Angular correlations of photons

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We study angular correlations of the radiation emitted by spatially extended sets of atoms. Such correlations result from the fact that the photons are identical quantum particles. We show that the symmetry (or the antisymmetry) alone of the wave function for identical, indistinguishable particles always leads to significant angular correlations and that in full agreement with our intuition, bosons tend to travel in the same direction while fermions tend to travel in opposite directions. In order to quantify these intuitive notions, we introduce—as a measure of angular correlations—the average value of the cosine of the angle between the directions of the momenta of two particles. The angular correlation coefficient is calculated for simple, model wave functions and the results are compared with those obtained from an exactly soluble model describing radiating harmonic oscillators. We find that even though the interaction modifies the exact values of the angular correlation coefficient, the gross features can be obtained from quantum statistics, i.e., from the symmetry of the many-photon wave functions. Moreover, the angular correlations as measured by the average cosine exhibit an almost universal character, at least for a small number of emitting atoms. Finally, we compare angular correlations obtained in quantum electrodynamics with the correlations resulting from the phenomenologically introduced randomness of the phases in the classical description of the electromagnetic field.

I. INTRODUCTION

The quantum nature of light manifests itself predominantly in the statistical properties of photon states. Usually, one studies the time correlations of photons and those studies have led recently to the discoveries of antibunching and squeezing which reveal and underscore the quantum nature of photons. In the present paper we consider a different aspect of quantum photon statistics, namely, the tendency of the photons to form collimated beams in space.

Angular correlations and the tendency of the radiating systems to emit photons in a narrow bundle were studied in several previous papers.[1–4] All these papers dealt with a large number of emitters, and the approximations used to evaluate the angular correlations of emitted photons took advantage of this fact. The angular correlations in the case of a small number of emitters were studied by Steudel and Richter[5] and Richter[6] with the use of the Markov approximation and more recently by Duncan and Stehle[7] and by Duncan, Mawhinney, and Stehle,[8] who found numerical solutions of the exact dynamical equations describing the spontaneous decay of a system of two-level atoms. Among other things, Duncan and Stehle demonstrated "the tendency of the photons to form a single ray."

The aim of our paper is to determine to what extent this tendency is due to the dynamics of radiation and to what extent it can be attributed just to the quantum statistics of the photons. Our paper is devoted exclusively to the study of the angular correlations of photons (or other identical particles; for example, neutrinos) produced by a small number of emitters. In contradistinction to previous work we rely on analytical results obtained with the use of two models.

The first model is based on a set of simplifying physical assumptions which replace specific dynamical equations and which, in essence, mean that we disregard all interactions except those directly responsible for the radiation of photons. Under these assumptions we are able to write down, in a general case, a model N-particle wave function and to derive an explicit expression for the angular correlations of the emitted particles. Unfortunately, the number of terms in this explicit formula grows with the number N of particles even faster than $N!$, and the evaluation of the correlation coefficient for a large N takes a lot of computer time. However, for the systems containing a few atoms ($N=2$, 3 and 4) that were considered by Steudel and Richter and by Duncan and Stehle the formulas become so simple that they can be easily evaluated by hand.

The second model studied in our paper is that of several harmonic oscillators distributed in space and interacting with the continuum of modes of the radiation field. Angular correlations of photons calculated in this case confirm qualitatively the results obtained within the first model. Moreover, when the spacing between the emitters is more than one wavelength, there is even a substantial, quantitative agreement between the full dynamical description and that employing the model wave function. Therefore, the simplified description based on our assumptions does seem to capture all essential features of

the problem.

Our results show that the bosonic nature of the emitted photons, embodied in the symmetry of their wave function, plays a decisive role in the formation of angular correlations. This is further underscored by the fact that for the fermions the antisymmetry of the wave function leads to the reversal of the angular correlations. In other words, fermions have a tendency to travel in opposite directions.

In Sec. IV, we use the same set of distributed emitters as in the preceding sections to calculate angular correlations in classical electrodynamics, assuming an ensemble of wave fields with random phases. We show that, even though the classical correlations have similar general features, they are weaker than their quantum counterparts.

II. SIMPLE PARAMETRIZATION OF ANGULAR CORRELATIONS

In order to measure the angular correlations of particles we introduce—as the correlation coefficient—the average value of the cosine of the angle between the momenta of two particles. Such an average can be calculated either from the wave function describing the N-photon state or from the intensity correlation function.

In the first case the angular correlation coefficient $\langle c \rangle$, i.e., the average value of the cosine is given by the formula

$$\langle c \rangle = \langle \mathbf{n}_i \cdot \mathbf{n}_j \rangle = \sum_{\lambda_i} \int d^{3N}k \, \Psi^*(\mathbf{k}_1,\lambda_1,\ldots,\mathbf{k}_N,\lambda_N) \times (\mathbf{n}_i \cdot \mathbf{n}_j) \Psi(\mathbf{k}_1,\lambda_1,\ldots,\mathbf{k}_N,\lambda_N) , \tag{2.1}$$

where $\Psi(\mathbf{k}_1,\lambda_1,\ldots,\mathbf{k}_N,\lambda_N)$ is the wave function in momentum space, the parameters λ_i label the spin or the polarization states, and $\mathbf{n}_i = \mathbf{k}_i / |\mathbf{k}_i|$.

In order to take into account only the truly two-particle correlations one should, in principle, subtract from the expression (2.1) the contribution resulting from a possible existence of a preferred direction in the one-particle emission. Such a corrected expression would read

$$\langle c \rangle = \langle \mathbf{n}_i \cdot \mathbf{n}_j \rangle - \langle \mathbf{n} \rangle \cdot \langle \mathbf{n} \rangle = \langle (\mathbf{n}_i - \langle \mathbf{n} \rangle) \cdot (\mathbf{n}_j - \langle \mathbf{n} \rangle) \rangle , \tag{2.2}$$

where $\langle \mathbf{n} \rangle$ is the average value of the direction $\mathbf{n}$ of the wave vector. Owing to the symmetry (or antisymmetry) of the wave function, the angular correlation coefficient $\langle c \rangle$ does not depend on the choice of the indices i and j in the formulas (2.1) and (2.2). In all the cases that we shall be dealing with in this paper, the average value of $\mathbf{n}$ vanishes. The vanishing of $\langle \mathbf{n} \rangle$ does not mean, of course, that the angular distribution of particles is isotropic. The general form (2.2) of the angular correlation coefficient $\langle c \rangle$ underscores its role as a measure of the uncertainty (or fluctuations) in the directions of momenta of emitted particles.

The second definition of the angular correlation coefficient (again in the simplest case when $\langle \mathbf{n} \rangle$ vanishes), based on the intensity correlation function, reads

$$\langle c \rangle = \frac{\int d^3k_1 d^3k_2 (\mathbf{n}_1 \cdot \mathbf{n}_2) \mathcal{I}(\mathbf{k}_1,\mathbf{k}_2)}{\int d^3k_1 d^3k_2 \mathcal{I}(\mathbf{k}_1,\mathbf{k}_2)} , \tag{2.3}$$

where the intensity correlation function $\mathcal{I}(\mathbf{k}_1,\mathbf{k}_2)$ is defined as the following expectation value of the normally ordered product of the particle number operators:

$$\mathcal{I}(\mathbf{k}_1,\mathbf{k}_2) = \langle \Psi | : a^\dagger(\mathbf{k}_1) a(\mathbf{k}_1) a^\dagger(\mathbf{k}_2) a(\mathbf{k}_2) : | \Psi \rangle .$$

The state vector $|\Psi\rangle$ describes the states of the complete system composed of particles and emitters.

The definition (2.3) will be used for the study of the time evolution of angular correlations. It coincides with the definition (2.1) when the state $|\Psi\rangle$ contains exactly N particles and when these particles are uncorrelated with their sources. The state of N initially excited sources, however, in its time evolution gives rise to a superposition of states with different numbers of particles, since for short times some emitters have not yet decayed. We expect that for times larger than the lifetime of the emitters the dominant term in this superposition will be the one with exactly N emitted particles. In Sec. IV we show that, due to dynamical effects, even for longer times the formulas (2.1) and (2.3) give different results.

III. ANGULAR CORRELATIONS CALCULATED FROM THE MODEL WAVE FUNCTION

In this section we use a model of the wave function describing N photons emitted by N excited atoms which are fixed at the positions $\mathbf{r}_1,\ldots,\mathbf{r}_N$. This wave function is deduced from the following assumptions.

(i) All emitted photons have the same energy because all emissions are due to the same atomic (or nuclear) transition and all linewidth effects are neglected.

(ii) Photons are emitted by pointlike sources, i.e., the atomic dimensions are much smaller than the wavelength of the emitted photons.

(iii) Every photon after its emission propagates freely and does not interact with the sources of radiation. There is also no interaction between the emitters.

In the Appendix we derive from assumptions (i)–(iii) the following form of the N-photon wave function in momentum space:

$$\Phi(\mathbf{n}_1,\ldots,\mathbf{n}_N) = \mathcal{N} \sum_{\text{perm } i} \exp(-ik\mathbf{n}_{i_1}\cdot\mathbf{r}_1 - \cdots - ik\mathbf{n}_{i_N}\cdot\mathbf{r}_N) = \mathcal{N} \sum_{\text{perm } A} \exp(-ik\mathbf{n}_1\cdot\mathbf{r}_{A_1} - \cdots - ik\mathbf{n}_N\cdot\mathbf{r}_{A_N}) , \tag{3.1}$$

where $\mathcal{N}$ is the normalization factor and k is the common length [assumption (i)] of the photon wave vectors. The equality of the two expressions for the angular photon wave function (3.1) follows from the fact that the sum of the permutations over the directions of the photon wave vectors is equivalent to the sum of the permutations over

the atom positions.

The N-photon wave function (3.1) describes the symmetrized combination of undistorted plane waves [assumption (iii)] originating at the atomic positions. Such a wave function can describe only the system of photons at times so large that practically all the atoms have emitted their photons.

For the wave function (3.1) the normalization constant $\mathcal{N}$ is defined by the following normalization condition:

$$1=\int d\Omega_1\cdots d\Omega_N|\Phi(\mathbf{n}_1,\ldots,\mathbf{n}_N)|^2$$
$$=\mathcal{N}^2\int d\Omega_1\cdots d\Omega_N\sum_{\text{perm }A}\sum_{\text{perm }B}\exp[ik\mathbf{n}_1\cdot(\mathbf{r}_{A_1}-\mathbf{r}_{B_1})+\cdots+ik\mathbf{n}_N\cdot(\mathbf{r}_{A_N}-\mathbf{r}_{B_N})]\ . \tag{3.2}$$

In the formula (3.2) the integrations $\int d\Omega_i$ over all directions can easily be performed and give the following expression for $\mathcal{N}$:

$$\mathcal{N}^{-2}=(4\pi)^N N!\sum_{\text{perm }A}S(R_{1A_1})\cdots S(R_{NA_N})\ , \tag{3.3}$$

where R_{AB} is the distance between the Ath and the Bth atom measured in units of $1/k$,

$$R_{AB}=|\mathbf{R}_{AB}|=kr_{AB}=k|\mathbf{r}_A-\mathbf{r}_B|\ , \tag{3.4}$$

and the function $S(R)$ is defined as

$$S(R)=\frac{\sin(R)}{R}=1-\frac{R^2}{6}+\frac{R^4}{120}+\cdots\ . \tag{3.5}$$

One summation over all permutations in formula (3.3) has been already performed because every term in the sum over the permutations of $A_1,\ldots,A_N$ was the same.

We can now use the expression (3.1) for the wave function Φ to calculate the average value $\langle c\rangle$ of the cosine,

$$\langle c\rangle=\int d\Omega_1\cdots d\Omega_N\Phi^*(\mathbf{n}_1,\ldots,\mathbf{n}_N)(\mathbf{n}_i\cdot\mathbf{n}_j)$$
$$\times\Phi(\mathbf{n}_1,\ldots,\mathbf{n}_N)\ . \tag{3.6}$$

This calculation is greatly facilitated by the observation that the vectors $\mathbf{n}_i$ standing next to the wave function Φ can be replaced by the appropriate derivatives. For example,

$$(\mathbf{n}_1\cdot\mathbf{n}_2)\Phi(\mathbf{n}_1,\ldots,\mathbf{n}_N)$$
$$=\mathcal{N}\sum_{\text{perm }A}(ik)^{-2}(\nabla_{A_1}\cdot\nabla_{A_2})$$
$$\times\exp(-ik\mathbf{n}_1\cdot\mathbf{r}_{A_1}-\cdots-ik\mathbf{n}_N\cdot\mathbf{r}_{A_N})\ . \tag{3.7}$$

Upon inserting this expression into (3.6) and after the integrations over the angles and the differentiations with respect to $\mathbf{r}$, we obtain the angular correlation coefficient as the following ratio of a double sum with respect to all permutations over a single sum:

$$\langle c\rangle=L/M\ , \tag{3.8}$$

$$L=-(N!)^{-1}\sum_{\text{perm }A}\sum_{\text{perm }B}\epsilon_P\rho_{A_1B_1}\cdot\rho_{A_2B_2}$$
$$\times S'(R_{A_1B_1})S'(R_{A_2B_2})$$
$$\times S(R_{A_3B_3})\cdots S(R_{A_NB_N})\ , \tag{3.9}$$

$$M=\sum_{\text{perm }A}\epsilon_P S(R_{1A_1})\cdots S(R_{NA_N})\ , \tag{3.10}$$

where $\boldsymbol{\rho}_{AB}$ is the unit vector in the direction of $\mathbf{R}_{AB}$ and the function $S'(R)$ is the derivative of $S(R)$,

$$S'(R)=\cos(R)/R-\sin(R)/R^2$$
$$=R/3-R^3/30+R^5/840+\cdots\ . \tag{3.11}$$

The sign factor ϵ_P has been introduced to cover also the case of fermions. For bosons it is always equal to 1, while for fermions it is equal to -1 when the permutation of the B indices is odd, as compared to the permutation of the A indices. In the second summation over all permutations in the formula (3.9) we need to consider only $N(N-1)/2$ different terms (the number of combinations of order 2 of N elements).

As we have already mentioned in Sec. II, the average value of $\mathbf{n}$ vanishes. This can be easily proved by the same method that was used to derive the formula (3.9). The expression for $\langle\mathbf{n}\rangle$ obtained in this manner will have the numerator made of terms linear in the $\boldsymbol{\rho}_{AB}$. In the sum over all permutations all such terms cancel pairwise, since they change sign under the interchange of the indices A and B.

Now, we shall apply formulas (3.8)–(3.10) for the angular correlation coefficient to various special cases. Our general formula for the angular correlation coefficient simplifies greatly when all the distances between the emitters are equal. This can happen, of course, only when $N=2$, 3, and 4, for the linear, triangular, and tetrahedral configurations. The results for bosons and for fermions are

$$\langle c\rangle_2^B=(S')^2/(1+S^2)\ , \tag{3.12}$$

$$\langle c\rangle_2^F=-(S')^2/(1-S^2)\ , \tag{3.13}$$

$$\langle c\rangle_3^B=(1+S)(S')^2/(1+3S^2+2S^3)\ , \tag{3.14}$$

$$\langle c\rangle_3^F=-(S')^2/(1+S-2S^2)\ , \tag{3.15}$$

$$\langle c\rangle_4^B=(1+2S+3S^2)(S')^2/(1+6S^2+8S^3+9S^4)\ , \tag{3.16}$$

$$\langle c\rangle_4^F=-(S')^2/(1+2S-3S^2)\ , \tag{3.17}$$

where we have introduced the superscripts B and F to denote the coefficients for bosons and fermions and the subscripts 2, 3, and 4 to denote the number of particles. In order to simplify the formulas, we have omitted the argument R of the functions $S(R)$ and $S'(R)$. The variation of the angular correlation coefficients, given by the

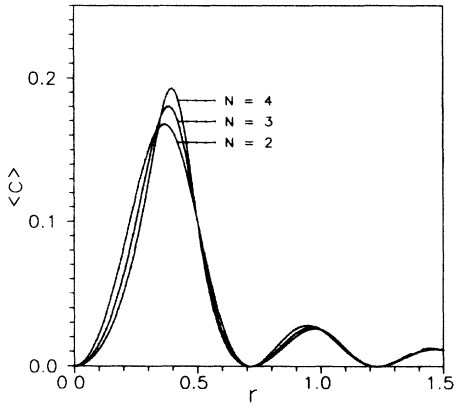

FIG. 1. Average cosine $\langle c \rangle$ calculated with the use of the model wave function for bosons as a function of the common interatomic distance r (in units of the photon wavelength) for linear, triangular, and tetrahedral configurations of N atoms ($N=2$, 3, and 4).

formulas (3.12)–(3.17), with the distances R between the atoms is shown in Fig. 1 for bosons and in Fig. 2 for fermions.

As was to be expected, bosons tend to travel in the same direction, since the average cosine is always positive ($\langle c \rangle^B > 0$), whereas fermions tend to travel in opposite directions ($\langle c \rangle^F < 0$). For bosons, the effect is largest when the distances between the emitters are close to a half of the wavelength of the emitted particles. For fermions, the effect is largest for small distances between the emitters.

For $N > 4$ not all the distances between the emitters can be made equal and the formulas for the coefficient $\langle c \rangle$ become more cumbersome. In addition, the number of terms grows so rapidly with N that we reach very quickly the stage when even large computers cannot handle the calculations in a reasonable amount of time. For example, when $N=12$, the numerator (3.9) contains more than 30 billion terms.

We calculated the value of $\langle c \rangle$, when $N=5$, 6, 7, and

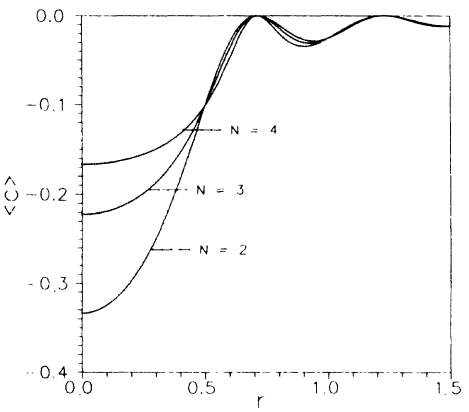

FIG. 2. Average cosine $\langle c \rangle$ for fermions as a function of the common distance r for the same configuration of emitters as in Fig. 1.

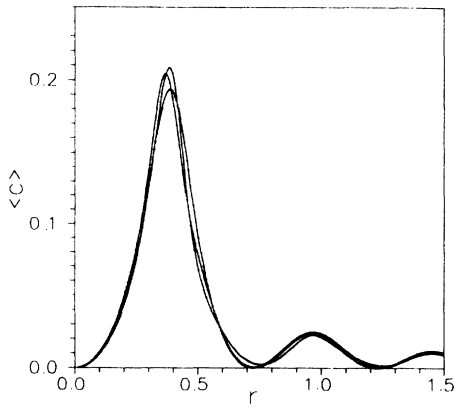

FIG. 3. Angular correlation coefficient for photons emitted by 5, 6, and 7 atoms, plotted in the same fashion as in Fig. 1.

8, for those distributions of atoms for which we expect maximal correlations, i.e., for those which most closely resemble the equidistant configurations studied for $N=2$, 3, and 4. For $N=5$, 6, and 7, we have calculated the angular correlation coefficient as a function of the typical distance between the atoms when they are located at the corners of two joined pyramids based on a triangle, a square, and a pentagon, respectively. The results are shown in Fig. 3.

We have also calculated the angular correlation coefficient for various configurations of eight atoms; for example, a cube, a twisted cube (top face rotated 45°), and two joined pyramids based on a hexagon. The results were very similar to those shown in Fig. 3 for 5, 6, or 7 atoms. Therefore, we can conclude that for our "almost equidistant" configurations the correlation coefficient practically does not depend on the number of atoms; the angular correlations of photons, measured by the average cosine, exhibit an almost universal character.

The tetrahedral configuration for $N=4$ has been recently studied also by Duncan and Stehle[9] and our values of the angular correlation coefficient reproduce the results of their fully dynamical calculations within 7%. This is, however, just a lucky coincidence, since, as we show in Sec. IV, where we study the influence of the dynamical effects with the use of an exactly soluble model, the influence of the dynamical effects on the angular coefficient is not that small.

IV. ANGULAR CORRELATIONS IN AN EXACTLY SOLUBLE MODEL

The exactly soluble model used by us to verify the results of the simplified analysis describes a set of N oscillators coupled to all modes of the radiation field in the dipole approximation. The Hamiltonian of this model has the form ($\hbar = 1 = c$)

$$H = \omega_0 \sum_{A=1}^{N} b_A^\dagger b_A + \int d^3k\, \omega a^\dagger(\mathbf{k}) a(\mathbf{k}) + \sum_{A=1}^{N} P_A \int d^3k\, g(\omega)[a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}_A} + a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}_A}] , \tag{4.1}$$

where P_A is the dimensionless momentum operator of the Ath oscillator,

$$P_A=-i(b_A-b_A^\dagger)\ , \tag{4.2}$$

and we have incorporated the coupling constant into the cutoff function $g(\omega)$. The Heisenberg equations of motion for the creation and annihilation operators in this model can be solved by the Laplace transformation. The resulting integrals can be performed analytically for properly chosen cutoff functions. In our study we have adopted the same form of this function,

$$[g(\omega)]^2=\frac{e^2}{8\pi^2\omega}\frac{\beta^2}{\omega^2+\beta^2}\ , \tag{4.3}$$

as in the papers by Rzążewski and Żakowicz[10] and Lewenstein and Rzążewski.[11]

The Laplace transforms of the Heisenberg equations of motion for the annihilation operators have the form

$$(z+i\omega)a(\mathbf{k},z)+i\sum_{A=1}^{N}g(\omega)e^{-i\mathbf{k}\cdot\mathbf{r}_A}P_A(z)=a^0(\mathbf{k})\ , \tag{4.4a}$$

$$(z+i\omega_0)b_A(z)+\int d^3k\,g(\omega)[a(\mathbf{k},z)e^{i\mathbf{k}\cdot\mathbf{r}_A}+a^\dagger(\mathbf{k},z)e^{-i\mathbf{k}\cdot\mathbf{r}_A}]=b_A^0\ , \tag{4.4b}$$

where $a^0(\mathbf{k})$ and b_A^0 are the annihilation operators at $t=0$ and z is the Laplace variable. Equations for the creation operators are obtained from Eqs. (4.4) by Hermitian conjugation (keeping z unchanged). After solving Eq. (4.4a) with respect to $a(\mathbf{k},z)$ and substituting the resulting expression into Eq. (4.4b), we obtain the following set of linear equations for b_A:

$$(z+i\omega_0)b_A(z)+i\sum_{B=1}^{N}P_B(z)f(r_{AB},z)=b_A^0+K(\mathbf{r}_A,z)\ , \tag{4.5}$$

where $f(r,z)$, defined as the following integral,

$$f(r,z)=\int d^3k[g(\omega)]^2\left[\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{z+i\omega}-\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{z-i\omega}\right]\ , \tag{4.6}$$

depends only on the length of $\mathbf{r}$. For the choice (4.3) of the cutoff function this integral can be explicitly evaluated giving

$$f(r,z)=\frac{e^2\beta^2(e^{-zr}-e^{-\beta r})}{2ir(\beta^2-z^2)}\ . \tag{4.7}$$

The operator $K(\mathbf{r},z)$ contains the initial photon creation and annihilation operators,

$$K(\mathbf{r},z)=-\int d^3k\,g(\omega)\left[a^0(\mathbf{k})\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{z+i\omega}-[a^0(\mathbf{k})]^\dagger\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{z-i\omega}\right]\ . \tag{4.8}$$

Since the electromagnetic field operators depend only on the operators of the momenta $P_A(z)$ of the oscillators, we shall solve Eqs. (4.4) and their conjugates for $P_A(z)$. The equations for $P_A(z)$ read

$$[z^2+\omega_0^2-2i\omega_0 f(0,z)]P_A(z)-2i\omega_0\sum_{B\,(\neq A)}P_B(z)f(r_{AB},z)=-J_A(z)-2\omega_0K(\mathbf{r}_A,z)\ , \tag{4.9}$$

where

$$J_A(z)=i(z-i\omega_0)b_A^0-i(z+i\omega_0)(b_A^0)^\dagger\ . \tag{4.10}$$

The solution of this set of equations has the form

$$P_A(z)=-\sum_{B=1}^{N}M_A^B(z)[J_B(z)+2\omega_0K(\mathbf{r}_B,z)]\ . \tag{4.11}$$

The matrix elements of the matrix $M_A^B(z)$ depend on the mutual distances between all the atoms and will be specified later. Upon substituting the expression (4.11) into Eq. (4.4a), we obtain the solution for the photon annihilation operators in terms of the initial value operators,

$$a(\mathbf{k},z)=\frac{1}{z+i\omega}\left[a^0(\mathbf{k})+\sum_{A=1}^{N}g(\omega)e^{-i\mathbf{k}\cdot\mathbf{r}_A}\sum_{B=1}^{N}M_A^B(z)[J_B(z)+2\omega_0K(\mathbf{r}_B,z)]\right]\ . \tag{4.12}$$

In our calculation of the intensity correlation functions (2.4) the relevant part of the solution (4.12) for the photon annihilation (creation) operator is the part proportional to the oscillator creation and annihilation operators taken at the initial time $t=0$, since we assume that there are no photons in the initial state. We shall call it the source part. The other two parts are the free part [the first term on the right-hand side in Eq. (4.12)] and the scattered part [coming from the operator $K(\mathbf{r}_A,z)$]. The source part $a_{\rm sr}(\mathbf{k},z)$ has the form

$$a_{\rm sr}(\mathbf{k},z)=\frac{1}{z+i\omega}\sum_{A,B}g(\omega)e^{i\mathbf{k}\cdot\mathbf{r}_A}M_A^B(z)J_B(z)\ . \tag{4.13}$$

In order to find the time dependence of these operators we must invert the Laplace transform,

$$a_{\rm sr}(\mathbf{k},t)=\frac{1}{2\pi i}\int dz\frac{1}{z+i\omega}e^{zt}\sum_{A,B}g(\omega)e^{i\mathbf{k}\cdot\mathbf{r}_A}M_A^B(z)J_B(z)\ , \tag{4.14}$$

where the integration contour runs parallel to the imaginary axis to the right of all singularities of the integrand.

The calculations simplify greatly when all the distances between the atoms are equal, because then in Eqs. (4.9) all the arguments of the functions $f(r_{AB},z)$ are equal. Thus, to find the matrix $M_A^B(z)$, we must invert a matrix all of

whose diagonal elements are equal to $H(z)$,

$$H(z)=z^2+\omega_0^2-2i\omega_0 f(0,z) \tag{4.15}$$

and also all of whose off-diagonal elements are equal. We will denote them by $G(z)$,

$$G(z)=-2i\omega_0 f(r,z)\ . \tag{4.16}$$

Such matrices can easily be inverted,

$$\begin{vmatrix} H & G & G & \cdots & G \\ G & H & G & \cdots & G \\ \vdots & & & & \vdots \\ G & G & G & \cdots & H \end{vmatrix}^{-1} = \begin{vmatrix} A & B & B & \cdots & B \\ B & A & B & \cdots & B \\ \vdots & & & & \vdots \\ B & B & B & \cdots & A \end{vmatrix}, \tag{4.17}$$

where the diagonal and the off-diagonal elements of both matrices are related by the formulas

$$A=\frac{N-1}{N}\frac{1}{H-G}+\frac{1}{N}\frac{1}{H+(N-1)G}\ , \tag{4.18}$$

$$B=-\frac{1}{N}\frac{1}{H-G}+\frac{1}{N}\frac{1}{H+(N-1)G}\ , \tag{4.19}$$

where N is the dimension of the matrix (the number of atoms).

Now we can easily perform the inverse Laplace transformation; we must only find the zeros of the denominators $H-G$ and $H+(N-1)G$. An additional contribution comes from the purely imaginary pole at the frequency $\omega=c|\mathbf{k}|$ in the formula (4.13).

In the weak-coupling approximation the zeros of the denominators in formulas (4.18) and (4.19) differ only by small corrections from their unperturbed values $z=\pm i\omega_0$. Since both functions $H(z)$ and $G(z)$ are real, the zeros come in mutually complex conjugate pairs, each pair representing one damped mode of radiation. The real and the imaginary parts of these corrections are the radiative level shifts and the widths of the damped modes of radiation.

The positions of the zeros of the denominators in the first order of perturbation theory with respect to e^2 are

$$z_{1,2}=\pm i(\omega_0-\Delta_0+\gamma C)-\gamma(1-S)\ , \tag{4.20}$$

$$z_{3,4}=\pm i(\omega_0-\Delta_0-\gamma(N-1)C)-\gamma(1+(N-1)S)\ , \tag{4.21}$$

where Δ_0 is the constant part (i.e., the part that does not depend on the distances between the atoms) of the frequency shift,

$$\Delta_0=\tfrac{1}{2}e^2\beta^3(\beta^2+\omega_0^2)^{-1}\ , \tag{4.22}$$

γ is the single oscillator decay rate,

$$\gamma=\tfrac{1}{2}e^2\beta^2\omega_0(\beta^2+\omega_0^2)^{-1}\ , \tag{4.23}$$

and

$$C=\cos(\omega_0 r)/\omega_0 r,\quad S=\sin(\omega_0 r)/\omega_0 r\ . \tag{4.24}$$

In what follows we shall restrict ourselves to the study of angular correlations of those photons whose frequency ω is equal to the frequency of the harmonic oscillators renormalized by the position independent shift Δ_0,

$$\omega=\omega_0-\Delta_0\ . \tag{4.25}$$

Under this resonance condition the contributions to $a_{\rm sr}(\mathbf{k},t)$ from the poles at the positions z_1 and z_3 can be disregarded, since they are smaller by the factor of e^2 than those from the remaining three poles at ω, z_2, and z_4. For the same reason one can neglect the part $i(z+i\omega_0)(b_A^0)^\dagger$ of $J_A(z)$. These approximations are essentially equivalent to the rotating-wave approximations but had we made this approximation in the Hamiltonian the functions $H(z)$ and $G(z)$ would have developed bothersome cuts in the complex plane.

At the resonance, after the integration over z, the leading part of the annihilation operator has the form

$$a_{\rm sr}(\mathbf{k},t)=e^{-i\omega t}g(\omega)D^{-1}\sum_{A,B}e^{i\mathbf{k}\cdot\mathbf{r}_A}m_A^B(t)b_B^0\ , \tag{4.26}$$

where the denominator D is equal to

$$D=2\gamma\omega_0 N(C+iS-i)[(N-1)(C+iS)+i] \tag{4.27}$$

and the matrix $m_A^B(t)$ has all of its diagonal elements equal to

$$a(t)=N[(N-2)(C+iS)+i]-(N-1)[(N-1)(C+iS)+i]e^{-i\gamma tC}e^{-\gamma t(1-S)}+(C+iS-i)e^{i\gamma t(N-1)C}e^{-\gamma t(NS+1-S)} \tag{4.28}$$

and all of its off-diagonal elements equal to

$$\begin{aligned} b(t)=&-N(C+iS)+[(N-1)(C+iS)+i] \\ &\times e^{-i\gamma tC}e^{-\gamma t(1-S)} \\ &+(C+iS-i)e^{i\gamma(N-1)C}e^{-\gamma t(NS+1-S)}\ . \end{aligned} \tag{4.29}$$

In order to calculate the average cosine of the angle between the directions of any two photons we shall now use the second definition (2.3) based on the intensity correlation function. We assume that all oscillators are initially excited to their first level. Substituting the expression (4.26) for the annihilation operators and the corresponding expression for the photon creation operators into the formula (2.4), we obtain

$$\mathcal{J}(\mathbf{n}_1,\mathbf{n}_2)=\text{const}\times\sum_{A,B,C,D,I,J,K,L}\cdots\sum e^{-ik\mathbf{n}_1\cdot(\mathbf{r}_A-\mathbf{r}_B)}e^{-ik\mathbf{n}_2\cdot(\mathbf{r}_C-\mathbf{r}_D)}[m_A^I(t)]^*m_B^J(t)[m_C^K(t)]^*m_D^L(t)\langle\Psi|(b_I^0)^\dagger(b_K^0)^\dagger b_J^0 b_L^0|\Psi\rangle\ . \tag{4.30}$$

The constant in front of this expression is irrelevant because it will disappear from the ratio (2.3) that determines the angular correlation coefficient. In order to evaluate this ratio we have to perform the integrations over $\mathbf{n}_1$ and $\mathbf{n}_2$ indicated in the formula (2.3) and to find the expectation value of the product of the oscillator creation and annihilation operators. The integrations over the angles are easily performed with the use of the techniques described in Sec. III with the following result:

$$\langle c\rangle(t)=L(t)/M(t)\ , \tag{4.31}$$

$$L(t)=-\sum_{A,B,C,D,I,J,K,L}\cdots\sum\rho_{AB}\cdot\rho_{CD}S'(R_{AB})S'(R_{CD})[m_A^I(t)]^*m_B^J(t)[m_C^K(t)]^*m_D^L(t)\langle\Psi|(b_I^0)^\dagger(b_K^0)^\dagger b_J^0 b_L^0|\Psi\rangle\ , \tag{4.32}$$

$$M(t)=\sum_{A,B,C,D,I,J,K,L}\cdots\sum S(R_{AB})S(R_{CD})[m_A^I(t)]^*m_B^J(t)[m_C^K(t)]^*m_D^L(t)\langle\Psi|(b_I^0)^\dagger(b_K^0)^\dagger b_J^0 b_L^0|\Psi\rangle\ . \tag{4.33}$$

If we assume that in the initial state all harmonic oscillators are singly excited, which corresponds to our previous calculations, the expectation value is equal to

$$\langle\Psi|(b_I^0)^\dagger(b_K^0)^\dagger b_J^0 b_L^0|\Psi\rangle=\delta_{IJ}\delta_{KL}+\delta_{IL}\delta_{JK}-2\delta_{IJ}\delta_{KL}\delta_{IK}\ . \tag{4.34}$$

The formulas (4.31)–(4.34), supplemented by the expressions (4.28) and (4.29) for the matrix elements $m_A^B(t)$, represent the final results for the angular correlation coefficient in our dynamical model. These results are represented in Fig. 4 for $N=2$, 3, and 4 and for different values of the time variable. As was to be expected for short times the angular correlation coefficient does not depend on the number of atoms, since for times much shorter than the lifetime the number of photons in the radiation field is very small. It is worth noticing that initially the angular correlation is practically the same as for $N=2$ in our previous calculation based on the model wave function. For distances of the order of one wavelength and larger the angular correlation coefficient does not vary with time and is almost the same for all N. We do not show the parts of the plots that correspond to small separations between the atoms, because for closely spaced atoms the one-dimensional harmonic-oscillator model without the dipole-dipole interactions does not represent correctly the real physical situation. The oscillatory behavior of the angular correlations found in this region was not present in our previous reports[12,13] in which we disregarded the position-dependent level shifts.

Our full dynamical calculations in the harmonic-oscillator model do confirm the gross features of the angular correlations found with the use of the model wave function, but these results differ in details. These differences are due to the fact that the model wave function does not include the photon exchanges between the emitters. Without those exchanges the matrix $m_A^B(t)$ would have been diagonal and the results would coincide. It is worth noticing that for times longer than the decay time γ^{-1} of the oscillator excitation the importance of the off-diagonal elements diminishes with increasing N. Thus, our basic assumption C from Sec. III would turn out to be better for a larger number of emitters.

V. COMPARISON OF QUANTUM AND STOCHASTIC CORRELATIONS

We attributed the angular correlations studied in the preceding sections to the quantum nature of the emitted particles. However, intensity correlations can also result from the averaging over an ensemble of classical fields. The need for such an averaging may arise, as has been shown already by Lord Rayleigh,[14] due to the randomness of the phases or due to the random distribution of the emitters. Since the evaluation of the quantum-mechanical expectation values may be expressed in the classical limit as an integration over the phase variable,[15] we might expect that the angular correlations studied in the preceding sections can be, to some extent, reproduced by the classical averaging over random phases.

In order to compare the classical case and the quantum case we will consider a collection of point sources of the classical electromagnetic waves of the same intensity, placed at the positions $\mathbf{r}_A$ and radiating at the same frequency. The spatial dependence of the wave produced by these sources is in the far zone given (up to an unimportant common factor) by the real part of the following expression:

$$F(\mathbf{n})=\sum_A\exp(ik\mathbf{n}\cdot\mathbf{r}_A+i\alpha_A)\ , \tag{5.1}$$

where $\mathbf{n}$ is the direction of observation and α_A is the phase of the Ath source. The classical analog $\mathcal{J}_{\rm cl}(\mathbf{n}_1,\mathbf{n}_2)$ of the quantum intensity correlation function (2.4) is given by the following average over the stochastic phases:

$$\begin{aligned}\mathcal{J}_{\rm cl}(\mathbf{n}_1,\mathbf{n}_2)&=\text{const}\times\int d\alpha_1\cdots\int d\alpha_N F^*(\mathbf{n}_1)F(\mathbf{n}_1)F^*(\mathbf{n}_2)F(\mathbf{n}_2)\\&=\sum_{A,B,C,D}e^{ik\mathbf{n}_1\cdot(\mathbf{r}_A-\mathbf{r}_B)}e^{ik\mathbf{n}_2\cdot(\mathbf{r}_C-\mathbf{r}_D)}(\delta_{AB}\delta_{CD}+\delta_{AD}\delta_{BC})\ .\end{aligned} \tag{5.2}$$

To compare this formula with the corresponding quantum expression (4.30), we must incorporate the dynamics of the radiation process by introducing the elements of the matrix $m_A^B(t)$ into the formula (5.2). Even after this correction is introduced, the classical result differs from its quantum counterpart, because the two terms within the last brackets in Eq. (5.2) replace the three terms appearing in the quantum expectation value (4.34). The

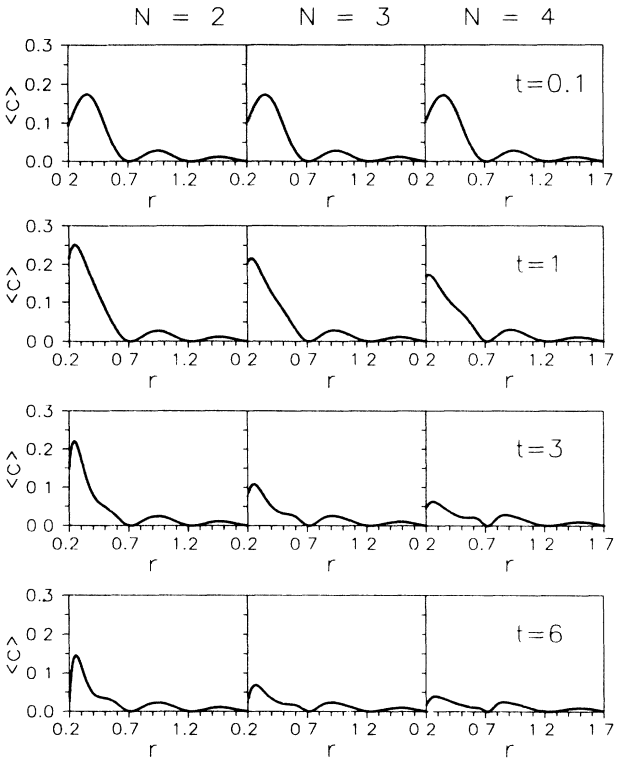

FIG. 4. Time evolution of the average cosine $\langle c \rangle$ plotted as a function of the common interatomic distance r (in units of the photon wavelength) for linear, triangular, and tetrahedral configurations of N atoms, calculated in the *quantum* model of radiating harmonic oscillators. Time is measured in units of the single oscillator decay rate γ^{-1}.

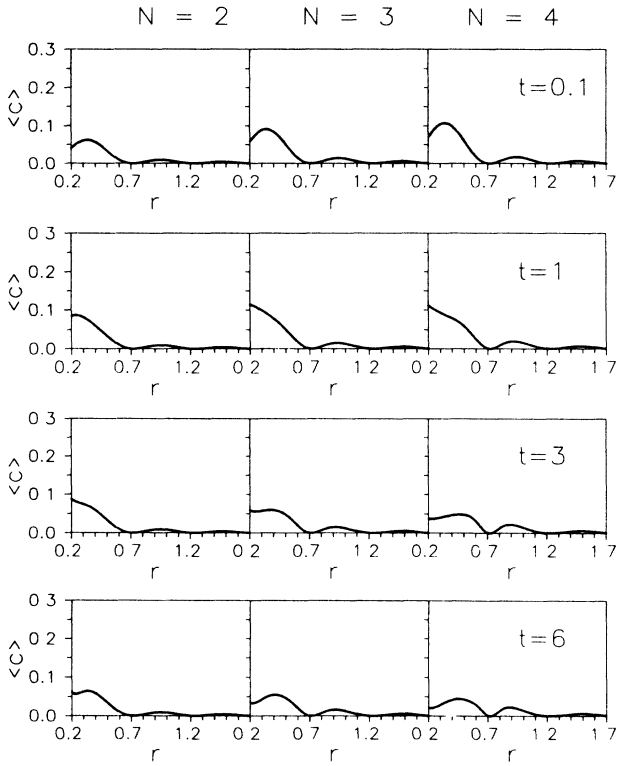

FIG. 5. Time evolution of the average cosine $\langle c \rangle$ plotted for the same situations as in Fig. 4 but calculated in the *classical* model of radiating harmonic oscillators.

term $-2\delta_{IJ}\delta_{KL}\delta_{IK}$, missing in the classical expression, represents the contribution when both detected photons come from the same emitter. As a result, the quantum correlation coefficient $\langle c \rangle$ is larger than the classical one. The difference between the classical calculations and the quantum calculations is substantial (cf. Figs. 4 and 5); the classical phase averaging gives the results scaled down by roughly a factor of 2.

VI. CONCLUSIONS

The main conclusion of this study is that the angular correlations of photons that are due to the spatial distribution of emitters are always positive, except for very small separations of the emitters (smaller than $\frac{1}{6}$ of the wavelength). Outside of this region, which is not correctly described by our simplified model, we have not found a single case where the angular correlation coefficient $\langle c \rangle$ would be negative. This conclusion, of course, is not valid for emitters that could individually radiate correlated photons. Such correlations are often a result of conservation laws (especially conservation of momentum and of angular momentum) as, for example, in the cases of a two-photon decay of an atomic excitation or of a scalar particle.

Correlations in physics arise from loss of information. In the quantum-mechanical description of emission processes angular correlations arise because photons are indistinguishable so that the question of which photon was emitted by a given atom cannot even be posed. In the classical description no correlation will be found unless we introduce some loss of information; typically the loss of information about the phases of the wave fields emitted by different sources. This phase averaging is automatic in the quantum description when we deal with N-photon states. We would like to stress, however, that the classical, phenomenological phase averaging does not reproduce fully the quantum results.

It is worth noting that in all cases studied in this paper the quantum effects enhance the correlations in space, whereas it is known that they decrease the correlations in time, leading to antibunching. We expect a similar phenomenon to occur for angular correlations when emitters are closely spaced and we intend to study this case with the help of a more realistic model.

Angular correlations of photons emitted by a set of atoms distributed in space have not so far been measured. We believe that with the present experimental techniques involving ion traps and atomic and molecular clusters, the measurement of angular correlations is now feasible.

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APPENDIX

In this Appendix we shall derive the form of the N-photon wave function according to our assumptions (i), (ii), and (iii). It will be convenient to use the Dirac (interaction) picture and to write down the final-state vector of our system made of atoms and photons as the product of the S operator and the initial-state vector describing the atoms in excited states and the electromagnetic field in its ground (vacuum) state. Using the Dyson formula for the S operator we obtain

$$|\Psi^f\rangle = T\exp\left[-i\int dt\, H_{\rm int}(t)\right]|\Psi^i\rangle\ , \tag{A1}$$

where the interacting Hamiltonian may be taken as describing either the relativistic coupling of the electromagnetic potential A_μ to the conserved current j^μ or the nonrelativistic coupling of the electromagnetic field to the electric and magnetic dipole and higher multipole atomic operators. In both cases the interaction Hamiltonian is linear in the photon creation and annihilation operators and this is the only property of the Hamiltonian that we shall use in our calculation.

Under our assumptions (ii) and (iii) the N-photon component $|\Psi^f\rangle^N$ of the final-state vector $|\Psi^f\rangle$ can be written in the form

$$|\Psi^f\rangle^N = \sum_{\lambda_i}\int d^{3N}k\, a^\dagger(\mathbf{k}_1,\lambda_1)\exp(-i\mathbf{k}_1\cdot\mathbf{r}_1)\cdots a^\dagger(\mathbf{k}_N,\lambda_N)\exp(-i\mathbf{k}_N\cdot\mathbf{r}_N)$$
$$\times(-i)^N\int dt_1\cdots\int dt_N\exp(i\omega_1 t_1+\cdots+i\omega_N t_N)T\{\mathbf{j}(t_1)\cdot\mathbf{f}(\mathbf{k}_1,\lambda_1)\cdots\mathbf{j}(t_N)\cdot\mathbf{f}(\mathbf{k}_N,\lambda_N)\}|\Psi^i\rangle\ , \tag{A2}$$

where the vectors $\mathbf{r}_A$ denote the positions of the emitters, the vectors $\mathbf{f}(\mathbf{k}_i,\lambda_i)$ denote the photon polarization vectors multiplied by the normalization factors arising from the Fourier expansion of field operators $\mathbf{f}(\mathbf{k}_i,\lambda_i)=(2\pi)^{-3/2}(2\omega)^{-1/2}\boldsymbol{\varepsilon}(\mathbf{n}_i,\lambda_i)$, and $T\{\ \}$ denotes the time-ordered product of all atomic operators. In the nonrelativistic version of the theory the current operators are to be replaced by the electric dipole operators. In order to extract the final state of the electromagnetic field, we must take the scalar product of the state vector (A2), with the state vector $|\Psi^f_{\rm at}\rangle$ describing the final state of the atomic system. The N-photon wave function $\Psi(\mathbf{k}_1,\lambda_1,\ldots,\mathbf{k}_N,\lambda_N)$ will appear as the coefficient multiplying the product of the photon creation operators in the formula

$$\langle\Psi^f_{\rm at}|\Psi^f\rangle^N = \sum_{\lambda_i}\int d^{3N}k\,\Psi(\mathbf{k}_1,\lambda_1,\ldots,\mathbf{k}_N,\lambda_N)$$
$$\times a^\dagger(\mathbf{k}_1,\lambda_1)\cdots a^\dagger(\mathbf{k}_N,\lambda_N)|\Omega\rangle\ , \tag{A3}$$

where $|\Omega\rangle$ denotes the vacuum of the electromagnetic field. By comparing expressions (A3) and (A2), we obtain the following formula for the photon wave function:

$$\Psi(\mathbf{k}_1,\lambda_1,\ldots,\mathbf{k}_N,\lambda_N) = (N!)^{-1}\sum_{\text{perm }i}\exp(-i\mathbf{k}_{i_1}\cdot\mathbf{r}_1-\cdots-i\mathbf{k}_{i_N}\cdot\mathbf{r}_N)$$
$$\times(-i)^N\int dt_1\cdots\int dt_N\exp(i\omega_1 t_1+\cdots+i\omega_N t_N)$$
$$\times\langle\Psi^f_{\rm at}|T\{\mathbf{j}(t_1)\cdot\mathbf{f}(\mathbf{k}_{i_1},\lambda_{i_1})\cdots\mathbf{j}(t_N)\cdot\mathbf{f}(\mathbf{k}_{i_N},\lambda_{i_N})\}|\Psi^i_{\rm at}\rangle\ . \tag{A4}$$

Under our simplifying assumption that the emitters do not interact among themselves, the expression in last line of the formula (A4) becomes equal to the product of the transition matrix elements for each atom. Each integration over time can then be separately performed leading to the product of the energy-conservation δ functions $\delta(\omega-E_i+E_f)$, where E_i and E_f are the initial and final energies of emitters. Thus the wave function $\Psi(\mathbf{k}_1,\lambda_1,\ldots,\mathbf{k}_N,\lambda_N)$ effectively reduces to a function $\Phi(\mathbf{n}_1,\lambda_1,\ldots,\mathbf{n}_N,\lambda_N)$ of the directions of momenta $\mathbf{n}_i$ and the polarization indices λ_i only. In this way we arrive at the following formula for the photon wave function, which can be used to calculate all the angular correlations:

$$\Phi(\mathbf{n}_1,\lambda_1,\ldots,\mathbf{n}_N,\lambda_N)$$
$$=\mathcal{N}\sum_{\text{perm }i}\mathbf{d}_1\cdot\boldsymbol{\varepsilon}(\mathbf{n}_{i_1},\lambda_{i_1})\cdots\mathbf{d}_N\cdot\boldsymbol{\varepsilon}(\mathbf{n}_{i_N},\lambda_{i_N})$$
$$\times\exp(-ik\mathbf{n}_{i_1}\cdot\mathbf{r}_1-\cdots--ik\mathbf{n}_{i_N}\cdot\mathbf{r}_N) \tag{A5a}$$

$$=\mathcal{N}\sum_{\text{perm }A}\mathbf{d}_{A_1}\cdot\boldsymbol{\varepsilon}(\mathbf{n}_1,\lambda_1)\cdots\mathbf{d}_{A_N}\cdot\boldsymbol{\varepsilon}(\mathbf{n}_N,\lambda_N)$$
$$\times\exp(-ik\mathbf{n}_1\cdot\mathbf{r}_{A_1}-\cdots-ik\mathbf{n}_N\cdot\mathbf{r}_{A_N})\ , \tag{A5b}$$

where $\mathcal{N}$ is the normalization constant and k is the common length of all the wave vectors $\mathbf{k}_i$. Formula (A5b) differs from the formula (A5a) only in the ordering of the terms which are being summed.

In what follows we shall simplify the calculations by dropping all the products $\mathbf{d}\cdot\boldsymbol{\varepsilon}$ from the photon wave function. This procedure is justified when in the expression (2.1) for the angular correlation coefficient one performs an additional summation over the photon polarizations and also an additional averaging over all possible orientations of the atomic dipoles (or in the terminology of quantum mechanics the averaging over the magnetic quantum numbers of the initial and of the final atomic states). Upon substituting the expression (A5) into the formula (2.1) for the angular correlation coefficient and after performing the summation over the polarizations and the averaging over the atomic orientations, all dependence on the vectors $\mathbf{d}$ and $\boldsymbol{\varepsilon}$ disappears. As a result, the angular correlations become completely independent of the photon polarizations. The same result would also ensue if photons were treated from the beginning as spinless particles, as in the work of Duncan and Stehle.[7]

The photon wave function without any polarization dependence is a function of $\mathbf{n}_i$'s only,

$$\begin{aligned}\Phi(\mathbf{n}_1,\ldots,\mathbf{n}_N) &= \mathcal{N}\sum_{\text{perm } i}\exp(-ik\mathbf{n}_{i_1}\cdot\mathbf{r}_1-\cdots-ik\mathbf{n}_{i_N}\cdot\mathbf{r}_N)\\ &= \mathcal{N}\sum_{\text{perm } A}\exp(-ik\mathbf{n}_1\cdot\mathbf{r}_{A_1}-\cdots-ik\mathbf{n}_N\cdot\mathbf{r}_{A_N})\,.\end{aligned} \tag{A6}$$

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