Eulerian formalism of linear beam-wave interactions

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A detailed account of a formal mathematical description of the interaction of relativistic charged particle beams with electromagnetic waves, within the frame of classical electrodynamics, is presented. The standard system of eight equations (Maxwell and Lorentz gauge conditions and fluid dynamics) in the four-vector potential A_μ and the four-vector current density j_μ is reduced, after linearization, to a canonical system of four coupled partial differential equations in the electromagnetic field perturbation δA_μ. Both electromagnetic and dynamical quantities are treated as fields, according to the Eulerian formalism. This new system is very general, and different beam-wave interactions are characterized by different fluid equilibria and boundary conditions for δA_μ and its derivatives. Finally, the equations are used, as an example, to study the dispersive characteristics of space-charge waves propagating in a cylindrical waveguide, along a relativistic electron beam confined by an axial magnetic field. The problem is treated in a fully relativistic way, for arbitrary values of the axial guide field and any degree of azimuthal symmetry.

I. INTRODUCTION

One of the central problems of relativistic electrodynamics[1-5] is the interaction of charged particles beams with electromagnetic waves. The physics of such interactions is very rich and a wide variety of complex phenomena arise, ranging from synchrotron and Čerenkov[6] radiation to free-electron laser,[7] cyclotron maser,[8] and other instabilities involving nonneutral plasmas, as discussed extensively by Davidson.[9] A large class of beam-wave interaction problems involve electromagnetic energies that are small compared to the particles' kinetic energy, and perturbation theory is appropriate to describe such linear beam-wave interactions. This category of problem will be the focus of our attention in this paper. Different formal mathematical descriptions of this type of interaction are possible, such as the Maxwell-Vlasov kinetic theory,[9] or the Maxwell-Euler fluid model. In this work, we consider the latter theory, which involves the manipulation of fields for both electromagnetic and dynamical quantities, and of operators as the electromagnetic wave propagator (d'Alembertian operator) or the fluid convective derivation, providing a compact and elegant mathematical framework to study these interactions.

One of the main objects of this work is to show that starting from the standard set of eight equations in the four-vector potential A_μ and the four-vector current density j_μ, we can obtain a canonical system of four coupled partial differential equations (PDE's) describing the evolution of the electromagnetic field perturbation δA_μ by linearizing the interaction equations. The compact set of PDE's derived in this manner involves the perturbed electromagnetic four-vector potential and the equilibrium fluid field components. Different specific problems are characterized by different fluid equilibria and boundary conditions for δA_μ and its derivatives. The initial set of eight equations consists of the four Maxwell equations with sources describing the evolution of the four-vector potential, the Lorentz gauge condition, which is equivalent to the conservation of charge or to the continuity equation, and three fluid equations of motion.

At this level, two main formal approaches can be used to solve this linear system of PDE's. On the one hand, one can expand δA_μ into known eigenmodes satisfying the appropriate boundary conditions, and study the coupling of these modes through the coupled PDE's. The other approach consists in solving directly these equations, then using the boundary conditions to determine the actual eigenvalues and eigenfunctions of the problem.[10]

This paper is organized as follows. In Sec. II, we linearize the standard system of eight interaction equations in A_μ and j_μ, and reduce it to a canonical system in the four-vector potential perturbation δA_μ. Section III is focused on the study of space-charge waves supported by an electron beam confined to an axial guide magnetic field, and is intended as an example of the use of the equations derived in Sec. II. Finally, conclusions are drawn in Sec. IV.

II. GENERAL FORMALISM

The purpose of this section is to give a detailed account of a formal mathematical description of the interaction of a relativistic electron beam with electromagnetic fields, within the frame of classical electrodynamics. A very large number of methods have been described in the literature, and there is, sometimes, some confusion about which equations and which variables should be used. For example, it is well known that the gauge condition, the conservation of charge, and the continuity equation are equivalent. Here, our objective is to reduce the linearized equations of interaction to the canonical system of four equations in the four-vector potential perturbation $\delta A_\mu(x_\nu)$.

We first briefly review the equations relevant to the problem. The interaction of charged particles with electromagnetic fields can be described, in the classical limit, by two sets of equations. On the one hand, there are Maxwell's two groups of equations,[1,2,11] governing the fields,

$$\nabla\times\mathbf{E}+\partial_t\mathbf{B}=\mathbf{0}\ , \tag{1}$$

$$\nabla\cdot\mathbf{B}=0\ , \tag{2}$$

and the group with sources

$$\nabla\cdot\mathbf{E}=\frac{1}{\epsilon_o}\rho\ , \tag{3}$$

$$\nabla\times\mathbf{B}-\frac{1}{c^2}\partial_t\mathbf{E}=\mu_o\mathbf{j}\ . \tag{4}$$

On the other hand, there are the equations governing the particles dynamics, which are given by the expression of the Lorentz force

$$d_t\mathbf{p}=-e(\mathbf{E}+\mathbf{v}\times\mathbf{B})\ , \tag{5}$$

and the continuity equation (charge or particles conservation)

$$\partial_t\rho+\nabla\cdot\mathbf{j}=0\ . \tag{6}$$

Here, $j_\mu\equiv(c\rho,\mathbf{j})=-en(c,\mathbf{v})$ is the four-vector current density, with n the particle density and $\mathbf{v}=c\boldsymbol{\beta}$ their velocities. The particles' momentum is given by $\mathbf{p}=\gamma m_o\mathbf{v}$, and their energy by $\gamma^{-2}=1-\beta^2$.

At this point, it is important to note that Maxwell's first group of equations [(1) and (2)] suggests the introduction of the four-vector potential $A_\mu\equiv(\phi/c,\mathbf{A})$, defined[1,2,11] such that

$$\mathbf{E}=-\nabla\phi-\partial_t\mathbf{A}\ , \tag{7}$$

$$\mathbf{B}=\nabla\times\mathbf{A}\ . \tag{8}$$

As a result, Eqs. (1) and (2) are automatically satisfied. If, in addition, we impose that the four-vector potential satisfies the Lorentz gauge condition

$$\frac{1}{c^2}\partial_t\phi+\nabla\cdot\mathbf{A}=0\ , \tag{9}$$

we see that the second group of equations is equivalent to

$$\left[\nabla^2-\frac{1}{c^2}\partial_t^2\right]\phi+\frac{1}{\epsilon_o}\rho=0\ , \tag{10}$$

$$\left[\nabla^2-\frac{1}{c^2}\partial_t^2\right]\mathbf{A}+\mu_o\mathbf{j}=0\ . \tag{11}$$

It should also be noted that the gauge condition (9) is equivalent to the continuity equation (6).

The equation of momentum transfer (5) implicitly satisfies energy conservation, as can be seen by taking the dot product of (5) by $\mathbf{p}$, to obtain

$$d_t\gamma=-\frac{e\mathbf{E}\cdot\mathbf{v}}{m_oc^2}\ . \tag{12}$$

Finally, using the definitions, Eq. (5) can be transformed to read, within the framework of a relativistic fluid model,

$$(\partial_t+\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{e}{m_o}\left[1-\frac{v^2}{c^2}\right]^{1/2}\left[-\nabla\phi-\partial_t\mathbf{A}+\mathbf{v}\times\nabla\times\mathbf{A}+\frac{\mathbf{v}}{c^2}(\nabla\phi+\partial_t\mathbf{A})\cdot\mathbf{v}\right]\ . \tag{13}$$

We thus obtain a closed system of eight equations with eight unknowns A_μ, n, and $\mathbf{v}$:

$$\left[\nabla^2-\frac{1}{c^2}\partial_t^2\right]\mathbf{A}=\mu_oen\mathbf{v}\ , \tag{14}$$

$$\left[\nabla^2-\frac{1}{c^2}\partial_t^2\right]\phi=\frac{1}{\epsilon_o}en\ , \tag{15}$$

$$\frac{1}{c^2}\partial_t\phi+\nabla\cdot\mathbf{A}=0\ , \tag{16}$$

together with Eq. (13).

We now focus on the linear analysis of the beam-field interaction. Any fluid field component $f(x_\nu)$ is written $f=f_0+\delta f$. The quantity f_0 refers to the beam self-consistent equilibrium in the external fields, while δf corresponds to the electromagnetic perturbation. We assume that for all fluid field components, we have $|\delta f|\ll|f_0|$. We can then linearize the equations presented above, with the result that

$$\left[\nabla^2-\frac{1}{c^2}\partial_t^2\right]\delta\mathbf{A}=\mu_o e(n_0\delta\mathbf{v}+\mathbf{v}_0\delta n)\ , \tag{17}$$

$$\left[\nabla^2-\frac{1}{c^2}\partial_t^2\right]\delta\phi=\frac{1}{\epsilon_o}e\delta n\ , \tag{18}$$

$$\frac{1}{c^2}\partial_t\delta\phi+\nabla\cdot\delta\mathbf{A}=0\ , \tag{19}$$

$$\begin{aligned}(\partial_t+\mathbf{v}_0\cdot\nabla)\delta\mathbf{v}+(\delta\mathbf{v}\cdot\nabla)\mathbf{v}_0=-\frac{e}{\gamma_0 m_o}\Bigg[&-\nabla\delta\phi-\partial_t\delta\mathbf{A}+\delta\mathbf{v}\times\mathbf{B}_0+\mathbf{v}_0\times\nabla\times\delta\mathbf{A}\\&-\frac{\mathbf{v}_0}{c^2}[\mathbf{E}_0\cdot\delta\mathbf{v}-(\nabla\delta\phi+\partial_t\delta\mathbf{A})\cdot\mathbf{v}_0]-\frac{\delta\mathbf{v}}{c^2}(\mathbf{E}_0\cdot\mathbf{v}_0)\\&-\frac{\gamma_0^2}{c^2}(\mathbf{v}_0\cdot\delta\mathbf{v})\left[\mathbf{E}_0+\mathbf{v}_0\times\mathbf{B}_0-\frac{\mathbf{v}_0}{c^2}(\mathbf{E}_0\cdot\mathbf{v}_0)\right]\Bigg]\ .\end{aligned} \tag{20}$$

Here the equilibrium electric and magnetic fields are $\mathbf{E}_0(x_\nu)$ and $\mathbf{B}_0(x_\nu)$, respectively. We shall now reduce this system by considering

$$n_0\delta\mathbf{v}=\frac{1}{\mu_o e}\left[\Box\delta\mathbf{A}-\boldsymbol{\beta}_0\Box\frac{\delta\phi}{c}\right]\ , \tag{21}$$

where

$$\Box\equiv\partial_\mu\partial^\mu\equiv\Delta^2-\frac{1}{c^2}\partial_t^2 \tag{22}$$

is the d'Alembertian operator (electromagnetic wave propagator). We have, on the other hand,

$$(\partial_t+\mathbf{v}_0\cdot\nabla)(n_0\delta\mathbf{v})=n_0(\partial_t+\mathbf{v}_0\cdot\nabla)\delta\mathbf{v}+\delta\mathbf{v}(\partial_t+\mathbf{v}_0\cdot\nabla)n_0\ , \tag{23}$$

and, after (21),

$$(\partial_t+\mathbf{v}_0\cdot\nabla)(n_0\delta\mathbf{v})=\frac{1}{\mu_o e}(\partial_t+\mathbf{v}_0\cdot\nabla)\left[\Box\delta\mathbf{A}-\boldsymbol{\beta}_0\Box\frac{\delta\phi}{c}\right]. \tag{24}$$

The first term on the right-hand side of Eq. (23) is given by (20):

$$\begin{aligned}n_0(\partial_t+\mathbf{v}_0\cdot\nabla)\delta\mathbf{v}+[(n_0\delta\mathbf{v})\cdot\nabla]\mathbf{v}_0=-n_0\frac{e}{\gamma_0 m_o}\Bigg[&-\nabla\delta\phi-\partial_t\delta\mathbf{A}+\delta\mathbf{v}\times\mathbf{B}_0+\mathbf{v}_0\times\nabla\times\delta\mathbf{A}\\&-\frac{\mathbf{v}_0}{c^2}[\mathbf{E}_0\cdot\delta\mathbf{v}-(\nabla\delta\phi+\partial_t\delta\mathbf{A})\cdot\mathbf{v}_0]-\frac{\delta\mathbf{v}}{c^2}(\mathbf{E}_0\cdot\mathbf{v}_0)\\&-\frac{\gamma_0^2}{c^2}(\mathbf{v}_0\cdot\delta\mathbf{v})\left[\mathbf{E}_0+\mathbf{v}_0\times\mathbf{B}_0-\frac{\mathbf{v}_0}{c^2}(\mathbf{E}_0\cdot\mathbf{v}_0)\right]\Bigg]\ ,\end{aligned} \tag{25}$$

while the second term can be derived from the equilibrium continuity equation

$$\partial_t n_0+\nabla\cdot(n_0\mathbf{v}_0)=0\Longleftrightarrow(\partial_t+\mathbf{v}_0\cdot\nabla)n_0=-n_0(\nabla\cdot\mathbf{v}_0)\ . \tag{26}$$

We thus have

$$(\partial_t+\mathbf{v}_0\cdot\nabla+\nabla\cdot\mathbf{v}_0)\left[\Box\delta\mathbf{A}-\boldsymbol{\beta}_0\Box\frac{\delta\phi}{c}\right]=\mu_0 e n_0(\partial_t+\mathbf{v}_0\cdot\nabla)\delta\mathbf{v}\ . \tag{27}$$

We now use Eq. (25) to obtain

$$(\partial_t+\mathbf{v}_0\cdot\nabla+\nabla\cdot\mathbf{v}_0)\left[\Box\delta\mathbf{A}-\boldsymbol{\beta}_0\Box\frac{\delta\phi}{c}\right]+\left[\left[\Box\delta\mathbf{A}-\boldsymbol{\beta}_0\Box\frac{\delta\phi}{c}\right]\cdot\nabla\right]\mathbf{v}_0$$

$$=-\mu_0 e n_0\frac{e}{\gamma_0 m_o}\left[-\nabla\delta\phi-\partial_t\delta\mathbf{A}+\delta\mathbf{v}\times\mathbf{B}_0+\mathbf{v}_0\times\nabla\times\delta\mathbf{A}-\frac{\mathbf{v}_0}{c^2}[\mathbf{E}_0\cdot\delta\mathbf{v}-(\nabla\delta\phi+\partial_t\delta\mathbf{A})\cdot\mathbf{v}_0]\right.$$

$$\left.-\frac{\delta\mathbf{v}}{c^2}(\mathbf{E}_0\cdot\mathbf{v}_0)-\frac{\gamma_0^2}{c^2}(\mathbf{v}_0\cdot\delta\mathbf{v})\left[\mathbf{E}_0+\mathbf{v}_0\times\mathbf{B}_0-\frac{\mathbf{v}_0}{c^2}(\mathbf{E}_0\cdot\mathbf{v}_0)\right]\right]. \tag{28}$$

At this point, we define the following parameters:

$$\boldsymbol{\Omega}_0=\frac{e\mathbf{B}_0(x_\nu)}{\gamma_0(x_\nu)m_0},\quad \frac{\omega_p^2}{c^2}=\mu_o\frac{n_0(x_\nu)e^2}{\gamma_0(x_\nu)m_o},\quad \boldsymbol{\beta}_0=\frac{\mathbf{v}_0(x_\nu)}{c},\quad \boldsymbol{\Lambda}_0=\frac{e\mathbf{E}_0(x_\nu)}{\gamma_0(x_\nu)m_o c}, \tag{29}$$

which are, respectively, the relativistic cyclotron frequencies in the equilibrium magnetic field, the relativistic beam plasma frequency, the normalized fluid equilibrium velocity field, and the normalized equilibrium electric field, governing the energy time scale. The formalism described here includes the most general case, where the dynamical quantities describing the fluid equilibrium state are functions of both space and time.

Upon replacement of every quantity $n_o\delta\mathbf{v}$ appearing on the right-hand side of Eq. (28) by the value defined in (21), we end up with the sought-after canonical system of four equations in the four-potential vector perturbation $\delta A_\mu\equiv(\delta\phi/c,\delta\mathbf{A})$:

$$\{\partial_t+\mathbf{v}_0\cdot\nabla+\nabla\cdot\mathbf{v}_0-\boldsymbol{\Lambda}_0\cdot\boldsymbol{\beta}_0-\boldsymbol{\beta}_0\boldsymbol{\Lambda}_0\cdot-\boldsymbol{\Omega}_0\times-\gamma_0^2[\boldsymbol{\Lambda}_0-\boldsymbol{\beta}_0(\boldsymbol{\Lambda}_0\cdot\boldsymbol{\beta}_0)+\boldsymbol{\beta}_0\times\boldsymbol{\Omega}_0]\boldsymbol{\beta}_0\cdot\}\left[\Box\delta\mathbf{A}-\boldsymbol{\beta}_0\Box\frac{\delta\phi}{c}\right]$$

$$+\left[\left[\Box\delta\mathbf{A}-\boldsymbol{\beta}_0\Box\frac{\delta\phi}{c}\right]\cdot\nabla\right]\mathbf{v}_0+\frac{\omega_p^2}{c^2}[-\nabla\delta\phi-\partial_t\delta\mathbf{A}+\mathbf{v}_0\times\nabla\times\delta\mathbf{A}+\boldsymbol{\beta}_0(\nabla\delta\phi+\partial_t\delta\mathbf{A})\cdot\boldsymbol{\beta}_0]=\mathbf{0}, \tag{30}$$

$$\frac{1}{c^2}\partial_t\delta\phi+\nabla\cdot\delta\mathbf{A}=0. \tag{31}$$

Note that we can easily identify the different terms in Eq. (30) as a beam-mode type operator coupled to an electromagnetic wave propagator, and a beam coupling term proportional to the beam density $\omega_p^2(x_\nu)$ and containing the ponderomotive force.[7,9] Equation (30) is written in terms of operators acting on the electromagnetic field; the dot and vectorial product symbols appearing there apply either explicitly to a vector (e.g., $\boldsymbol{\Lambda}_0\cdot\boldsymbol{\beta}_0$), or implicitly to the four-wave vector (e.g., $\boldsymbol{\beta}_0\boldsymbol{\Lambda}_0\cdot$) as do the space-time derivatives.

At this point, different beam-wave interactions are characterized by different fluid equilibria and different boundary conditions for δA_μ. Two main formal approaches can be used to solve the canonical system derived above. On the one hand, one can expand δA_μ into known eigenmodes satisfying the appropriate boundary conditions, and study the coupling of these modes through the coupled PDE's describing the evolution of the four-vector potential perturbation. The other approach consists in solving directly these equations, then using the boundary conditions to determine the actual eigenvalues and eigenfunctions of the problem.

The reduction in the number of unknown field components to a minimum of four clearly warrants the terminology "canonical system." However, it should also be noted that the degree of the system derived above (third-order PDE's) is higher than that of the system described by Eqs. (13)–(16) (second-order PDE's). In addition, we wish to remark that the mathematical formalism exposed here generalizes a procedure which is virtually of universal use for the study of any particular beam-wave interaction, within the framework of a linear relativistic fluid model.[9] The equations presented above form a compact system of PDE's describing the evolution of the four-vector potential perturbation. Starting the electromagnetic stability analysis of a given beam-wave interaction from this system allows one to avoid the lengthy algebraic manipulations generally involved in the conventional mathematical procedure, therefore reducing the risk of errors to a minimum; the mathematical consistency of the perturbation formalism is also guaranteed by these equations. In Sec. III which is intended as an example of the use of these equations, we study the propagation of space-charge waves supported by an electron beam confined by an axial guide magnetic field.

III. BEAM MODES WITH FINITE GUIDE FIELD

Here, we study waves supported by an electron beam, confined by an axial guide magnetic field $\hat{\mathbf{z}}B_\parallel$. We start from the master system of equations governing the evolution of the four-vector potential perturbation δA_μ (30) and (31). Since this section is intended as an example of the use of the equations derived in Sec. II, the fluid equilibrium studied here is the simplest possible in this case; in particular, we neglect the equilibrium self-fields produced by the beam space charge. The field-fluid equilibrium assumptions are then

$$\mathbf{B}_0=\hat{\mathbf{z}}B_\parallel , \tag{32}$$

which corresponds to the externally applied guide magnetic field, and

$$\boldsymbol{\beta}_0=\hat{\mathbf{z}}\beta_\parallel , \tag{33}$$

which describes the axial drift velocity of the electron beam. In addition, there is no externally applied electric field, therefore $\mathbf{E}_0=\mathbf{0}$. The radial beam density profile is

$$n_0(r)=\begin{cases} n_0 & \text{for } r<r_b \\ 0 & \text{for } a>r>r_b , \end{cases} \tag{34}$$

where r_b is the beam radius. Note that because the equilibrium fluid velocity field is purely axial, the continuity equation is satisfied everywhere, including at the edge of the beam. The equations of evolution of the four-vector potential are, in cylindrical coordinates, for the radial component,

$$\begin{aligned}(\partial_t+v_\parallel\partial_z)&\left[\Box\delta A_r-\frac{1}{r^2}(\delta A_r+2\partial_\theta\delta A_\theta)\right]\\ &+\Omega_\parallel\left[\Box\delta A_\theta-\frac{1}{r^2}(\delta A_\theta-2\partial_\theta\delta A_r)\right]\\ &+\frac{\omega_p^2}{c^2}[-(\partial_t+v_\parallel\partial_z)\delta A_r+\partial_r(v_\parallel\delta A_z-\delta\phi)]=0 ;\end{aligned} \tag{35}$$

for the azimuthal component,

$$\begin{aligned}(\partial_t+v_\parallel\partial_z)&\left[\Box\delta A_\theta-\frac{1}{r^2}(\delta A_\theta-2\partial_\theta\delta A_r\right]\\ &-\Omega_\parallel\left[\Box\delta A_r-\frac{1}{r^2}(\delta A_r+2\partial_\theta\delta A_\theta)\right]\\ &+\frac{\omega_p^2}{c^2}\left[-(\partial_t+v_\parallel\partial_z)\delta A_\theta+\frac{1}{r}\partial_\theta[v_\parallel\delta A_z-\delta\phi]\right]=0 ;\end{aligned} \tag{36}$$

for the axial component,

$$(\partial_t+v_\parallel\partial_z)\Box\left[\delta A_z-\beta_\parallel\frac{\delta\phi}{c}\right]-\frac{\omega_p^2}{\gamma_\parallel^2c^2}(\partial_z\delta\phi+\partial_t\delta A_z)=0 ; \tag{37}$$

and finally, for the Lorentz gauge condition,

$$\frac{1}{c^2}\partial_t\delta\phi+\frac{1}{r}\partial_r(r\delta A_r)+\frac{1}{r}\partial_\theta\delta A_\theta+\partial_z\delta A_z=0 . \tag{38}$$

We shall demonstrate that this system of four coupled differential equations in the four-potential vector δA_μ admits solutions of the following form:

$$\delta A_r(r,\theta,z,t)=\left[\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\frac{J_l(\chi r)}{\chi r}\right]\times\exp[i(\omega t-kz+l\theta)] , \tag{39}$$

$$\delta A_\theta(r,\theta,z,t)=i\left[\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\frac{J_l'(\chi r)}{l}\right]\times\exp[i(\omega t-kz+l\theta)] , \tag{40}$$

$$\delta A_z(r,\theta,z,t)=\mathcal{C}J_l(\chi r)\exp[i(\omega t-kz+l\theta)] , \tag{41}$$

$$\delta\phi(r,\theta,z,t)=c\mathcal{D}J_l(\chi r)\exp[i(\omega t-kz+l\theta)] . \tag{42}$$

Note that the terms in $\mathcal{B}$ are similar to the usual vacuum transverse electric (TE) modes, while δA_z and $\delta\phi$ are similar to usual vacuum transverse magnetic (TM) modes. In the first case, the boundary conditions at the waveguide wall ($r=a$) would yield $\chi=\chi_{ln}'/a$, and in the second case $\chi=\chi_{ln}/a$, where χ_{ln}' and χ_{ln} are the nth zeros of J_l' and J_l, respectively. The axial magnetic field creates hybrid modes resulting from the mixing of vacuum TE and TM modes. We start with the gauge equation (38); using Eqs. (39)–(42), we have

$$\frac{i\omega}{c}\mathcal{D}J_l(\chi r)+\frac{1}{r}d_r\left[r\left[\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\frac{J_l(\chi r)}{\chi r}\right]\right]-\frac{l}{r}\left[\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\frac{J_l'(\chi r)}{l}\right]-ik\mathcal{C}J_l(\chi r)=0 . \tag{43}$$

The terms in $\mathcal{B}$ cancel out, and (43) reduces to

$$i\left[\frac{\omega}{c}\mathcal{D}-k\mathcal{C}\right]J_l(\chi r)+\frac{\mathcal{A}}{r}[d_r(r)-l]J_{l-1}(\chi r)=0 . \tag{44}$$

At this point, we use the following formula:[12]

$$J_{l-1}(\chi r)=J_l'(\chi r)-l\frac{J_l(\chi r)}{\chi r} ,$$

which yields

$$i\left[\frac{\omega}{c}\mathcal{D}-k\mathcal{C}\right]J_l(\chi r)+\mathcal{A}\left[\frac{1}{r}d_r[rJ_l'(\chi r)]-\frac{1}{\chi}\frac{l^2}{r^2}J_l(\chi r)\right]=0 . \tag{45}$$

The terms in $\mathcal{A}$ correspond to the usual Bessel differential equation, and we end up with

$$\left[i\left[\frac{\omega}{c}\mathcal{D}-k\mathcal{C}\right]-\chi\mathcal{A}\right]J_l(\chi r)=0 . \tag{46}$$

Equation (46) must hold for any value of r, which finally yields

$$i\left[\frac{\omega}{c}\mathcal{D}-k\mathcal{C}\right]-\chi\mathcal{A}=0 . \tag{47}$$

We now examine the $\hat{z}$ component of the equation of the four-vector potential (37). Remembering the definition of the d'Alembertian operator, or electromagnetic wave propagator, in cylindrical coordinates

$$\Box\equiv\left[\frac{1}{r}\partial_r(r\partial_r)+\frac{1}{r^2}\partial_\theta^2+\partial_z^2-\frac{1}{c^2}\partial_t^2\right] ,$$

and using the conventional Bessel differential equation,[13] we easily obtain

$$\left[\frac{\omega}{c}-\beta_\parallel k\right]\left[\frac{\omega^2}{c^2}-k^2-\chi^2\right](\mathcal{C}-\beta_\parallel\mathcal{D})-\frac{\omega_{p0}^2}{\gamma_\parallel^2c^2}\left[\frac{\omega}{c}\mathcal{C}-k\mathcal{D}\right]=0 . \tag{48}$$

At this point, we need to evaluate the quantities

$$\Box_r=\Box\delta A_r-\frac{1}{r^2}(\delta A_r+2\partial_\theta\delta A_\theta) \tag{49}$$

and

$$\Box_\theta=\Box\delta A_\theta-\frac{1}{r^2}(\delta A_\theta-2\partial_\theta\delta A_r) . \tag{50}$$

We have, by definition,

$$\Box_r=\left[\left[\frac{1}{r}d_r(rd_r)+\frac{\omega^2}{c^2}-k^2-\frac{l^2+1}{r^2}\right]\left[\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\frac{J_l(\chi r)}{\chi r}\right]+\frac{2l}{r^2}\left[\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\frac{J_l'(\chi r)}{l}\right]\right]\exp[i(\omega t-kz+l\theta)] . \tag{51}$$

We group the terms in $\mathcal{A}$ and $\mathcal{B}$ together to obtain

$$\Box_r=\exp[i(\omega t-kz-l\theta)]\left\{\mathcal{A}\left[\frac{1}{r}d_r(rd_r)+\frac{\omega^2}{c^2}-k^2-\frac{l^2-1}{r^2}\right]J_{l-1}(\chi r)+\mathcal{B}\left[\frac{1}{r}d_r\left[rd_r\frac{J_l(\chi r)}{\chi r}\right]+\left[\frac{\omega^2}{c^2}-k^2-\frac{l^2+1}{r^2}\right]\frac{J_l(\chi r)}{\chi r}+\frac{2}{r^2}J_l'(\chi r)\right]\right\} . \tag{52}$$

We now use the identity[13]

$$\frac{1}{r^2}J_l'(\chi r)=\frac{1}{r}d_r\left[\frac{J_l(\chi r)}{\chi r}\right]+\frac{1}{r^2}\left[\frac{J_l(\chi r)}{\chi r}\right] \tag{53}$$

to obtain

$$\Box_r=\exp[i(\omega t-kz+l\theta)]\left\{\left[\frac{\omega^2}{c^2}-k^2-\chi^2\right]\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\left[d_r^2\left[\frac{J_l(\chi r)}{\chi r}\right]+\frac{3}{r}d_r\left[\frac{J_l(\chi r)}{\chi r}\right]+\left[\frac{\omega^2}{c^2}-k^2+\frac{1-l^2}{r^2}\right]\left[\frac{J_l(\chi r)}{\chi r}\right]\right]\right\} . \tag{54}$$

Finally, we make use of the following Bessel differential equation:[14]

$$y''+\frac{1-2\alpha}{x}y'+\left[\beta^2+\frac{\alpha^2-p^2}{x^2}\right]y=0 , \tag{55}$$

for $y(x)=x^\alpha Z_p(\beta x)$. In our case, $l=p$, $\beta=\chi$, $\alpha=-1$, and $Z\equiv J$. We then end up with

$$\Box_r=\left[\frac{\omega^2}{c^2}-k^2-\chi^2\right]\left[\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\frac{J_l(\chi r)}{\chi r}\right]\exp[i(\omega t-kz+l\theta)] . \tag{56}$$

After similar algebraic manipulations, we also obtain

$$\Box_\theta=\left[\frac{\omega^2}{c^2}-k^2-\chi^2\right]i\left[\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\frac{J_l'(\chi r)}{l}\right]\exp[i(\omega t-kz+l\theta)] . \tag{57}$$

We can now proceed with the $\hat{r}$ and $\hat{\theta}$ components of the wave equation, (35) and (36). Making use of (56) and (57), the $\hat{r}$-component equation now reads

$$i(\omega-v_{\parallel}k)\left[\frac{\omega^2}{c^2}-k^2-\chi^2-\frac{\omega_{p0}^2}{c^2}\right]\left[\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\frac{J_l(\chi r)}{\chi r}\right]+i\Omega_{\parallel}\left[\frac{\omega^2}{c^2}-k^2-\chi^2\right]\left[\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\frac{J_l'(\chi r)}{l}\right]$$
$$+\frac{\omega_{p0}^2}{c^2}(v_{\parallel}\mathcal{C}-c\mathcal{D})\chi J_l'(\chi r)=0\ , \qquad (58)$$

while the $\hat{\theta}$ component is given by

$$(\omega-v_{\parallel}k)\left[\frac{\omega^2}{c^2}-k^2-\chi^2-\frac{\omega_{p0}^2}{c^2}\right]\left[\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\frac{J_l'(\chi r)}{l}\right]+\Omega_{\parallel}\left[\frac{\omega^2}{c^2}-k^2-\chi^2\right]\left[\mathcal{A}J_{l-1}(\chi r)+\mathcal{B}\frac{J_l(\chi r)}{\chi r}\right]$$
$$-\frac{\omega_{p0}^2}{c^2}(v_{\parallel}\mathcal{C}-c\mathcal{D})\frac{il}{r}J_l(\chi r)=0\ . \qquad (59)$$

In the $\hat{r}$-component equation, we use the identity[14]

$$\frac{J_l(\chi r)}{\chi r}=\frac{1}{l}[J_{l-1}(\chi r)-J_l'(\chi r)] \qquad (60)$$

to obtain two equations:

$$(\omega-v_{\parallel}k)\left[\frac{\omega^2}{c^2}-k^2-\chi^2-\frac{\omega_{p0}^2}{c^2}\right]\left[\mathcal{A}+\frac{\mathcal{B}}{l}\right]+\Omega_{\parallel}\left[\frac{\omega^2}{c^2}-k^2-\chi^2\right]\mathcal{A}=0\ , \qquad (61)$$

$$\left[(\omega-v_{\parallel}k)\left[\frac{\omega^2}{c^2}-k^2-\chi^2-\frac{\omega_{p0}^2}{c^2}\right]-\Omega_{\parallel}\left[\frac{\omega^2}{c^2}-k^2-\chi^2\right]\right]\mathcal{B}+i\frac{\omega_{p0}^2}{c^2}(v_{\parallel}\mathcal{C}-c\mathcal{D})l\chi=0\ , \qquad (62)$$

corresponding to terms in $J_{l-1}(\chi r)$ and $J_l'(\chi r)$, respectively. The second equation (62) is of particular interest since it couples the quasi-TE modes $(\mathcal{A},\mathcal{B})$ to the quasi-TM modes $(\mathcal{C},\mathcal{D})$ through the beam plasma frequency ω_{p0}. Proceeding in the same manner with the $\hat{\theta}$-component equation (59), and using the identity (60) to express $J_l'(\chi r)$ as a function of $J_{l-1}(\chi r)$ and $J_l(\chi r)/\chi r$, we obtain two equations that are identical to (61) and (62).

The dispersion relation $D(\omega,k,\chi)=0$ for waves propagating inside the electron beam is obtained by taking the determinant of the system of four equations [(47), (48), (61), and (62)] in the four unknowns $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. We obtain

$$\left[\omega-kv_{\parallel}+\frac{\omega_{p0}}{\gamma_{\parallel}}\left[\frac{\frac{\omega^2}{c^2}-k^2}{\frac{\omega^2}{c^2}-k^2-\chi^2}\right]^{1/2}\right]\left[\omega-kv_{\parallel}-\frac{\omega_{p0}}{\gamma_{\parallel}}\left[\frac{\frac{\omega^2}{c^2}-k^2}{\frac{\omega^2}{c^2}-k^2-\chi^2}\right]^{1/2}\right]$$
$$\times\left[\omega-kv_{\parallel}+\Omega_{\parallel}\left[\frac{\frac{\omega^2}{c^2}-k^2-\chi^2}{\frac{\omega^2}{c^2}-k^2-\chi^2-\frac{\omega_{p0}^2}{c^2}}\right]\right]\left[\omega-kv_{\parallel}-\Omega_{\parallel}\left[\frac{\frac{\omega^2}{c^2}-k^2-\chi^2}{\frac{\omega^2}{c^2}-k^2-\chi^2-\frac{\omega_{p0}^2}{c^2}}\right]\right]$$
$$+\chi^2\frac{\omega_{p0}^2}{\gamma_{\parallel}^2}\left[\frac{(\omega-kv_{\parallel})^2}{\frac{\omega^2}{c^2}-k^2-\chi^2}\right]=0\ , \qquad (63)$$

where the four different types of space-charge waves appear clearly: the fast and slow ordinary modes couple to the fast and slow extraordinary (cyclotron) modes through the term proportional to $\chi^2\omega_{p0}^2$. In the limiting case $B_{\parallel}\rightarrow\infty$, this result is similar to that derived by Trotman.[15] We now outline the mathematical procedure followed to obtain the complete solution to this problem. Note that at this point χ remains unknown. To determine χ, we must solve the vacuum wave equation and match the solutions at the beam edge ($r=r_b$). Outside the beam ($a>r>r_b$), the plasma frequency is zero, and we have

$$\Box\delta A_r-\frac{1}{r^2}(\delta A_r+2\partial_\theta\delta A_\theta)=0\ , \qquad (64)$$

$$\Box\delta A_\theta-\frac{1}{r^2}(\delta A_\theta-2\partial_\theta\delta A_r)=0\ , \qquad (65)$$

$$\Box \delta A_z = 0 , \tag{66}$$

$$\Box \delta \phi = 0 , \tag{67}$$

together with the gauge condition

$$\frac{1}{c^2}\partial_t \delta\phi + \frac{1}{r}\partial_r(r\delta A_r) + \frac{1}{r}\partial_\theta \delta A_\theta + \partial_z \delta A_z = 0 . \tag{68}$$

The general solution is

$$\delta A_\mu(r,\theta,z,t) = \delta A_\mu(r)\exp[i(\omega t - kz + l\theta)] , \tag{69}$$

with

$$\delta A_r(r) = \frac{1}{\chi r}[\mathcal{W} J_l(\chi r) + \mathcal{X} Y_l(\chi r)] , \tag{70}$$

$$\delta A_\theta(r) = \frac{1}{l}[\mathcal{W} J_l'(\chi r) + \mathcal{X} Y_l'(\chi r)] , \tag{71}$$

which are similar to vacuum TE modes, and

$$\delta A_z(r) = \frac{\omega}{ck}[\mathcal{U} J_l(\chi r) + \mathcal{V} Y_l(\chi r)] , \tag{72}$$

$$\delta\phi(r) = c[\mathcal{U} J_l(\chi r) + \mathcal{V} Y_l(\chi r)] , \tag{73}$$

which correspond to vacuum TM modes. Note that in this region of space, the modified Bessel functions of the first kind, $Y_l(\chi r)$, must be included in the general solution because $r > r_b \neq 0$. In addition, ω, k, and χ are constrained by the vacuum dispersion relation

$$\frac{\omega^2}{c^2} - k^2 - \chi^2 = 0 . \tag{74}$$

At this point, we have found general solution to the 4-D wave equation in two distinct regions of space: region 1, outside the electron beam ($a > r > r_b$), and region 2, inside the electron beam ($r_b > r$). The corresponding solutions are

$$\delta A_{\mu 1}(r,\theta,z,t) = \delta A_{\mu 1}(r)\exp[i(\omega_1 t - k_1 z + l_1\theta)] ,$$

where the radial dependence of $\delta A_{\mu 1}$ is described by Eqs. (70)–(73) for $\chi \equiv \chi_1$, and ω_1, k_1, and χ_1 must satisfy the vacuum dispersion relation

$$\frac{\omega_1^2}{c^2} - k_1^2 - \chi_1^2 = 0 ,$$

and, inside the beam,

$$\delta A_{\mu 2}(r,\theta,z,t) = \delta A_{\mu 2}(r)\exp[i(\omega_2 t - k_2 z + l_2\theta)] ,$$

where the $\delta A_{\mu 2}$ are given by Eqs. (39)–(42) for $\chi \equiv \chi_2$, together with the beam dispersion relation (63)

$$D(\omega_2, k_2, \chi_2) = 0 .$$

The boundary conditions are the following. At the beam edge ($r = r_b$), all the components of the four-vector potential must be continuous, except the radial component,

$$\Delta\delta A_\theta(r = r_b) = 0 , \tag{75}$$

$$\Delta\delta A_z(r = r_b) = 0 , \tag{76}$$

$$\Delta\delta\phi(r = r_b) = 0 , \tag{77}$$

to avoid infinite field components. In addition, because the cylindrical distribution of surface charges and currents cannot contribute to the discontinuity of the following field components, we have

$$\Delta\delta E_\theta(r = r_b) = 0 , \tag{78}$$

$$\Delta\delta E_z(r = r_b) = 0 , \tag{79}$$

$$\Delta\delta B_z(r = r_b) = 0 . \tag{80}$$

We note, however, that conditions (78) and (79) are automatically satisfied by (75)–(77). Finally, at the waveguide wall ($r = a$), the following field components must be zero:

$$\delta E_\theta(r = a) = 0 , \tag{81}$$

$$\delta E_z(r = a) = 0 , \tag{82}$$

$$\delta B_r(r = a) = 0 . \tag{83}$$

Again, we note that condition (83) is automatically satisfied by (81) and (82). At this point, we have two series of independent boundary conditions: four at the beam edge and two at the waveguide wall, and eight field amplitudes ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ inside the beam, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{X}$ outside the beam). The remaining two boundary conditions are obtained by considering the surface charge and current densities at the beam edge[15] that generate discontinuities of δE_r and δB_θ. We have

$$\Delta\delta E_r(r = r_b) = -\frac{1}{\epsilon_o} e n_0 \delta r , \tag{84}$$

$$\Delta\delta B_\theta(r = r_b) = -\mu_o e n_0 v_\parallel \delta r , \tag{85}$$

where $\delta r(\theta,z,t)$ is the beam edge perturbation induced by the electromagnetic waves. This quantity can be evaluated by considering

$$n_0 \delta v_r = \frac{1}{\mu_0 e}\left[\Box\delta A_r - \frac{1}{r^2}(\delta A_r + 2\partial_\theta \delta A_\theta)\right] . \tag{86}$$

We have, by definition,

$$\delta v_r(r = r_b) = (\partial_t + \mathbf{v}_0 \cdot \nabla)\delta r , \tag{87}$$

which yields

$$\delta r(\theta,z,t) = -\frac{i\left[\dfrac{\omega^2}{c^2} - k^2 - \chi_2^2\right]}{\mu_o e n_0(\omega - k v_\parallel)} \times \left[\mathcal{A} J_{l-1}(\chi_2 r_b) + \mathcal{B}\frac{J_l(\chi_2 r_b)}{\chi_2 r_b}\right] \times \exp[i(\omega t - kz + l\theta)] . \tag{88}$$

Here, note that the continuity of δA_θ, δA_z, and $\delta\phi$ at the beam edge [(75)–(77)] immediately yields the following relations:

$$k_1 = k_2 = k , \tag{89}$$

$$l_1 = l_2 = l , \tag{90}$$

$$\omega_1 = \omega_2 = \omega , \tag{91}$$

which correspond to the fact that continuity must hold any z, θ, and t, respectively. We can then eliminate the amplitudes and obtain a relation between ω, k, χ_1, and χ_2, of the form

$$B(\omega,k,\chi_1,\chi_2)=0 \ , \tag{92}$$

which includes the geometrical factors of the problem such as the beam radius r_b and the waveguide radius a. Equation (92) and the two dispersion relations in vacuum (74) and inside the beam (63) form a system of three nonlinear equations in k, χ_1, and χ_2. For a given value of the frequency ω, we can determine the wave number k and the radial intensity profile of the electromagnetic waves propagating along the electron beam. Note that in the case where the beam fills the waveguide ($r_b=a$), only discrete values of ω and k are allowed.[15] We also point out that the respective amplitudes of the other vacuum modes propagating outside the electron beam can be determined by an additional set of boundary conditions at the beam edge and at the waveguide wall.

From the form of the beam dispersion relation (63), we also notice that generally two different values of χ_2 are allowed, reflecting the birefringence of the magnetized electron beam.

Finally, the exact form of the boundary equation (92) can be obtained by writing down the eight independent boundary conditions and eliminating the eight field amplitudes.

IV. CONCLUSIONS

We have presented a detailed account of a formal mathematical description of the interaction of relativistic charged particle beams with electromagnetic waves, within the frame of classical electrodynamics. This description of beam-wave interactions is quite general and can be used as a new canonical system of third-order PDE's describing the self-consistent evolution of the electromagnetic perturbation described in terms of four-vector potential in the linear regime. The formalism is Eulerian in the sense that the (now implicit) fluid dynamical quantities are treated as continuous space-time fields, on an equal footing with their electromagnetic counterparts. The limitations of this model are the following. On the one hand, only small (linear) electromagnetic perturbations can be considered, as is customary in the analysis of the dispersive and stability properties of beam-wave systems; on the other hand, these studies are limited to macroscopic fluid instabilities, as opposed to the microinstabilities studied within the framework of kinetic theory.[9]

These equations are then used, as an example, to study the dispersive characteristics of space-charge waves propagating in a cylindrical waveguide along a relativistic electron beam confined by an axial magnetic field. The problem is studied in a fully relativistic way for arbitrary values of the axial guide field and any degree of azimuthal symmetry. In the limiting case of an infinite guide field, we recover the result derived by Trotman.[15] Finally, we plan to expand on this theory in a forthcoming paper, by treating the problem of optical guiding in a free-electron laser (Ref. 16) within the framework of the formalism exposed in this paper.

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