Singular continuous quasienergy spectrum in the kicked rotator with separable perturbation: Possibility of the onset of quantum chaos

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We prove that the quasienergy spectrum of the kicked quantum rotator model with separable potential that has been recently introduced by Combescure [J. Stat. Phys. **59**, 679 (1990)] is singularly continuous under certain conditions. The time evolution of this system is numerically investigated in detail.

I. INTRODUCTION

Recently a great deal of work has been devoted to time-dependent quantum systems beyond the usual perturbation theory.[1] The motivation of these papers was to investigate to what extent chaotic behavior displayed by the classical time-dependent systems is present in their quantum counterparts. Along this line of research the numerically, most thoroughly, investigated model is the so-called kicked quantum rotor that corresponds to the classical map of Chirikov and Taylor.[2] The result obtained for it was, however, disappointing: It appeared that for small coupling constants (which by the way, were large enough for the classical system that was completely chaotic) this system displays recurrent behavior associated with the pure pointness of its quasienergy spectrum.[3] The nonrecurrent pattern appeared only if the driving pulses were rationally connected with the internal frequency of the free rotor.[3] These "quantum resonances" that are connected with the absolutely continuous part of the quasienergy spectrum represent, however, completely different behavior than the expected chaotic one. They simply describe the resonant energy accumulation inside the system. The chaotic behavior (if any) is expected to take place for nonresonant frequencies only.

These desultory numerical results (known also as the quantum suppression of chaos) obtained a clear heuristic explanation in the work of the Fishman, Grempel, and Prange,[4] who showed that there is a close connection between the behavior of the quantum rotor and the Anderson localization in disordered solids. Using this connection it can be easily shown that the recurrent and resonance patterns in the quantum rotor correspond to localized and extended states in the Anderson model, respectively. Concerning the chaotic behavior of the rotor, it is expected that *it must be connected with the singular continuous component of the Floquet operator* and hence with the appearance of exotic states in the corresponding Anderson model. The singular continuous spectrum is, however, a subject that is mathematically very subtle. Trying to prove its existence for the quantum rotor one's success is contingent upon the possibility of the Fishman-Grempel-Prange model. The point is that the similarity between the quantum rotor and the Anderson model (no matter how illustrative it may be) does not endure a rigorous mathematical analysis and cannot be used if such fragile objects like singular continuous spectrum are concerned.

On the other hand, there are two arguments that make the search for the singular continuous states in the quantum rotor quite optimistic. The first is an abstract mathematical result obtained by Casati and Guarneri,[5] which showed that for a "generic" potential there is some continuous spectrum left even in the nonresonant case. They were, however, not able to prove that this spectrum is in fact singular continuous, although they conjectured that it is. The biggest disadvantage of their result is, however, that it holds only for a generic potential and one is in fact not able to check whether a given potential belongs to this generic class.

The second argument for the benefit of the singular continuous states comes from the numerical results.[3,6] It has been shown that the recurrent nonresonance patterns that take place for small couplings are replaced by a nonrecurrent behavior as soon as the coupling becomes strong. These results have been interpreted by some authors as a manifestation of the singular continuous spectral component of the Floquet operator.[3,6] The arguments are, however, not completely convincing. First of all, it is not quite clear that the behavior is nonrecurrent. It is possible that we have a recurrent pattern with a very long period and hence with localized states. The computer results do not offer the possibility of distinguishing which spectral type is in fact present. The second difficulty is connected with the fact that it is not clear how the singular continuous states would manifest themselves in the course of the computer simulation. In the ideal case one expects, of course, a nonrecurrent pattern. But the real result represents an interplay between the presence of exotic states on the one side and the unavoidable computer errors on the second side. Now taking into account the extreme complexity of the singularly continuous eigenfunctions (note that this spectrum lives on something like a Cantor set[7]), it is possible that the numerical errors win after few iterations and that the expected nonrecurrent pattern will be replaced by an oscillatory one.

In this situation it appears to be of interest to investigate a more simple model in which the mathematical as well as the numerical analysis go hand in hand up to the end. Our aim here is to demonstrate how one can "see" the presence of the singular continuous states in a standard numerical simulation of the time evolution of the system. For this purpose we chose a simplification of the quantum rotor model introduced recently by Combescure.[8] In this model the local potential of the standard rotor is replaced with a separable potential of rank 1. The advantage of this procedure is that the spectral properties of the corresponding Floquet operator can be thoroughly analyzed and in particular, the presence of the singularly continuous component can be *rigorously* proven. We describe the corresponding results in Sec. II. Section III contains numerical results. We conclude the Introduction by quoting from a paper by Casati *et al.*:[1] "Mathematical rigor (in quantum chaos) though desirable, is a rare occurrence: only few are the landmarks in this terra incognita."

II. MATHEMATICAL PRELIMINARIES

Before proceeding further we remind the reader of the basic mathematical results concerning the instability of a time-dependent quantum system. Let $H(t)$ denote a time-dependent quantum Hamiltonian defined on a Hilbert space $\mathcal{H}$. We will suppose that $H(t)$ has the form

$$H(t)=H_0+V(t) \ , \tag{2.1}$$

with H_0 being a self-adjoint operator, bounded from below, with a discrete spectrum. The potential $V(t)$ is assumed to be time periodic

$$V(t)=V(t+T), \quad T>0, \quad t\in\mathbb{R} \ , \tag{2.2}$$

and such that the resulting Hamiltonian $H(t)$ is reasonably defined. The dynamics of the corresponding quantum system is described by an evolution operator $U(t)$, which solves the time-dependent Schrödinger equation

$$\begin{aligned} &i\partial_t U(t)=H(t)U(t) \ , \\ &U(0)=1 \ . \end{aligned} \tag{2.3}$$

To investigate the stability or instability of the system, it is sufficient to investigate the stroboscopic picture of the time evolution at time t_ν which is equal to the integer multiple of the period T

$$t_\nu=\nu T \ . \tag{2.4}$$

This "stroboscopic" evolution is governed by the one-cycle Floquet operator U

$$U=U(T) \ , \tag{2.5}$$

$$\psi(\nu T)=U^\nu\psi(0) \ . \tag{2.6}$$

[$\psi(0)$ and $\psi(\nu T)$ denote the wave function at times 0 and νT, respectively, $\nu=1,2,\ldots$].[1]

The spectral nature of U is of central importance for the time evolution of the system. In order to illustrate this statement, let us compute the probability $p_{n,m}(\nu)$ to excite the nth state of H_0 after ν cycles to the mth state,

$$p_{n,m}(\nu)=|P_{n,m}(\nu)|^2 \ , \tag{2.7}$$

with $P_{n,m}$ being the probability amplitude

$$P_{n,m}(\nu)=\langle n|U^\nu|m\rangle \ . \tag{2.8}$$

Decomposing the Floquet operator U into its eigenvectors

$$U|\omega\rangle=e^{i\omega}|\omega\rangle \ , \tag{2.9}$$

we get

$$p_{n,m}(\nu)=\sum_\omega e^{i\nu\omega}\langle n|\omega\rangle\langle\omega|m\rangle \ . \tag{2.10}$$

Hence the transition $n\to m$ is possible only if at least one quasienergy state $|\omega\rangle$ connects these two states, i.e., if $\langle n|\omega\rangle\langle\omega|m\rangle\neq 0$ for at least one ω. Now the importance of the spectral nature of U becomes apparent: If the spectrum of U is pure point (PP), all the quasienergy states $|\omega\rangle$ are localized. Consequently, they can connect only a few states of the original Hamiltonian H_0 and a strong recurrence is expected (this is the essence of the well-known theorem of Hogg and Hubermann[9]). In the case of continuous spectrum of U, the quasienergy states $|\omega\rangle$ are not normalizable. They must therefore connect an infinite number of the original states $|n\rangle$ leading in such a way to nonrecurrent (unstable) evolution. Summarizing this heuristic argument, we can say that the pure pointness of the Floquet operator means stability, while the occurrence of the continuous spectrum implies instability of the system. Therefore the only promising systems (from the points of view of the quantum chaos) are those with a continuous quasienergy spectrum. We will discuss these in more detail. Let us note that in classical systems the appearance of the continuous spectrum is a guarantee of mixing.

From the abstract point of view we can divide the continuous spectrum into two parts: the absolutely continuous (AC) and the singularly continuous (SC). Let us assume that the spectrum of U is purely AC. We get then for the probability amplitude

$$P_{n,m}(\nu)=\int e^{i\omega\nu}d\mu_{n,m}(\omega)=\int e^{i\omega\nu}f_{n,m}(\omega)d\omega \ , \tag{2.11}$$

with $f_{n,m}(\omega)\in L^1(0,2\pi)$. (The spectral measure $d\mu_{n,m}$ is continuous with respect to the Lebesgue measure.) Using now the Lebesgue lemma[10] we find

$$\lim_{\nu\to\infty} P_{n,m}(\nu)=0 \tag{2.12}$$

for all n,m. Consequently, the probability of finding the system after ν oscillations at a state m tends to zero as $\nu\to\infty$. In other words, this means that the system is continuously accelerated and excites to higher and higher states. The mean energy is supposed to grow very quickly with time. This type of behavior is usually associated with some kind of resonance phenomena. For example, for the kicked quantum rotator with resonance frequency, the energy growth is quadratic in time.

In the singularly continuous case the probability $p_{n,m}(\nu)$ decreases very slowly toward zero as $\nu\to\infty$. We find, in this case,

$$P_{n,m}(\nu)=\int e^{i\omega\nu}d\mu_{n,m}(\omega) \ , \tag{2.13}$$

with a measure $d\mu_{n,m}$ being singular with respect to the Lebesgue measure. This implies that (see the RAGE theorem[11])

$$\lim_{M\to\infty}\sum_{\nu=0}^{M} p_{n,m}(\nu)=\infty \ , \tag{2.14}$$

together with

$$\lim_{M\to\infty}\frac{1}{M}\sum_{\nu=0}^{M} p_{n,m}(\nu)=0 \ . \tag{2.15}$$

The system now spends an infinite amount of time in the "lower" states and the "return probability" (2.14) diverges. On the average, however, the system escapes any fixed state which leads to a vanishing Cesaro mean (2.15). The evolution along the states $|n\rangle$ is now "recurrently pulsing" with larger and larger "amplitude." The energy growth is slow and mimics the diffusive acceleration known from classical chaotic systems. This type of quasienergy spectrum is assumed to be responsible for the "true" quantum chaotic evolution.

Let us now return to the model. It is described by a time-dependent Hamiltonian

$$H(t)=H_0+\Delta(t)P \ , \tag{2.16}$$

where H_0 is any self-adjoint operator with a pure-point spectrum. A particular case is the kinetic-energy operator of the free rotor

$$H_0=-\frac{1}{2}\frac{\partial^2}{\partial\theta^2} \ , \tag{2.17}$$

defined on Hilbert space

$$\mathcal{H}=L^2(0,2\pi) \ , \tag{2.18}$$

with periodic boundary conditions. P denotes a separable potential of rank 1

$$P=|f\rangle\langle f|, \quad f\in\mathcal{H} \tag{2.19}$$

and $\Delta(t)$ is a periodic sequence of kicks

$$\Delta(t)=\sum_{\nu\in\mathbb{Z}}\delta(t-\nu T) \ . \tag{2.20}$$

In this case the Floquet operator is given by a quantum map

$$U=e^{-iH_0T}e^{-iPT} \ , \tag{2.21}$$

the spectral properties of which can be simply analyzed.

Notation. In what follows we will use the following notation. We will say that $f\in L^2(0,2\pi)$ belongs to L if it has summable Fourier coefficients, i.e., if

$$f(x)=\sum_n a_n e^{inx} \ ,$$

with

$$\sum_n |a_n|<\infty \ .$$

Theorem 1 (Combescure).

(i) Suppose that $f\in L^2(0,2\pi)\cap L$. Then the operator U has pure point spectrum for almost every T.

(ii) If $f\in L^2(0,2\pi)$ but $f\notin L$, then (a) the spectrum of U is pure point if T/π is a rational number and (b) the spectrum of U is purely continuous if T/π is a Diophantine number (i.e., an irrational number that is poorly approximated by rational numbers).

Combescure based the proof of this theorem on her generalization of the results of Simon and Wolff that concern the stability of the dense pure-point spectra of self-adjoint operators under rank-1 perturbations.[12] Part (i) of the theorem holds for all T which are rational multiples of π and for almost every T which is an irrational multiple of π (see Ref. 8 for more details).

For our purposes we need to investigate the continuous spectrum in more details and prove that it is in fact *purely singularly continuous.*

Theorem 2. Assume that the conditions of part (ii) of Theorem 1 are fulfilled with T/π being a Diophantine number. Then the spectrum of U is purely singularly continuous.

Proof. We know from the Theorem 1 that the spectrum of U is continuous. It is therefore sufficient to prove that its absolutely continuous part is empty. For this reason we use the fact that P is a rank-1 operator, which enables us to express $\exp(-iPT)$ as

$$\exp(-iPT)=1+\frac{1}{\|f\|^2}[\exp(-i\|f\|^2T)-1]P \ . \tag{2.22}$$

Inserting this formula into (2.21) we get

$$U=U_0+R \ , \tag{2.23}$$

where U_0 is the free evolution operator

$$U_0=\exp(-iH_0T) \ , \tag{2.24}$$

and R is an operator of rank 1. We use now the scattering theory. Let us assume that the spectrum of U is absolutely continuous. In this case the wave operators

$$\Omega_{\pm}=s\lim_{\nu\to\pm\infty}U^{\nu}U_0^{-\nu} \tag{2.25}$$

exist (because of the rank-1 perturbation R, see, for instance, Refs. 13 and 14 for the proof), and hence the absolutely continuous spectrum of U_0 contains the absolutely continuous spectrum of U

$$\sigma_{AC}(U)\subset\sigma_{AC}(U_0) \ . \tag{2.26}$$

On the other hand, we know that $\sigma_{AC}(U_0)=\varnothing$ (the operator U_0 can be trivially diagonalized) and therefore $\sigma_{AC}(U)=\varnothing$.

Let us now proceed to numerical simulations in order to illustrate the above theorems.

III. NUMERICAL INVESTIGATIONS

Here we use a simple representation of the state $|f\rangle$ in terms of the unperturbed basis $|n\rangle$ of H_0

$$|f\rangle=\sum_{n=-N}^{N}a_n|n\rangle \ , \tag{3.1}$$

with

$$a_n = |n|^{-\gamma} . \quad (3.2)$$

Studying the sums

$$S_1 = \sum_{n=-N}^{N} |a_n|^2 \quad (3.3)$$

and

$$S_2 = \sum_{n=-N}^{N} |a_n| , \quad (3.4)$$

one can easily see that in the limit of an infinite number of basis states ($N \to \infty$), the state f belongs to $L^2(0,2\pi)$ for $\gamma > 0.5$. For $0.5 < \gamma < 1.0$ the sum S_2 goes in the considered limit to infinity. Hence $|f\rangle$ tends to a vector which does not belong to L and one should expect the singularly continuous spectrum to start to manifest itself in the time evolution of the system. Due to Theorem 1, we will denote the cases with $\gamma > 1.0$ and $0.5 < \gamma < 1.0$ as the cases (i) and (ii), respectively.

The solution for the Schrödinger equation for the considered model can be found by expanding the wave function in the unperturbed basis

$$|\psi(t)\rangle = \sum_{n=-N}^{N} c_n(t)|n\rangle . \quad (3.5)$$

The unknown coefficients are governed by a recursion formula based on the quantum map (2.21)

$$U = e^{-iH_0T}[1 + (e^{-iTS_1} - 1)P/S_1] \quad (3.6)$$

according to

$$c_n(\nu+1) = e^{-iE_nT}\left[c_n(\nu) + Aa_n \sum_{m=-N}^{N} a_m c_m(\nu) \right] , \quad (3.7)$$

with the eigenvalues of the free rotor

$$E_n = n^2/2 \quad (3.8)$$

and

$$A = (e^{-iTS_1} - 1)/S_1 . \quad (3.9)$$

Here $c(\nu)$ denotes the coefficients just before the νth kick. Now the quantum mapping (3.7) can be iterated numerically for any given initial condition $c(0)$. In the current investigations we initially localized the particle in the center of the unperturbed basis, i.e., $c_0(0) = 1$ and $c_n(0) = 0$ for all other n.

The time autocorrelation function $\mathcal{A}(t)$ (Ref. 1)

$$\mathcal{A}(t) = \lim_{\tau\to\infty} \left| \frac{1}{2\tau} \int_{-\tau}^{\tau} \langle \psi(s) U(t+s,0) | \psi(0) \rangle ds \right| \quad (3.10)$$

can be easily expressed with the help of expression (2.7) as $\mathcal{A}(t) = (p_{0,0})^{1/2}$ with

$$p_{0,0}(\nu) = |\langle 0 | U^\nu | 0 \rangle|^2 . \quad (3.11)$$

It can be expected from Eqs. (2.11)–(2.15) that the different qualitative nature of the PP or SC quasienergy spectrum can be seen also in the numerical results, calculated from (3.11). From Eq. (2.13) one has to state that the autocorrelation is the Fourier transformation of a singular continuous measure. However, due to the very weak time decrease (2.14)–(2.15), the detailed time behavior of the autocorrelation function is rather unpredictable.

Before proceeding to the numerical results let us say few words about the time evolution of the mean energy. For the rank 1 perturbation is the excitation of basis states given after the first kick by

$$c_n(1) \cong Aa_n a_0, \quad n \neq 0 , \quad (3.12)$$

and is therefore essentially determined by the distribution of the state f over the basis. From here we have to state a quite uncommon property of the model. The averaged energy

$$\langle E \rangle = \sum_n n^2/2|c_n|^2 \quad (3.13)$$

is after the first kick proportional to $\sum_n |n|^{2(1-\gamma)}$, which means that the energy diverges for $\gamma < 1.5$ with the number of basis states tending to infinity. However, one need not be concerned about the type of divergence. The point is that we are not measuring the observable H_0. Roughly speaking, in a real experiment the measurement is associated with an observable that is different from the operator H_0. We measure, in fact, the energy from some chosen interval, which depends on the apparatus used in the experiment. The measured energy is therefore given by

$$\langle E \rangle = \langle \psi | \hat{E} | \psi \rangle , \quad (3.14)$$

with the observable $\hat{E}$ given by

$$\hat{E} = \sum_{n=n_1}^{n_2} \frac{n^2}{2} |n\rangle\langle n| , \quad (3.15)$$

and with n_1 and n_2 dependent on the energy interval, which has to be measured (see, for instance, Ref. 15). In this sense one need not be anxious about the infinite matrix elements of H_0 which appear for $N \to \infty$.

In order to see some nonrecurrent behavior in the numerical calculations one is forced to choose a large number N. This number must be large enough in order to avoid a domination of the rescattered flux which comes from the reflection on the borders in the basis at $\pm N$. One expects, however, that for rational ratios T/π the wave function will be localized (the spectrum is pure point) and hence rather independent on the size of N.

In accordance with the standard investigations of the kicked rotator (see, for example, Ref. 9), the crucial parameter for the dynamics has been chosen as

$$x = T/4\pi . \quad (3.16)$$

For $\gamma > 1$ this parameter should not play any role for the qualitative features of the numerical solutions because according to case (i) of the theorem the spectrum is always pure point. But for $0.5 < \gamma < 1$ one expects localization phenomena for rational values of x and delocalization for irrational (Diophantine) values. In the practical calculations the parameter x has been in the irrational case chosen as the golden mean $x=(\sqrt{5}-1)/2=0.618\ldots$. The example of a rational x has been realized by $x=\frac{6}{10}$, which is close to the golden mean.

The Schrödinger equation has been solved up to 500 kicks with a total number of basis states up to $2N+1=5001$. The calculations have been performed at IBM-AT compatible personal computers (CPU 80286/386) in Joint Institute for Nuclear Research, Dubna and at the computer facilities of Gesellschaft für Schwerionenforschung Darmstadt, Darmstadt within an accuracy of 16 valid digits. This guarantees a conservation of the norm after the maximum kick number within an uncertainty of 10^{-14}. As an additional check we considered the time reverse after a short time, using the operator U^{-1}, and found the same degree of accuracy. It is worth noting at this point that the easy time reversibility of the system is present in the rational as well as in the irrational case, and persist even for large times. This is in sharp contrast with the classical chaotic systems where the time reversibility is lost after a rather short time due to the exponentially decreasing autocorrelation function (see, for instance, Ref. 16). In the quantum case, however, the autocorrelation function is recurrent (for x rational) or extremely slowly decreasing (for x irrational), which means an easy time reversibility even after long-time evolution.[16] (See Fig. 5.)

In Fig. 1 the averaged energy $\langle E\rangle$ for case (ii) ($\gamma=0.7$) has been drawn versus the kick number for $x=0.6$ (dashed curve) and $x=0.618\ldots$ (solid curve). In excellent agreement with the theorems from Sec. II one can establish the qualitative difference: for rational x quantum interference ensures a periodic behavior of the averaged energy, not exceeding an upper limit, but for the irrational case the energy grows nonrecurrently in the considered time interval. The increase in energy is not monotonic. One can observe plateaulike structures, which have been discussed also for the usually perturbed rotator with strong coupling.[3,6] The plateaus are connected with the "pulsing" spreading of the initial state during the evolution. From the global point of view one can see a more or less linear increase which is a direct manifestation of the "energy diffusion" due to the "random-walk-like" spreading of the wave function, which is typical for the singularly continuous case (see, for instance, Ref. 17).

In Fig. 2, we present the results for $\gamma=3.0$ [case (i)]. The averaged energy is a periodic function of time for both values of x as a consequence of the pure pointness of the quasienergy spectrum. Considering the comparatively small magnitude of the energies, one can state a strong localization of the quasienergy functions.

In Fig. 3 the magnitudes of the expansion coefficients $|c|$ of the wave function are shown for $\gamma=0.7$ and x equal to the golden mean. The diagrams shows the coefficients for sequence of time (the kick number is indi-

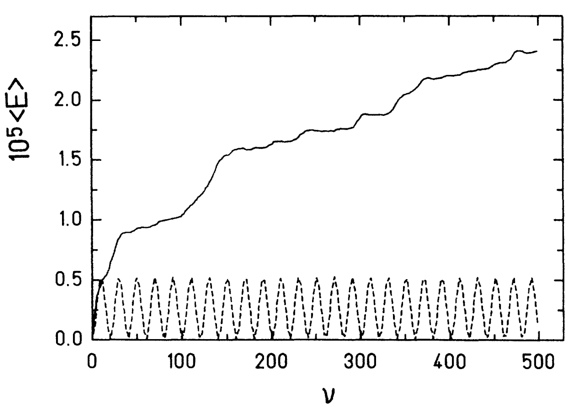

FIG. 1. Averaged energy of the kicked rotator vs kick number for case (ii) of Theorem 1 ($\gamma=0.7$). Solid curve is for an irrational frequency ratio (x is the golden mean) with a singular continuous quasienergy spectrum; dashed line is for a rational x value close to the golden mean ($x=0.6$) with a pure-point quasienergy spectrum. The rotator basis covers 5001 states ($N=2500$).

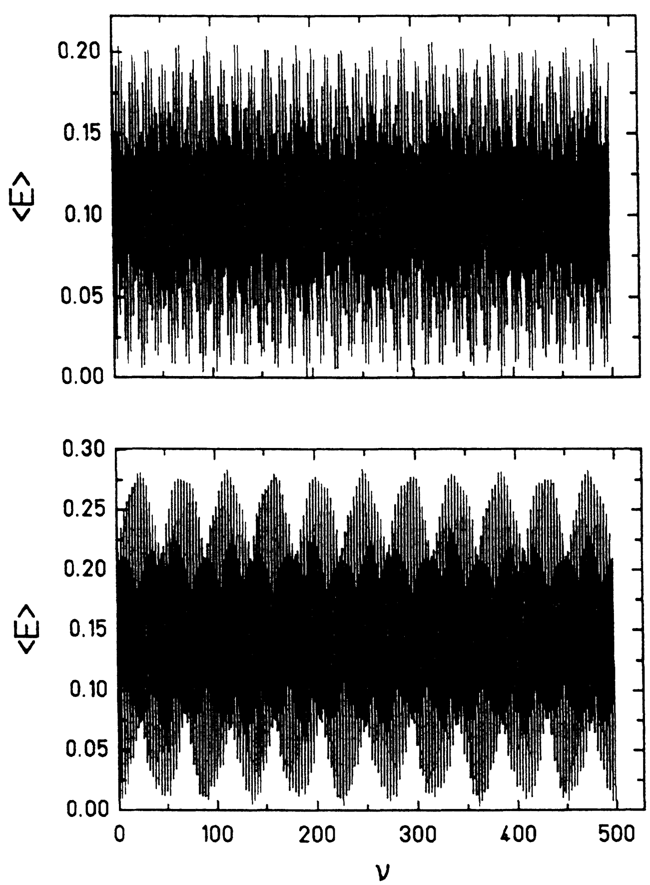

FIG. 2. Same as in Fig. 1 but for case (i) of Theorem 1 ($\gamma=3.0$) with a pure-point quasienergy spectrum independent of the frequency ratio x. The irrational (rational) cases are plotted on the upper (lower) part.

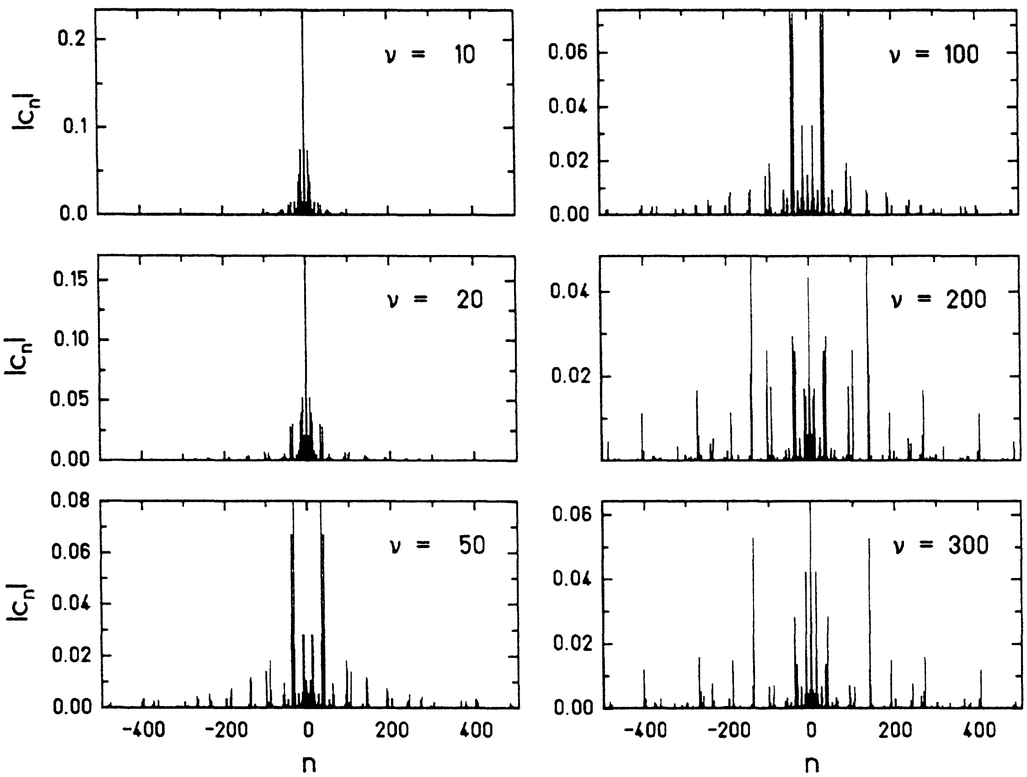

FIG. 3. Time evolution of the wave packet in the rotator basis for the case (ii) with x equal to the golden mean (solid curve of Fig. 1). The growing kick number is indicated inside the figure. Please note that the scale of the diagrams changes with time and that only a part of the basis states are incorporated.

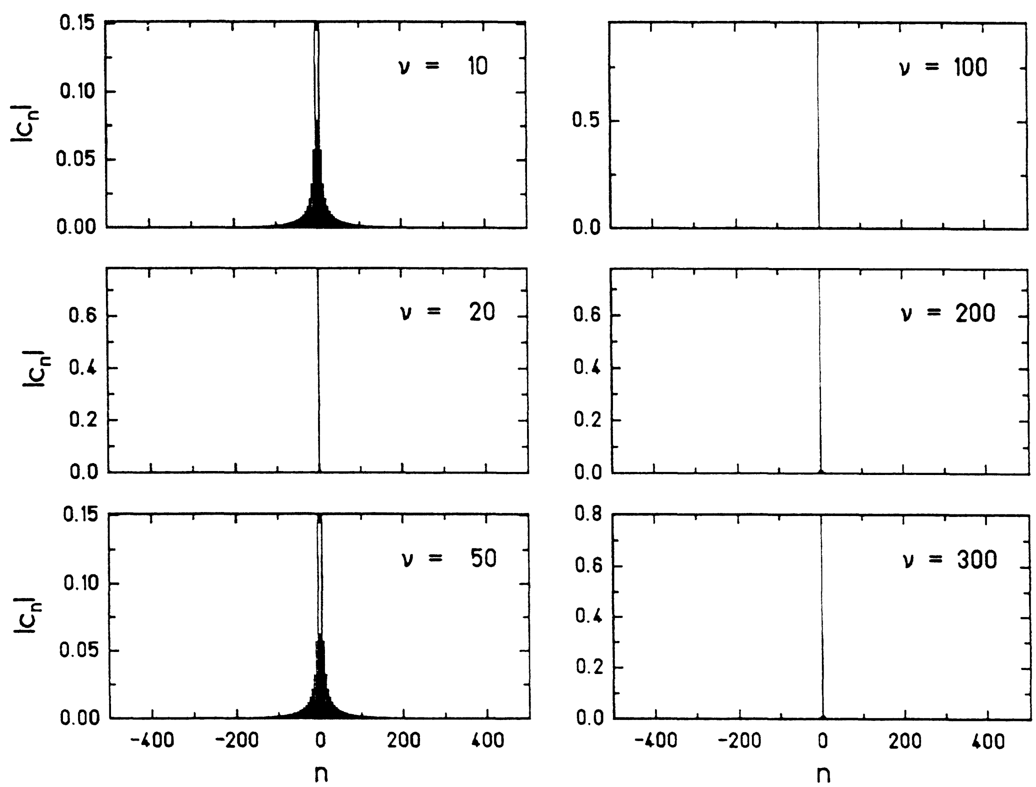

FIG. 4. Same as in Fig. 3 but for the rational case ($x = 0.6$, dashed line of Fig. 1).

cated inside the figures). Instead of a monotonic spread of the wave function over the unperturbed states, one can see a pulsing spread which is directly related to the observed plateaulike structure of the averaged energy (Fig. 1). For comparison, we demonstrate in Fig. 4 the same results for $x=0.6$. The observed localization is obviously a consequence of the pure-point quasienergy spectrum. The analogous calculations for case (i) (not shown in the figures) yield qualitatively the same results, but with even stronger localization.

Figure 5 shows the autocorrelation function $\mathcal{A}(\nu)$ for cases (ii) with $\gamma=0.7$. The slow and irregular decrease that is characteristic for a singular continuous quasienergy spectrum is demonstrated (x equal to the golden mean) and compared with the strong recurrence which appears for the pure-point case ($x=0.6$). Because of the delicate mathematical nature of the autocorrelation function [see Eqs. (2.13)–(2.15) and (3.11)], it is very difficult to comment in detail on its calculated long-time behavior. Regardless, one can see a series of "bumps" with decreasing height. The height of the bumps will finally decrease to zero. This type of behavior should be expected from the complementary results on the averaged energy and the wave function. The return probability (2.14) and the corresponding Cesaro mean (2.15) are plotted on the Fig. 6.

IV. CONCLUDING REMARKS

We have demonstrated the occurrence of a singular continuous spectrum of the Floquet operator for the periodically kicked quantum rotor with a separable perturbation. In contrast to previous papers on a similar subject, we were able to base our statements on rigorous mathematical proofs. The exotic quasienergy eigenstates accompanying the singular continuous spectrum have been manifested in standard numerical calculations applying a special version of the separable interaction. For T being an irrational multiple of π (strictly speaking, a Diophantine number), the numerical calculations exhibit qualitatively all the expected features like the plateaulike structures in the growing time-dependent mean energy of the rotor, the nonrecurrent behavior of the wave function, and a decreasing—however, very weakly and irregularly—autocorrelation function.

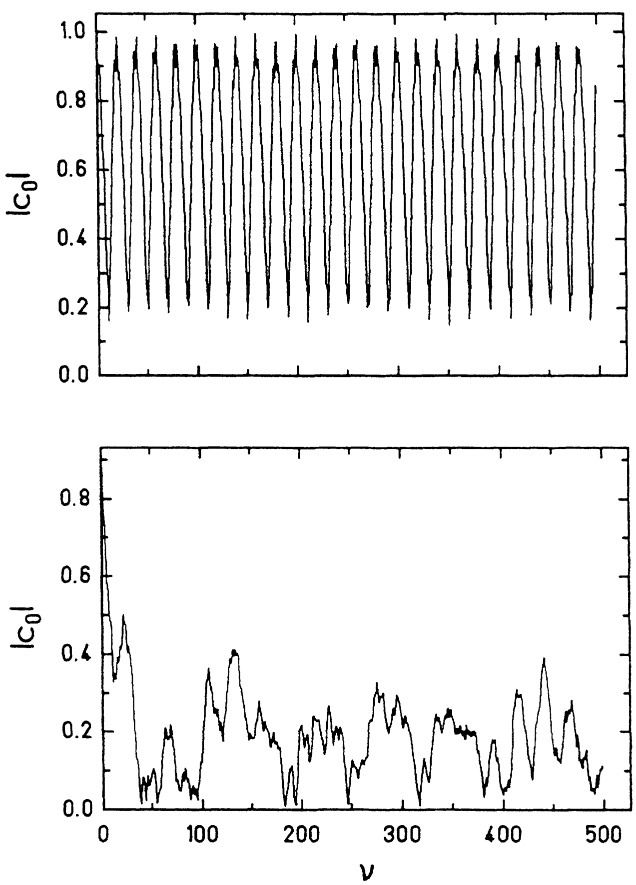

FIG. 5. Autocorrelation function in dependence of the kick number for case (ii) of Theorem 1. Lower part for the irrational frequency ratio (x is the golden mean) with a singular continuous quasienergy spectrum; upper part of the rational x value close to the golden mean ($x=0.6$) with a pure-point quasienergy spectrum. The other parameters are the same as in Fig. 1.

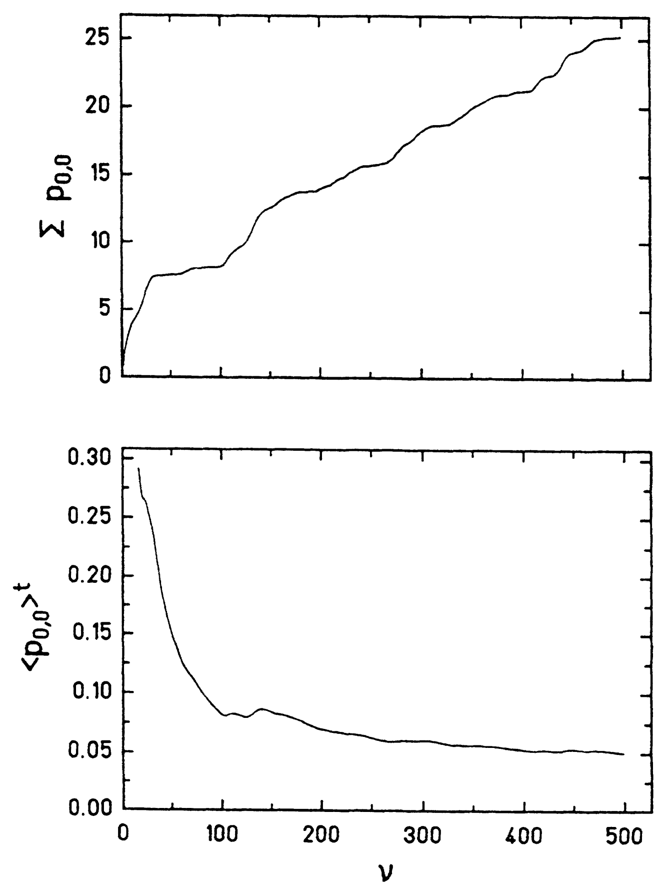

FIG. 6. The "return probability" (2.14) is plotted on the upper part of the figure and is compared with the Cesaro mean (2.15) in its lower part. x is equal to the golden mean.

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