq. (9a), gives

$$c\tau_d=\frac{\lambda_p}{2\pi\phi_0}\frac{k_0^2}{k_{p0}^2}[\exp(2\pi\phi_0)-1] , \quad (12)$$

where $|L^{-1}|\simeq k_p\phi_0$ has been used. Inserting this into Eq. (9a) gives a maximum frequency upshift of

$$k(\tau_d)=k_0\exp(\pi\phi_0) . \quad (13)$$

Notice, in the limit $2\pi\phi_0<1$, one has $c\tau_d\simeq\lambda_p(k_0^2/k_{p0}^2)$ and $k(\tau_d)\simeq k_0(1+\pi\phi_0)$. Clearly, to achieve large frequency shifts $\Delta k(\tau_d)>k_0$, a large-amplitude plasma wave $2\pi\phi_0>1$ (with peak[7,8] $|\delta n|\geq n_0$) is required.

The above nonlinear analysis has assumed that the background ambient plasma density is uniform. However, if the ambient density is a slowly increasing function of the propagation distance $c\tau$, then it may be possible to increase the detuning distance.[5] The above results may be readily modified to include the τ dependence in the quantities $k_{p0}(\tau)$ and $L(\tau)$. One finds that $k(\tau)$ and $\Theta(\tau)$ are given by

$$\frac{k(\tau)}{k_0}=\left[1-\frac{c}{k_0^2}\int_0^\tau d\tau'\frac{k_{p0}^2(\tau')}{L(\tau')}\right]^{1/2} \quad (14a)$$

$$\Theta(\tau)=Lk-L_0k_0-\int_0^\tau d\tau\, k\partial L/\partial\tau+i\ln(k/k_0)+\Theta_0 , \quad (14b)$$

where $L_0=L(0)$. In terms of the normalized electrostatic potential of the wake field ϕ, Eq. (14a) may be used to give an expression for the laser frequency ω as a function of the propagation distance $c\tau$,

$$\frac{\omega(\tau)}{\omega_0}\simeq\left[1-\frac{\omega_{p0}^2}{\omega_0^2}\int_0^{c\tau}d(c\tau)\frac{\partial}{\partial\zeta}(1+\phi)^{-1}\right]^{1/2} , \quad (14c)$$

where $\omega_0=\omega(\tau=0)$.

As the background plasma density increases, several effects occur: (i) the plasma wavelength decreases, (ii) the rate at which the laser frequency evolves is changed, and (iii) the group velocity of the laser pulse decreases. To account for these effects, it is necessary to consider the evolution of the relative distance $\Delta\zeta$ between the center of the laser pulse and the position on the plasma wave for which $\phi=0$. The rate of change of $\Delta\zeta$ as a function of the propagation distance is given by

$$\frac{\partial\Delta\zeta}{\partial\tau}=l\frac{\partial\lambda_p}{\partial\tau}-(v_p-v_g) , \quad (15)$$

where l is the number of plasma periods behind the end of the driving beam (producing the wake field with wavelength $\lambda_p=2\pi/k_p$) at which the laser pulse is located. Furthermore, it will be assumed that the phase velocity of the plasma wave is constant $v_p=v_{p0}$ and $v_g/c\simeq1-k_p^2/(2k^2)$, where k is given by Eq. (14a). Assuming a small linear increase in the ambient density, $k_p^2(\tau)=k_{p0}^2(1+c\tau/L_n)$, and assuming that the frequency shift remains small, $k^2\simeq k_0^2+\delta k^2$, Eq. (15) may be solved to determine the conditions required to have no phase slippage, i.e., $\partial\Delta\zeta/\partial\tau=0$. Setting $\partial\Delta\zeta/\partial\tau=0$ implies the following two conditions (assuming $l=1$):

$$L_n^{-1}=-L^{-1}k_{p0}^2/k_0^2 , \quad (16a)$$

$$\lambda_{p0}L_n^{-1}/2=1-v_{p0}/c-k_{p0}^2/(2k_0^2) . \quad (16b)$$

Equation (16a) gives the rate at which the ambient density must be linearly increased, $L_n>0$. (Recall that $L<0$ corresponds to frequency upshifting of the laser pulse.) Equation (16b) indicates the restrictions necessary on the plasma wave "equilibrium" in order for there to be no phase slippage. For example, the limit $v_{p0}/c=1-k_{p0}^2/k^2$ implies that $L=-\lambda_{p0}$ (where $L^{-1}=-\partial\phi/\partial\zeta$). Satisfying both Eqs. (16a) and (16b) to achieve no slippage may not be possible in practice. However, such an increase in density may result in an increase in the detuning distance and thus a larger frequency upshift.

As a final remark, it should be noted that a large amplitude plasma wave may be used not only to upshift the frequency of a trailing laser pulse, but also to optically guide a trailing pulse. The diffractive properties of a laser pulse in a plasma wave are determined by the effective index of refraction[8] $\eta\simeq1-(k_{p0}^2/2k^2)(1+\phi)^{-1}$ (this expression assumes that the radial profiles of the plasma wave and laser pulse are broad compared to λ_p). For optical guiding to be possible, it is necessary for the radial profile of η to exhibit a maximum on axis, $\partial\eta/\partial r<0$. This is the case if the laser pulse is located at a position on the plasma wave at which $\phi>0$ (assuming $\partial\phi/\partial r<0$). Hence, it may be possible for a plasma wave to simultaneously upshift the frequency and optically guide the laser pulse provided the pulse is phased at a position where $\partial\phi/\partial\zeta>0$ and $\phi>0$.

IV. DISCUSSION

The analytic theory presented above describes how a laser pulse becomes modified due to variations in the plasma through which the pulse propagates. In particular, this theory is used to examine the process by which a plasma wave wake field upshifts the frequency of a laser pulse. The linear theory presented in Sec. II describes how arbitrary variations in the plasma parameter $k_p^2(\zeta,\tau)$ lead to shifts in both the frequency and wave number, as indicated by Eqs. (4a) and (4b). The validity of the linear theory requires that these shifts are sufficiently small in comparison to the initial frequency and wave number. It was shown that temporal plasma variations lead to frequency shifts, whereas spatial plasma variations lead to wave-number shifts. The possibility of upshifting the frequency of the laser pulse using a plasma wave has been examined and it is found that maximum frequency shifts result when $v_p=v_g$. Positive frequency shifts require phasing the laser pulse such that it is centered at a position of decreasing density or, more precisely, at a position where $\partial\delta k_p^2/\partial\zeta<0$. Physically, these shifts may be interpreted as arising from variations in the local phase velocity of the radiation field. Local phase velocity variations allow the individual phase peaks in the radiation field to move relative to one another, thus changing the radiation frequency and wave number.

The nonlinear theory presented in Sec. III is valid for large frequency shifts occurring in laser pulses propagating in underdense plasmas ($k_p^2/k^2\ll1$). In the region where the gradient in the electrostatic potential of the plasma wave is approximately linear, the evolution of the laser frequency is given by Eq. (14c). The amplitude of the radiation field evolves in such a manner as to keep the laser power constant as the pulse propagates. The pulse energy increases logarithmically with increasing frequency, as a result of the pulse length increasing. For a uniform plasma (ϕ independent of τ), the frequency $\omega(\tau)$ asymptotically scales as the square root of the propagation distance. Large frequency shifts $\Delta\omega\gtrsim\omega_0$ require propagation over large distances $c\tau\gtrsim\lambda_p\omega_0^2/(2\pi\phi_0\omega_{p0}^2)$. Assuming that the laser pulse and the plasma wave may be maintained over a sufficiently long distance, the process of frequency upshifting is limited by phase detuning. It may be possible to increase the detuning distance by slowly increasing the ambient plasma density as a function of the propagating distance. Furthermore, it may be possible to phase the laser pulse at a position on the plasma wave (where $\partial\phi/\partial\zeta > 0$ and $\phi > 0$) such that the plasma wave both upshifts the laser frequency and optically guides the laser pulse.

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