Large-order perturbation expansion of three-dimensional Coulomb systems from four-dimensional anharmonic oscillators

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The three-dimensional Coulomb system can be mapped onto the four-dimensional harmonic oscillator. Additional perturbations of the type gr^p $[r=(x^2+y^2+z^2)^{1/2}]$ translate into anharmonic perturbations $\lambda(\vec{x}^2)^{p+1}$ $(\vec{x}^2=x_1^2+x_2^2+x_3^2+x_4^2)$ to the oscillator. We use this observation to relate the large-order behavior of perturbation series of the perturbed Coulomb systems to well-known large-order formulas for anharmonic oscillators. In this way, the leading behavior of Coulomb systems can be understood by simple scaling and symmetry arguments within the oscillator systems. Applications to more physical Stark- and Zeeman type perturbations are briefly discussed.

I. INTRODUCTION

The large-order behavior of perturbation series in Coulomb systems has attracted much interest in the literature. Especially well studied are the Stark and Zeeman effects.[1] To gain an analytic insight into the latter, Avron[2] considered a simplified model Hamiltonian in which the Zeeman potential in a uniform magnetic field, $V \propto (x^2+y^2)$, is replaced by the unphysical, but easier to handle, rotationally symmetric potential $V \propto r^2 = x^2+y^2+z^2$,

$$H = \frac{\mathbf{p}^2}{2} - \frac{1}{r} + gr^2. \tag{1}$$

Using WKB techniques he was able to derive the large-order behavior of the perturbation series for all energy levels. For the ground-state energy, $E(g) = -\frac{1}{2} + \sum_{k=1} E_k g^k$, he finds

$$E_k \underset{k\to\infty}{\sim} E_k^{\text{asy}} = -\frac{64}{\pi^3}\left(-\frac{8}{\pi^2}\right)^k \Gamma(2k+2). \tag{2}$$

The structure of this result, $E_k^{\text{asy}} \propto (-a)^k \Gamma(pk+\beta)$, is completely analogous to the corresponding Bender-Wu formulas[3-7] for anharmonic oscillators. In their calculation, the parameters p and β in the leading factorial growth factor can be deduced by very simple scaling and symmetry arguments. In Avron's work, on the other hand, it is not as simple to see how the leading behavior $\Gamma(2k+2) \approx 2k(2k)!$ emerges.

The purpose of this paper is to point out that it was not really necessary to redo the calculation by Avron from the beginning. Ever since Schrödinger's[8] early observation in 1941 and Kustaanheimo and Stiefel's more recent classical work[9] which was quantized via path integrals by Duru and Kleinert,[10] it has been known and exploited in many ways that Coulomb systems in *three* dimensions can be mapped onto oscillator systems in *four* dimensions. Applying the same method to the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2} - \frac{1}{r} + gr^p \tag{3}$$

it goes over into that of an anharmonic oscillator

$$h = \frac{\vec{p}^2}{2} + \frac{\omega^2}{2}\vec{x}^2 + \lambda(\vec{x}^2)^{p+1}. \tag{4}$$

(Bold letters are three-vectors; letters with arrows denote four-vectors.)

Our dimensionless variables in Eq. (3) measure lengths in units of the Bohr radius $a_0 = \hbar^2/me^2$ and energies in units of $me^4/\hbar^2$ where m and e are respectively (reduced) mass and charge of the electron. The dimensionless coupling g is related to the physical coupling F via $F = ge^2/a_0^{p+1}$.

By using known Bender-Wu formulas[3-5,7] for Eq. (4), we shall derive quite directly the large-order estimates for the Hamiltonian (3). As an additional result, we shall use a similar relation between Eq. (3) and Eq. (4) when *both* Hamiltonians are studied in *two* dimensions to give the large-order behavior also in that case. It should be mentioned that the same idea has appeared before in calculations of large-order estimates for the Stark Hamiltonian,[11] as will be briefly sketched in the discussion. The present work is a systematic model study intended to prepare the grounds for a subsequent analysis of the physically more relevant Zeeman effect, to be published separately.[12]

II. THE MAPPING

The power of mapping three-dimensional (D^C=3) Coulomb problems onto equivalent four-dimensional

($D = 4$) oscillator systems became most transparent with the path-integral quantization of the pure Coulomb system by Duru and Kleinert.[10] The basic result is that, using a path-dependent time reparametrization in the path integral together with a Kustaanheimo-Stiefel transformation from three- to four-dimensional space, the $D^C = 3$ Coulomb Green function can be expressed in terms of the $D = 4$ harmonic oscillator Green function. In the present context we do not need the whole machinery involved in this treatment. All we shall make use of here is the relation between the energy spectra of the two systems. Consider therefore the two Schrödinger equations

$$H\psi = E\psi, \quad h\phi = \epsilon\phi, \tag{5}$$

with the Hamiltonians in Eqs. (3) and (4). The result can be stated as follows. The two systems are equivalent if their parameters are related by

$$\omega^2 = -E/2, \tag{6}$$

$$\lambda = g/4^{p+1}. \tag{7}$$

The set of all bound-state wave functions of H is related to all oscillator wave functions of a fixed energy

$$\epsilon = \epsilon(\omega, \lambda) = 1. \tag{8}$$

These equations have to be supplemented by further relations between the dimensions D^C (Coulomb) and D (anharmonic oscillator) and the angular momentum quantum numbers ℓ^C and ℓ, to be compiled in Table I. Equation (8) will be central to our calculation. The Bender-Wu results for anharmonic oscillator systems give us the oscillator energies $\epsilon_n(\omega, \lambda)$, for various quantum numbers n, when expanded in powers of λ. The energies E_n of the Coulomb systems are found by solving the equation

$$\epsilon_n\left(\sqrt{-E_n/2}, \frac{g}{4^{p+1}}\right) = 1 \tag{9}$$

for E_n. It is then obvious that the Bender-Wu formulas for the large-order behavior of the perturbation expansions of ϵ_n in powers of $\lambda = g/4^{p+1}$ should determine also the behavior of the expansions of $E_n(g)$ in powers of g.

Let us begin by deriving the relations (6)–(8) between the two Schrödinger systems. We consider a general spherical symmetric perturbation $V(r)$ and the radial part of the D^C-dimensional Coulomb problem

$$\left[-\frac{1}{2}\left(\partial_r^2 + \frac{D^C - 1}{r}\partial_r - \frac{L^2}{r^2}\right) - \frac{Z}{r} + V(r)\right]\hat{\psi} = E\hat{\psi}, \tag{10}$$

where L is the angular momentum operator satisfying

$$L^2 Y_{\ell^C \mathbf{m}} = \ell^C(\ell^C + D^C - 2)Y_{\ell^C \mathbf{m}}, \tag{11}$$

with $Y_{\ell^C \mathbf{m}}$ being hyperspherical harmonics in D^C dimensions. The substitution

$$\hat{\psi} = r^{-(D^C-1)/2}\varphi \tag{12}$$

transforms this into

$$\left(-\frac{1}{2}\partial_r^2 + [\ell^C(\ell^C + D^C - 2) + (D^C - 1)(D^C - 3)/4]\frac{1}{2r^2} - \frac{Z}{r} + V(r)\right)\varphi = E\varphi. \tag{13}$$

A further change of variables,

$$r = x^2, \quad \varphi(r) = x^{1/2}\hat{\phi}(x), \tag{14}$$

leads to

$$\left(-\frac{1}{2}\partial_x^2 + [4\ell^C(\ell^C + D^C - 2) + (D^C - 1)(D^C - 3) + 3/4]\frac{1}{2x^2} + 4(-E)x^2 + 4x^2 V(x^2)\right)\hat{\phi} = 4Z\hat{\phi}. \tag{15}$$

Consider now the new centrifugal barrier in (15) for $D^C = 3$:

$$\frac{4\ell^C(\ell^C + 1) + 3/4}{2x^2}. \tag{16}$$

Identifying $\ell^C = \ell/2$ this becomes

$$\frac{\ell(\ell + 2) + 3/4}{2x^2}, \tag{17}$$

i.e., the centrifugal barrier in (13) with $D^C \longrightarrow D = 4$ and $\ell^C \longrightarrow \ell = 2\ell^C$. Hence, Eq. (15) can be interpreted as the radial part of the *four*-dimensional Schrödinger equation (after a further trivial rescaling)

$$\left[-\frac{1}{2}\vec{\partial}^2 + \frac{\omega^2}{2}\vec{\mathbf{x}}^2 + \left(\frac{\vec{\mathbf{x}}}{2}\right)^2 V\left(\left(\frac{\vec{\mathbf{x}}}{2}\right)^2\right)\right]\phi = \epsilon\phi, \tag{18}$$

with

$$\omega^2 = -E/2, \quad \epsilon = Z. \tag{19}$$

For $V(r) = gr^p$ we obtain the transformed potential

TABLE I. Relations between the spatial dimensions D^C, D, and the angular momentum quantum numbers ℓ^C, ℓ in the equivalence (5)–(8).

Coulomb (r^p)	Anharmonic oscillator [$(\vec{\mathbf{x}}^2)^{p+1}$]
$D^C = 2$	$D = 2$
$D^C = 3$	$D = 4$
⋮	⋮
D^C	$D = 2(D^C - 1)$
ℓ^C	$\ell = 2\ell^C$

$\tilde{V}(|\vec{x}|) = \lambda(\vec{x}^2)^{p+1}$ with

$$\lambda = \frac{g}{4^{p+1}}. \tag{20}$$

These are exactly the equations announced in Eqs. (6)–(8).

In the case $D^C = 2$, Eq. (16) becomes

$$\frac{4(\ell^{C\,2} - 1/4) + 3/4}{2x^2} = \frac{4\ell^{C\,2} - 1/4}{2x^2}. \tag{21}$$

With $\ell^C = \ell/2$ we therefore find again the *two*-dimensional centrifugal barrier and (18) is replaced by its *two*-dimensional version. The relation between the parameters in Eqs. (19) and (20) remains the same.

In the general case, the relation is $D = 2(D^C - 1)$ and $\ell = 2\ell^C$ (see Table I).

III. SOLUTION OF THE IMPLICIT RELATION BETWEEN E AND ϵ

As observed above the crucial point of the equivalence (5) is the implicit relation between E and ϵ in Eq. (8).

A. Exact energies

First of all, Eq. (8) may be used to generate (numerically) precise values of the energies $E(g)$ from known values of $\epsilon(\omega, \lambda)$.[13] From Eq. (4) it is easy to read off that $\epsilon(\omega, \lambda)$ scales as

$$\epsilon(\omega, \lambda) = \omega\hat{\epsilon}(\hat{\lambda}), \quad \hat{\lambda} = \lambda/\omega^{2+p}, \tag{22}$$

where $\hat{\epsilon}$ and $\hat{\lambda}$ are reduced energy and coupling. Using Eqs. (6)–(8), this implies a parametric representation for $E = E(g)$:

TABLE II. Perturbation coefficients for the anharmonic oscillator systems with $p = 1, \ldots, 5$ in $D = 2$ and $D = 4$ dimensions.

k	$D = 2$ ϵ_k	$D = 4$ ϵ_k
	$p = 1$	
1	2	6
2	−9	−39
3	89	540
4	$-\frac{5013}{4}$	$-\frac{41433}{4}$
5	$\frac{88251}{4}$	242208
	$p = 2$	
1	6	24
2	−249	−1752
3	28821	322176
4	$-\frac{23700261}{4}$	−96401928
5	$\frac{7459161843}{4}$	40978267968
	$p = 3$	
1	24	120
2	−11400	−110760
3	21808560	341494800
4	−102141774120	−2246022566760
5	959191470808464	27036786901714800
	$p = 4$	
1	120	720
2	−826440	−10052640
3	36358384560	694963877760
4	−5600598555541800	−145167171564124800
5	2262348623667536526480	73563718336492244820480
	$p = 5$	
1	720	5040
2	−89540640	−1286681760
3	118399164428160	2610699387166080
4	−781951013486796585600	−2307683664253646521920 0
5	17524674002198720891773370880	64651309130725069911851218 1760

$$E = -\frac{2}{\hat{\epsilon}(\hat{\lambda})^2}, \tag{23}$$

$$g = 4^{p+1}\frac{\hat{\lambda}}{\hat{\epsilon}(\hat{\lambda})^{2+p}}. \tag{24}$$

Certainly, an analytic solution of these equations for $E(g)$ is impossible. For our purpose this will not be necessary. We shall only have to relate the power series of $\hat{\epsilon}(\hat{\lambda})$ and $E(g)$ with each other.

B. Low-order perturbation theory

In this work, we shall concentrate on the perturbation theory for the ground-state energies, to be denoted shortly by E and ϵ, omitting the ground-state subscript $n = 0$. The starting point is the asymptotic expansion for anharmonic oscillators (4),

$$\epsilon = \omega \sum_{k=0} \epsilon_k \left(\frac{\lambda}{\omega^{2+p}}\right)^k, \tag{25}$$

with the coefficients ϵ_k given in Table II. The solution of Eq. (8), $\epsilon = 1$, yields then the expansion for the Coulomb problem

$$E = \sum_{k=0} E_k g^k. \tag{26}$$

In order to express the *low*-order coefficients E_k in terms of ϵ_k, it is convenient to rewrite Eq. (8) as

$$1 - \epsilon_0\omega = \omega\left[\epsilon_1\lambda\left(\frac{1}{\omega^{2+p}}\right) + \epsilon_2\lambda^2\left(\frac{1}{\omega^{2+p}}\right)^2 + \cdots\right]. \tag{27}$$

Inserting the ansatz

$$\omega = \frac{1}{\epsilon_0}(1 + a_1\lambda + a_2\lambda^2 + \cdots), \tag{28}$$

it is easy to see that the coefficients in the resulting expansion of the right-hand side, $b_1\lambda + b_2\lambda^2 + b_3\lambda^3 + \cdots$, depend on the unknown a_k as follows:

$$b_k = b_k(a_1, \ldots, a_{k-1}). \tag{29}$$

It is then straightforward to solve Eq. (27) recursively for $a_k, k = 1, 2, 3, \ldots$, starting with $k = 1$. This gives

$$\begin{aligned} a_1 &= -b_1 = -\epsilon_0^{p+2}(\epsilon_1/\epsilon_0), \\ a_2 = -b_2(a_1) &= -[\epsilon_0^{3+2p}\epsilon_2 - (1+p)a_1\epsilon_0^{1+p}\epsilon_1] \\ &= -\epsilon_0^{2(p+2)}[(\epsilon_2/\epsilon_0) + (1+p)(\epsilon_1/\epsilon_0)^2], \end{aligned} \tag{30}$$

$$\begin{aligned} a_3 &= -b_3(a_1, a_2) \\ &= -\epsilon_0^{3(p+2)}[(\epsilon_3/\epsilon_0) + (3p+4)(\epsilon_1/\epsilon_0)(\epsilon_2/\epsilon_0) \\ &\quad + (\tfrac{3}{2}p^2 + \tfrac{7}{2}p + 2)(\epsilon_1/\epsilon_0)^3], \end{aligned}$$

and so on. Obviously the a_k depend on all coefficients ϵ_ℓ with $\ell \le k$. The expansion of E follows now simply from Eq. (6),

$$\begin{aligned} E = -2\omega^2 &= -\frac{2}{\epsilon_0^2}[1 + 2a_1\lambda + (2a_2 + a_1^2)\lambda^2 + \cdots] \\ &= E_0 + E_1 g + E_2 g^2 + \cdots \end{aligned} \tag{31}$$

with $\lambda = g/4^{p+1}$. As a trivial check we note that, by inserting $\epsilon_0 = D/2$, the lowest expansion term in Eq. (31),

$$E_0 = -2/\epsilon_0^2 = -2/(D^C - 1)^2, \tag{32}$$

reproduces the correct ground-state energy of the unperturbed Coulomb potential, as it should [$E_0 = -2$ for $D^C = 2(= D)$; $E_0 = -\frac{1}{2}$ for $D^C = 3(D = 4)$]. For the next coefficients, the elementary but tedious algebraic manipulations yield, for general p,

$$\begin{aligned} E_1 &= 4\epsilon_0^{p-1}\epsilon_1/4^{p+1}, \\ E_2 &= \epsilon_0^{2p}[2(2p+1)\epsilon_1^2 + 4\epsilon_2\epsilon_0]/4^{2(p+1)}, \\ E_3 &= \epsilon_0^{3p}[2(3p^2+5p+2)\epsilon_1^3\epsilon_0 + 12(p+1)\epsilon_1\epsilon_2\epsilon_0^2 + 4\epsilon_3\epsilon_0^3]/4^{3(p+1)}, \\ E_4 &= \epsilon_0^{4p}[(\tfrac{32}{3}p^3 + 32p^2 + \tfrac{94}{3}p + 10)\epsilon_1^4\epsilon_0^2 + 8(4p^2+9p+5)\epsilon_1^2\epsilon_2\epsilon_0^3 + 2(4p+5)(2\epsilon_1\epsilon_3 + \epsilon_2^2)\epsilon_0^4 + 4\epsilon_4\epsilon_0^5]/4^{4(p+1)}, \\ E_5 &= \epsilon_0^{5p}[(\tfrac{125}{6}p^4 + \tfrac{275}{3}p^3 + \tfrac{895}{6}p^2 + \tfrac{319}{3}p + 28)\epsilon_1^5\epsilon_0^3 + (\tfrac{250}{3}p^3 + 300p^2 + \tfrac{1070}{3}p + 140)\epsilon_1^3\epsilon_2\epsilon_0^4 \\ &\quad + 2(25p^2 + 65p + 42)(\epsilon_1^2\epsilon_3 + \epsilon_1\epsilon_2^2)\epsilon_0^5 + 4(5p+7)(\epsilon_1\epsilon_4 + \epsilon_2\epsilon_3)\epsilon_0^6 + 4\epsilon_5\epsilon_0^7]/4^{5(p+1)}. \end{aligned} \tag{33}$$

(It is advisable to use an algebraic computer program like, e.g., REDUCE or RIEMANN.) It is easy to show that for general anharmonic oscillators $\epsilon_1 = \Gamma(p+1+D/2)/\Gamma(D/2) = (D/2)(D/2+1)\cdots(D/2+p)$, i.e., $\epsilon_1 = (p+1)!$ for $D = 2$ and $\epsilon_1 = (p+2)!$ for $D = 4$. Inserting this in Eq. (33) gives

$$E_1 = \begin{cases} (p+1)!/4^p & (D^C = 2) \\ (p+2)!/2^{p+1} & (D^C = 3), \end{cases} \tag{34}$$

in agreement with a direct first-order perturbation cal-

culation for the Coulomb systems.

In Table II we have compiled further low-order coefficients ϵ_k for anharmonic oscillators $[(\vec{x}^2)^{p+1}$ anharmonicities with $p = 1, \ldots, 5$ and $D = 2, 4]$ up to $k = 5$. Inserting these values in Eq. (33), we find perfect agreement with the values of E_k calculated directly from the Coulomb Hamiltonian (3) by means of the recursion relations derived in the Appendix. Some results of this computation are shown in Table III for the most interesting cases $p = 1, 2$.

C. Large-order perturbation theory

We now turn to the calculation of the asymptotic behavior of E_k as $k \to \infty$. To perform the inversion of Eq. (8), we shall make use of the corresponding known large-order formulas for anharmonic oscillators.

Let us briefly recall their derivation. For our purposes, Langer's[14] framework is most convenient, which starts with the calculation of the imaginary part of the energy for small *negative* coupling (i.e., the decay rate of

TABLE III. Perturbation coefficients for the Coulomb systems with $p = 1, 2$ in $D^C = 2$ and $D^C = 3$ dimensions.

$p = 1, \ D^C = 2$
E(1)=1/2
E(2)=−3/64
E(3)=21/1024
E(4)=−987/65536
E(5)=15555/1048576
E(6)=−600243/33554432
E(7)=13526613/536870912
E(8)=−2770115403/68719476736
E(9)=79111307439/1099511627776
E(10)=−4978189192881/35184372088832
$p = 1, \ D^C = 3$
E(1)=3/2
E(2)=−3/2
E(3)=27/4
E(4)=−795/16
E(5)=3843/8
E(6)=−5583
E(7)=9543339/128
E(8)=−1141062999/1024
E(9)=18769071555/1024
E(10)=−1343699301873/4096
$p = 2, \ D^C = 2$
E(1)=3/8
E(2)=−159/1024
E(3)=17967/65536
E(4)=−15522195/16777216
E(5)=5189052801/1073741824
E(6)=−4896676641339/137438953472
E(7)=3094900497137871/879609302208
E(8)=−20233178231139761499/4503599627370496
E(9)=208085588278258599984 45/288230376151711744
E(10)=−526934854653695435660650 89/36893488147419103232
$p = 2, \ D^C = 3$
E(1)=3
E(2)=−129/4
E(3)=5451/4
E(4)=−6609975/64
E(5)=734589303/64
E(6)=−880224055389/512
E(7)=169960252839003/512
E(8)=−1316458151745974019/16384
E(9)=391398896113218866535/16384
E(10)=−1124700335952727250306379/131072

the metastable system). For oscillators in D dimensions with anharmonicity $\lambda(\vec{x}^2)^{p+1}$ the result is (using WKB techniques[4,5] or instanton methods[7])

$$\mathrm{Im}\,\epsilon(\omega,\lambda) \underset{\lambda\to 0^-}{\sim} \omega\gamma\left(\frac{1}{(a|\lambda|/\omega^{2+p})^{1/p}}\right)^{D/2} \times e^{-1/(a|\lambda|/\omega^{2+p})^{1/p}} \times[1+c_1(a|\lambda|/\omega^{2+p})^{1/p}+\cdots], \tag{35}$$

where

$$a=\frac{1}{2}\left(\frac{\Gamma(2+2/p)}{\Gamma(1+1/p)^2}\right)^p \tag{36}$$

and

$$\gamma=\frac{(2a)^{D/2p}}{\Gamma(D/2)}=\frac{1}{\Gamma(D/2)}\left(\frac{\Gamma(2+2/p)}{\Gamma(1+1/p)^2}\right)^{D/2}. \tag{37}$$

For $p=1$ (i.e., $|\vec{x}|^4$ perturbation), also c_1 is known analytically:[15]

$$c_1=-\frac{20+54D+21D^2}{72}\quad(p=1) \tag{38}$$
$$=-(\tfrac{95}{72},\tfrac{53}{18},\tfrac{371}{72},\tfrac{143}{18},\ldots)\quad(D=1,2,3,4,\ldots).$$

The connection with the perturbation coefficients ϵ_k is given by the dispersion relation

$$\omega\epsilon_k/(\omega^{2+p})^k=\frac{1}{\pi}\int_{-\infty}^{0}d\lambda\frac{\mathrm{Im}\,\epsilon(\omega,\lambda+i0)}{\lambda^{k+1}}, \tag{39}$$

where the integral runs on top of a cut along the negative λ axis in the complex coupling-constant plane, signalizing the metastability for $\lambda<0$. Inserting the small-λ expansion (35), one finds the following large-k behavior of the perturbation coefficients [recall Eq. (25)]

$$\epsilon_k \underset{k\to\infty}{\sim} -\frac{p}{\pi}\gamma(-a)^k\Gamma\left(pk+\frac{D}{2}\right)\left(1+\frac{c_1}{pk}+\cdots\right). \tag{40}$$

We now derive the corresponding formula for E_k. In principle we could start with (40) and go through the steps (27)–(31). We found it, however, more convenient to work with the imaginary part (35).[16] Adding the real part of ϵ (which, perturbatively, does not take into account the metastability), we have then to solve (with λ small and negative)

$$1=\epsilon(\omega,\lambda)=\omega\sum_{k=0}\epsilon_k\left(\frac{\lambda}{\omega^{2+p}}\right)^k+i\,\mathrm{Im}\,\epsilon(\omega,\lambda). \tag{41}$$

The strategy is to determine (recall $g=\lambda/4^{p+1}$)

$$\omega(g)\equiv x(g)+iy(g) \tag{42}$$

and from this $E=-2\omega^2=-2(x^2-y^2)-4ixy$. We note that obviously $x\neq 0$, so that it is immediately clear that y is of the same smallness as $\mathrm{Im}\,\epsilon$, i.e., it is exponentially small as $g\propto\lambda\to 0$. We may therefore keep only terms linear in y and find

$$1=x\left(\epsilon_0+\epsilon_1\frac{\lambda}{x^{2+p}}+\epsilon_2\frac{\lambda^2}{x^{2(2+p)}}+\cdots\right) +iy\left(\epsilon_0-\frac{(1+p)\epsilon_1\lambda}{x^{2+p}}-\frac{(3+2p)\epsilon_2\lambda^2}{x^{4+2p}}-\cdots\right) +ix\gamma\left(\frac{x^{1+2/p}}{(a|\lambda|)^{1/p}}\right)^{D/2}\exp\left(-\frac{x^{1+2/p}}{(a|\lambda|)^{1/p}}\right) \times[1+c_1(a|\lambda|/x^{2+p})^{1/p}+\cdots]. \tag{43}$$

The real part of this equation agrees with Eq. (27) and has been solved before,

$$x=\frac{1}{\epsilon_0}(1+a_1\lambda+a_2\lambda^2+\cdots), \tag{44}$$

with $a_1,a_2,\ldots$, given in Eq. (30). The solution of the imaginary part is trivial

$$y=-\frac{x}{\epsilon_0}\gamma\left(\frac{x^{1+2/p}}{(a|\lambda|)^{1/p}}\right)^{D/2}\exp\left(-\frac{x^{1+2/p}}{(a|\lambda|)^{1/p}}\right) \times\left(1+c_1(a|\lambda|/x^{2+p})^{1/p}+\frac{(1+p)(\epsilon_1/\epsilon_0)\lambda}{x^{2+p}}+\cdots\right). \tag{45}$$

Going to the imaginary part of E, and inserting x from Eq. (44), we obtain

$$\mathrm{Im}\,E=-4xy=\frac{4}{\epsilon_0^3}\gamma\left(\frac{1}{(\epsilon_0^{2+p}a|\lambda|)^{1/p}}\right)^{D/2} \times\exp\left(-[1/(\epsilon_0^{2+p}a|\lambda|)^{1/p}]\{1+(1+2/p)a_1\lambda+(1+2/p)(a_1^2/p+a_2)\lambda^2+\cdots\}\right) \times\left\{1+c_1(\epsilon_0^{2+p}a|\lambda|)^{1/p}+(1+p)(\epsilon_1/\epsilon_0)\epsilon_0^{2+p}\lambda+\left[2+\frac{D}{2}\left(1+\frac{2}{p}\right)\right]a_1\lambda+\cdots\right\}. \tag{46}$$

For $p\geq 2$, all terms in curly brackets are subleading corrections which we shall neglect for the moment. Only for $p=1$, the correction $\propto\lambda$ in the exponential contributes to the prefactor:

$$e^{-(1/\epsilon_0^3a|\lambda|)[1+3a_1\lambda+O(\lambda^2)]} = e^{3a_1/\epsilon_0^3a}e^{-1/(\epsilon_0^3a|\lambda|)}[1+O(\lambda)]. \tag{47}$$

Inserting $a_1=-\epsilon_0^3(\epsilon_1/\epsilon_0)$ from Eq. (30) and $a=3$ for

$p = 1$ from Eq. (36), the additional prefactor becomes $\exp(-\epsilon_1/\epsilon_0)$.

Thus, recalling $\lambda = g/4^{p+1}$, we find for the leading behavior of the imaginary part, for general p,

$$\mathrm{Im}\,E(g) \underset{g\to 0^-}{\sim} \gamma^* \left(\frac{1}{(a^*|g|)^{1/p}}\right)^{D/2} e^{-1/(a^*|g|)^{1/p}} \times[1 + c^*_1(a^*|g|)^{1/p} + \cdots], \tag{48}$$

and, via the dispersion relation (39), for the large-order behavior of the perturbation coefficients

$$E_k \underset{k\to\infty}{\sim} -\frac{p}{\pi}\gamma^*(-a^*)^k \Gamma(pk + D/2)\left(1 + \frac{c^*_1}{pk} + \cdots\right), \tag{49}$$

with the growth parameters

$$a^* = \frac{1}{2^p}\left(\frac{\epsilon_0}{2}\right)^{2+p} a \tag{50}$$

and

$$\gamma^* = \begin{cases} (4\gamma/\epsilon_0^3)e^{-\epsilon_1/\epsilon_0}, & p = 1 \\ (4\gamma/\epsilon_0^3), & p > 1. \end{cases} \tag{51}$$

Recall that $\epsilon_0 = D/2 = D^C - 1$ and that a and γ denote the growth parameters for anharmonic oscillators given before in Eqs. (36) and (37). Their numerical values and those of a^* and γ^* can be read off conveniently from Table IV. Furthermore, we have $\epsilon_1(p = 1) = D(D+2)/4$ so that, for $p = 1$, $\epsilon_1/\epsilon_0 = (D+2)/2 = D^C$.

For $p = 1$, also the first-order correction c^*_1 can be calculated analytically. Expanding the exponential in Eq. (46), we find a total correction

$$1 + c_1(\epsilon_0^3 a|\lambda|) + 2(\epsilon_1/\epsilon_0)\epsilon_0^3\lambda + (2 + \tfrac{3}{2}D)a_1\lambda - \frac{3(a_1^2 + a_2)}{\epsilon_0^3 a}|\lambda| = 1 + c^*_1(\epsilon_0^3 a|g|/4^2), \tag{52}$$

with (using $a = 3$ for $p = 1$)

$$\begin{aligned} c^*_1 &= c_1 - \frac{2(\epsilon_1/\epsilon_0)}{a} - \frac{(2+\frac{3}{2}D)a_1}{\epsilon_0^3 a} - \frac{3(a_1^2+a_2)}{\epsilon_0^6 a^2} \\ &= c_1 - \tfrac{2}{3}(\epsilon_1/\epsilon_0) + \frac{(2+\frac{3}{2}D)(\epsilon_1/\epsilon_0)}{3} \\ &\quad - \frac{(\epsilon_1/\epsilon_0)^2 - (\epsilon_2/\epsilon_0) - 2(\epsilon_1/\epsilon_0)^2}{3} \\ &= c_1 + \frac{D}{2}(\epsilon_1/\epsilon_0) + \frac{1}{3}\left[\left(\frac{\epsilon_1}{\epsilon_0}\right)^2 + \frac{\epsilon_2}{\epsilon_0}\right]. \end{aligned} \tag{53}$$

Inserting $\epsilon_1/\epsilon_0 = (D+2)/2$, $\epsilon_2/\epsilon_0 = -(D+2)(2D+5)/4$ (compare Table II) and c_1 from Eq. (38), we obtain

$$\begin{aligned} c^*_1 &= c_1 + \frac{2D^2 + D - 6}{12} \\ &= -\frac{9D^{C\,2} + 6D^C - 1}{18} \\ &= \begin{cases} -\frac{53}{18} + \frac{1}{3} = -\frac{47}{18} = -2.6111\ldots & (D^C = 2) \\ -\frac{143}{18} + \frac{5}{2} = -\frac{49}{9} = -5.4444\ldots & (D^C = 3). \end{cases} \end{aligned} \tag{54}$$

For general $p \geq 2$, analyzing the different powers of λ in (46), we find that

$$c^*_n = c_n \qquad \text{for } n < p - 1 \tag{55}$$

and

$$c^*_{p-1} = c_{p-1} - \frac{1 + 2/p}{a}(\epsilon_1/\epsilon_0), \tag{56}$$

with the correction coming from the expansion of the exponential in (46). All higher-order coefficients c^*_n with $n \geq p$ are given by more complicated expressions similar to (53) for $p = 1$. In particular, the first correction term is

$$c^*_1 = c_1 - \frac{2}{a}(\epsilon_1/\epsilon_0) = c_1 - \frac{\pi^2}{16}(\epsilon_1/\epsilon_0) \quad (p = 2) \tag{57}$$

for $p = 2$ and $c^*_1 = c_1$ for all $p \geq 3$.

TABLE IV. Using G, it is straightforward to obtain the values of $a = \frac{1}{2}G^p$, $\gamma = G^{D/2}/\Gamma(D/2)$ and to compute the parameters a^*, γ^* in Eq. (49). D^C and $D = 2(D^C - 1)$ are the dimensions of the Coulomb and equivalent oscillator systems, respectively.

p	$G \equiv \Gamma(2+\frac{2}{p})/\Gamma(1+\frac{1}{p})^2$	
1	6	
2	$8/\pi$	$\approx 2.546\,479\,089$
3	$10\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})^2$	$\approx 1.886\,822\,268$
4	$12\sqrt{\pi}/\Gamma(\frac{1}{4})^2$	$\approx 1.618\,057\,804$
5	$14\Gamma(\frac{2}{5})/\Gamma(\frac{1}{5})^2$	$\approx 1.473\,451\,345$

$a^* = \frac{1}{2}(\frac{D}{4})^2(\frac{D}{8}G)^p = \begin{cases} \frac{1}{8}(\frac{G}{4})^p & \text{for } D^C = 2 \\ \frac{1}{2}(\frac{G}{2})^p & \text{for } D^C = 3 \end{cases}$

$\gamma^* = 4(\frac{2}{D})^3 G^{D/2}/\Gamma(D/2) = \begin{cases} 4G & \text{for } D^C = 2 \\ \frac{1}{2}G^2 & \text{for } D^C = 3 \end{cases}$ if $p \neq 1$. For $p = 1$, there is an additional factor $\exp(-D^C)$.

Let us conclude this section by summarizing the main results. Explicitly, we have for $p = 1$,

$$E_k \underset{k\to\infty}{\sim} -\frac{24}{\pi}e^{-2}\left(-\frac{3}{16}\right)^k \Gamma(k+1)\left(1-\frac{47}{18}\frac{1}{k}+\ldots\right) \quad (D^C = 2), \quad (58)$$

$$E_k \underset{k\to\infty}{\sim} -\frac{18}{\pi}e^{-3}\left(-\frac{3}{2}\right)^k \Gamma(k+2)\left(1-\frac{49}{9}\frac{1}{k}+\ldots\right) \quad (D^C = 3), \quad (59)$$

and for $p = 2$

$$E_k \underset{k\to\infty}{\sim} -\frac{64}{\pi^2}\left(-\frac{1}{2\pi^2}\right)^k \Gamma(2k+1) \quad (D^C = 2), \quad (60)$$

$$E_k \underset{k\to\infty}{\sim} -\frac{64}{\pi^3}\left(-\frac{8}{\pi^2}\right)^k \Gamma(2k+2) \quad (D^C = 3). \quad (61)$$

Equation (61) agrees with Avron's[2] result (2) obtained from a direct WKB analysis of the decay rate of the Coulomb system (if we correct a minor error in his formula for E_k—Avron's decay rate is correct).

IV. COMPARISON WITH NUMERICAL RESULTS

Let us now test the accuracy of the leading large-order formulas (49)–(51) for the Coulomb systems, $E_k^{\text{asy}} \equiv -(p/\pi)\gamma^*(-a^*)^k\Gamma(pk + D/2)$, by comparing it with the exact coefficients E_k in high order (up to $k = 500$ for $p = 1$ and $k = 200$ for $p = 2,3,4,5$). These can be generated quite easily from the recursion relation derived in the Appendix. Some care is necessary with floating underflow problems which are easily overlooked. To avoid any numerical tricks (like appropriate rescalings) we have changed from a VAX 11/780 computer (with a floating-point range 0.29×10^{-38}–$0.17 \times 10^{+39}$) to a CRAY-XMP 2/4 computer (with a range $0.13 \times 10^{-2449} - 0.78 \times 10^{+2450}$) where no such problems occur. Since single (64-bit) and double (128-bit) precision calculations agree at least up to 11 digits (implying that around 3–4 digits of the initial accuracy are lost during the recursions), also roundoff errors are under control. Some numerical values are given in Table V for $p = 1$. As a first graphical test we have plotted in Fig. 1 for $p = 1,\ldots,5$ the ratios

$$R_0 \equiv \frac{E_k}{E_k^{\text{asy}}} = 1 + \frac{\gamma^*_1}{k} + \cdots \quad (62)$$

versus k in (a) $D^C = 2$ and (b) $D^C = 3$ dimensions. (For clarity, we have erased in these figures the quite erratic behavior for $k \lesssim 4$.) We see that for $p \geq 3$ the asymptotic region is reached very rapidly. The magnitude of the correction terms $\gamma^*_1/k \equiv c^*_1/pk$ can be read off from Fig. 2, which shows the more sensitive ratios

$$R_1 \equiv (R_0 - 1)k = \gamma^*_1 + O(1/k) \quad (63)$$

versus $1/k$. The finite limits for $1/k \to 0$ verify the correctness of the asymptotic formulas (49)–(51) and provide a rough graphical estimate of γ^*_1. More precise values can be obtained by means of numerical extrapolation procedures. We have used a modified Neville table[15] based on the recursions

TABLE V. Comparison of exact perturbation coefficients E_k with the leading large-order estimate (49) for the Coulomb potential plus gr perturbation ($p = 1$) in $D^C = 2,3$ dimensions ($R_0 \equiv E_k/E_k^{\text{asy}}$, $R_1 \equiv (R_0 - 1)k = \gamma^*_1 + O(1/k)$, $R_1^{(1)} \equiv$ first modified Neville extrapolant [see Eq. (64)]).

k	$-E_k$		$-E_k^{\text{asy}}$		R_0	R_1	$R_1^{(1)}$
			$p = 1,\ D^C = 2$				
10	1.414 886 467	−1	2.014 875 234	−1	0.7022	−2.9778	−3.046 384
20	6.255 848 488	3	7.254 750 204	3	0.8623	−2.7538	−2.613 084
30	3.867 004 057	10	4.247 864 373	10	0.9103	−2.6898	−2.613 805
40	6.549 555 458	18	7.017 272 730	18	0.9333	−2.6661	−2.611 782
50	1.330 237 196	28	1.404 788 298	28	0.9469	−2.6535	−2.611 383
100	1.875 100 243	85	1.925 764 898	85	0.9737	−2.6309	−2.611 135
150	5.174 782 254	153	5.266 918 834	153	0.9825	−2.6240	−2.611 118
200	3.205 435 817	229	3.247 995 793	229	0.9869	−2.6207	−2.611 114
			$p = 1,\ D^C = 3$				
10	3.280 515 874	8	6.566 100 040	8	0.4996	−5.0039	−5.428 698
20	3.577 983 429	22	4.846 277 511	22	0.7382	−5.2341	−5.221 952
30	3.707 550 668	38	4.497 785 366	38	0.8243	−5.2708	−5.560 945
40	9.151 876 481	55	1.055 160 376	56	0.8673	−5.3062	−5.457 668
50	2.520 500 733	74	2.821 286 483	74	0.8934	−5.3306	−5.448 892
100	1.034 311 215	177	1.093 176 081	177	0.9462	−5.3848	−5.444 759
150	6.149 830 450	290	6.379 672 771	290	0.9640	−5.4041	−5.444 525
200	7.272 058 065	411	7.474 388 976	411	0.9729	−5.4140	−5.444 476

TABLE VI. Higher-order corrections $(1+\gamma^*_1/k+\gamma^*_2/k^2+\gamma^*_3/k^3+\cdots)$ to the leading large-order formula for $p=1$ and $D^C=2,3$ (MNE stands for modified Neville extrapolation, $\delta_n \equiv |[\gamma^*_n-\gamma^*_n(\mathrm{MNE})]/\gamma^*_n|$). In the entries for γ^*_n (MNE), the numbers in parentheses indicate the roundoff errors in the last digit. The numbers in square brackets are estimates of the error propagation.

n	b_n	c_n	δc_n	c^*_n	γ^*_n	γ^*_n (MNE)	δ_n
				$p=1,\ D^C=2$			
1	$\frac{53}{6}$	$-\frac{53}{18}$	$\frac{1}{3}$	$-\frac{47}{18}$	$-\frac{47}{18}$	−2.611 111 111 117(8)	2.3×10^{-12}
2	$\frac{227}{4}$	$-\frac{1277}{648}$	$\frac{1}{9}$	$-\frac{1205}{648}$	$-\frac{1205}{648}$	−1.859 567 897 5[100](6)	2.0×10^{-9}
3	$\frac{46513}{72}$	$-\frac{336437}{34992}$	$\frac{1163}{1944}$	$-\frac{315503}{34992}$	$-\frac{380573}{34992}$	−10.876 001 5[260](1)	1.2×10^{-7}
				$p=1,\ D^C=3$			
1	$\frac{143}{6}$	$-\frac{143}{18}$	$\frac{5}{2}$	$-\frac{49}{9}$	$-\frac{49}{9}$	−5.444 444 444 46(4)	3.7×10^{-12}
2	224	$\frac{4321}{648}$	$-\frac{425}{72}$	$\frac{62}{81}$	$\frac{503}{81}$	6.209 876 550 0[350](8)	1.1×10^{-9}
3	$\frac{259021}{72}$	$-\frac{667673}{34992}$	$\frac{3395}{1296}$	$-\frac{72001}{4374}$	$-\frac{99163}{4374}$	−22.671 013 0[800](2)	1.1×10^{-7}

TABLE VII. Modified Neville extrapolations of the higher-order corrections $(1+\gamma^*_1/k+\gamma^*_2/k^2+\gamma^*_3/k^3+\cdots)$ to the leading large-order formulas for the Coulomb and corresponding anharmonic oscillator systems with $p=2,3,4,5$.

p	n	γ^*_n (Coulomb)	γ_n (anharmonic oscillator)
		$D^C=2$	$D=2$
2	1	−1.850 550 825 2	10^{-13}
	1 [Eq. (72)]	−1.850 550 825 204 3	
	1	−0.201 533 262 64	−0.201 533 262 63
3	2	−1.160 032 763	0.163 255 602
	2 [Eq. (72)]	−1.160 032 776	
	1	−0.229 074 464 48	−0.229 074 464 4
4	2	0.026 237 70	0.026 237 6
	3	−0.721 322	0.099 357 9
	3 [Eq. (72)]	−0.721 270	
	1	−0.230 615 004 24	−0.230 615 004 24
	2	0.004 163 240	0.004 163 244
5	3	0.017 132	0.017 130
	4	−0.407 456	0.057 253
	4 [Eq. (72)]	−0.407 189	
		$D^C=3$	$D=4$
2	1	−3.701 101 650 41	10^{-12}
	1 [Eq. (72)]	−3.701 101 650 409	
	1	−0.443 373 177 85	−0.443 373 177 9
3	2	−2.561 822 689	0.746 398 31
	2 [Eq. (72)]	−2.561 822 63	
	1	−0.458 148 928 75	−0.458 148 928 7
4	2	0.219 487 53	0.219 487 5
	3	−2.091 441 3	0.370 45
	3 [Eq. (72)]	−2.091 43	
	1	−0.428 285 007 89	−0.428 285 007 89
	2	0.110 085 859	0.110 085 862
5	3	0.054 000 9	0.054 000 2
	4	−1.356 8	0.268 8
	4 [Eq. (72)]	−1.356 7	

$$(R_1^{(n+1)})_k = (R_1^{(n)})_{k+1} - \frac{2n+2}{2n+1}\frac{[(R_1^{(n)})_{k+2} - (R_1^{(n)})_{k+1}][(R_1^{(n)})_{k+1} - (R_1^{(n)})_k]}{(R_1^{(n)})_{k+2} - 2(R_1^{(n)})_{k+1} + (R_1^{(n)})_k}, \quad (R_1^{(0)})_k = (R_1)_k, \tag{64}$$

which eliminate two successive powers of $1/k$ at each step, i.e., $(R_1^{(n)})_k = \gamma^*_1 + O(1/k^{2n+1})$. Using our double-precision (128-bit) data for E_k up to $k = 200$ we have determined the values compiled in Tables VI and VII.

For $p = 1$, the extrapolated values for γ^*_1 in Table VI agree very well with the analytical result (54). The convergence properties of the first extrapolants $(R_1^{(1)})_k$ can be inspected in the last column of Table V. To illustrate graphically how much the leading asymptotic estimate is improved by including the γ^*_1 correction, we have plotted in Fig. 3 the ratio

$$\tilde{R}_0 = \frac{E_k}{E_k^{\text{asy}}[1 + \gamma^*_1/k]} = 1 + O(1/k^2). \tag{65}$$

For $p = 2$, our numerical results in Table VII are extremely well reproduced by

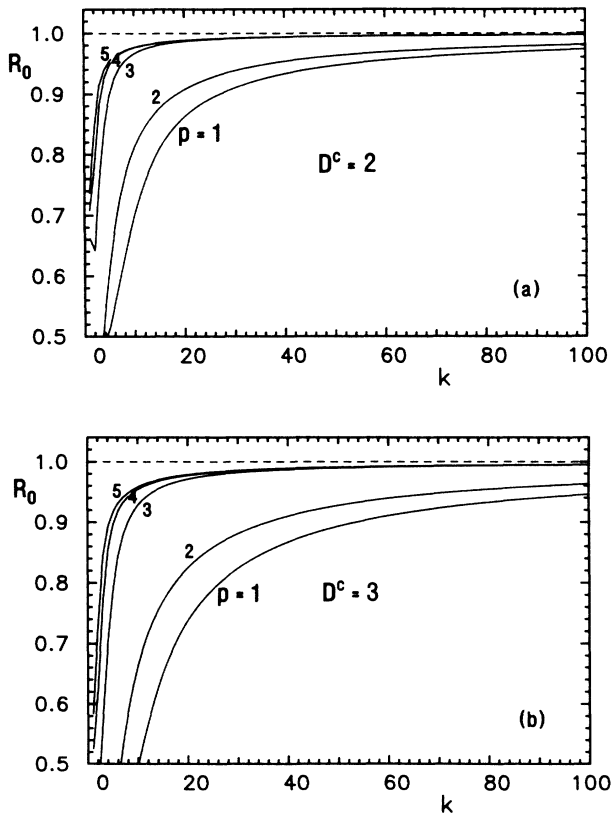

FIG. 1. The ratio $R_0 \equiv E_k/E_k^{\text{asy}} = 1 + \gamma^*_1/k + \cdots$ for Coulomb systems with perturbation $\propto r^p, p = 1, 2, \ldots, 5$. E_k^{asy} is the leading asymptotic result (49), and E_k are the exact coefficients calculated by means of the recursion in the Appendix. For clarity, we have omitted the quite erratic behavior for $k \lesssim 4$.

$$\gamma^*_1 = c^*_1/2 = \begin{cases} -3\pi^2/16 & (D^C = 2) \\ -3\pi^2/8 & (D^C = 3). \end{cases} \tag{66}$$

These predictions are implied by Eq. (57) under the assumption of a vanishing first-order correction, $\gamma_1 = c_1/2 = 0$, for the $|x|^6$ anharmonic oscillator in $D = 2$ and $D = 4$ dimensions. Recently this latter property has indeed been observed for $D = 1, \ldots, 4$ with an accuracy better than 10^{-12}.[17]

Let us now turn to higher-order corrections which, for $\text{Im}\,E(g)$ in (48), generically have the form

$$1 + c^*_1(a^*|g|)^{1/p} + c^*_2(a^*|g|)^{2/p} + \cdots. \tag{67}$$

Via the dispersion relation (39) this yields a correction

$$R_0 = 1 + c^*_1/(pk + D/2 - 1) + c^*_2/[(pk + D/2 - 1)(pk + D/2 - 2)] + \cdots \tag{68}$$

to the leading large-order behavior (49) which can be reexpanded as

$$R_0 = 1 + \gamma^*_1/k + \gamma^*_2/k^2 + \cdots, \tag{69}$$

with

$$\begin{aligned} \gamma^*_1 &= c^*_1/p, \\ \gamma^*_2 &= [c^*_2 - (D/2 - 1)c^*_1]/p^2, \\ \gamma^*_3 &= [c^*_3 - (D - 3)c^*_2 + (D/2 - 1)^2 c^*_1]/p^3, \\ &\vdots \end{aligned} \tag{70}$$

In terms of γ^*_n, the relations (55) and (56) between Coulomb and oscillator systems become

$$\gamma^*_n = \gamma_n \quad \text{for } n < p - 1, \tag{71}$$

$$\gamma^*_{p-1} = \gamma_{p-1} - \frac{2+p}{ap^p}(\epsilon_1/\epsilon_0), \tag{72}$$

with more complicated formulas for $n \geq p$.

In the case $p = 1$, the higher-order corrections γ_n for the $|x|^4$ anharmonic oscillator have been determined *numerically* for all D up to $n = 6$ with very high precision.[15] More precisely, although the analysis in Ref. 15 is based on series extrapolations, the correction is given for the imaginary part of the ground-state energy ϵ in the form

$$1 + \sum_{n=1} c_n(a|\lambda|)^n = \exp\left(-\sum_{n=1} b_n|\lambda|^n\right), \tag{73}$$

with

$$b_n \equiv \frac{1}{3n2^n}\sum_{l=0}^{n+1} b_{nl}\left(\frac{D}{2}\right)^l \tag{74}$$

and integral numbers b_{nl}. For $l = 0$ and $l = n, n+1$ these integers are known *analytically* from the $D \to 0$ and $D \to \infty$ limit, respectively. All other b_{nl} up to $n = 6$ are determined numerically. We have checked that, for $D = 2$, the resulting rational numbers $c_n, n = 1, \ldots, 6$ agree precisely with the 15 digit numbers computed in Ref. 18 along completely different lines. [If we correct a printing error in Table II of Ref. 15: $b_{60} = 30220880 \to 30220800$ (compare Table I in Ref. 15). The constants $b^{(n)}$ in Ref. 18 and our c_n are related by $b^{(n)} = 6^n c_n$.] Thus, besides $c_1(D)$ given in (38) also $c_2(D), \ldots, c_6(D)$ are presumably exact. In fact, $c_2(D = 2)$ agrees with the analytical value which was derived likewise in Ref. 18 (see also Ref. 19). If this special case is combined with the results coming from the $D \to 0$ and $D \to \infty$ limits, all coefficients for $n = 2$ in (74) are fixed and

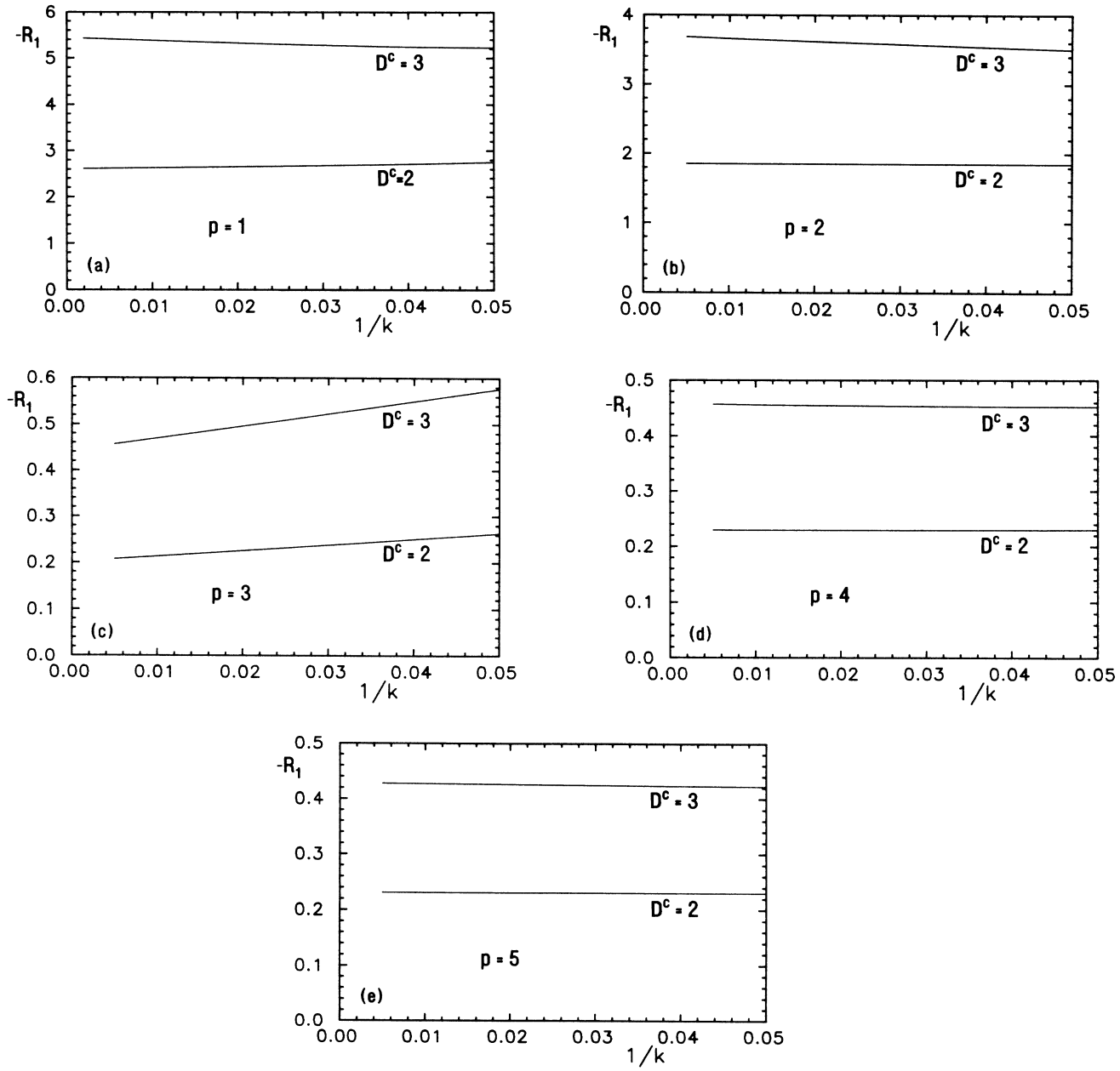

FIG. 2. The ratio $-R_1 \equiv -(E_k/E_k^{\rm asy} - 1)k = -\gamma^*_1 - \gamma^*_2/k + \cdots$ vs $1/k$ for r^p perturbations of the Coulomb system in $D^C = 2, 3$ with (a) $p = 1$, (b) $p = 2$, (c) $p = 3$, (d) $p = 4$, and (e) $p = 5$. These plots are very sensitive tests of the asymptotic formula (49). The extrapolated y values to $1/k = 0$ give $-\gamma^*_1$, and the asymptotic slope is an estimate for $-\gamma^*_2$. Notice the very good agreement in (a) with the analytical result (54) for $p = 1, -\gamma^*_1 = 2.6111\ldots, 5.4444\ldots$ in $D^C = 2, 3$. In the other cases, see Table VII for more precise values.

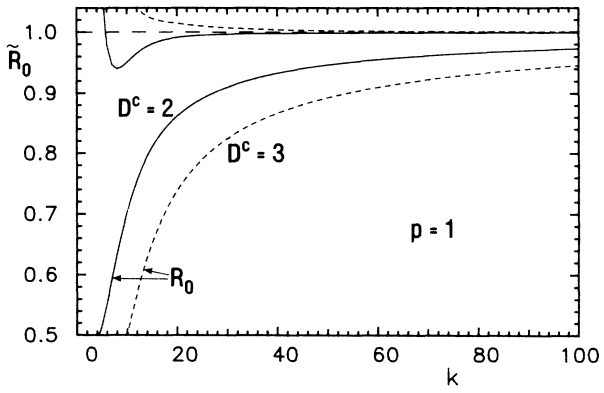

FIG. 3. The ratio $\tilde{R}_0 \equiv E_k/[E_k^{asy}(1 + \gamma^*_1/k)]$ for $p = 1$ in $D^C = 2$ (—) and $D^C = 3$ (- - -) dimensions including the theoretical $1/k$ correction ($\gamma^*_1 = -\frac{47}{18}, -\frac{49}{9}$ for $D^C = 2, 3$). For comparison, also the zeroth-order ratio $R_0 \equiv E_k/E_k^{asy}$ is shown.

$$c_2(D) = \frac{441D^4 + 792D^3 - 3660D^2 - 8352D - 2480}{10368} \qquad (p = 1)$$

$$= (-\tfrac{13259}{10368}, -\tfrac{1277}{648}, -\tfrac{3371}{10368}, \tfrac{4321}{648}, \ldots) \qquad (D = 1, 2, 3, 4, \ldots) \quad (75)$$

may also be considered as an analytically exact result, for all D. For $D = 2, 4$, we have compiled the constants b_n and c_n up to $n = 3$ in the first two columns of Table VI. The corresponding correction terms for the Coulomb system are then given by $c^*_n = c_n + \delta c_n$, where the δc_n follow from higher-order corrections similar to Eq. (53). For instance, the next term is for $p = 1$ [$\hat{\epsilon}_i = (3/a)^i \epsilon_i/\epsilon_0$]

$$c^*_2 = c_2 + c_1\left[\tfrac{1}{3}\hat{\epsilon}_1^2 + \left(\frac{D}{2} - 1\right)\hat{\epsilon}_1 + \tfrac{1}{3}\hat{\epsilon}_2\right] + \frac{D^2}{8}\hat{\epsilon}_1^2 + D(\tfrac{1}{6}\hat{\epsilon}_1^3 - \tfrac{5}{12}\hat{\epsilon}_1^2 + \tfrac{1}{6}\hat{\epsilon}_1\hat{\epsilon}_2 - \tfrac{1}{6}\hat{\epsilon}_2) + \tfrac{1}{18}\hat{\epsilon}_1^4 - \tfrac{10}{27}\hat{\epsilon}_1^3 + \tfrac{1}{9}\hat{\epsilon}_1^2\hat{\epsilon}_2 + \tfrac{1}{3}\hat{\epsilon}_1^2 - \tfrac{5}{9}\hat{\epsilon}_1\hat{\epsilon}_2 + \tfrac{1}{18}\hat{\epsilon}_2^2 + \frac{1}{3}\hat{\epsilon}_2 - \frac{1}{9}\hat{\epsilon}_3. \qquad (76)$$

Inserting $a = 3$ and the ϵ_i from Table II, we obtain the numbers shown in the third and fourth columns of Table VI. Finally, using (70) we find the γ^*_n in the fifth column, which can be compared directly with our modified Neville extrapolations (MNE) in the sixth column. The numbers in parentheses indicate the uncertainty in the last digit due to roundoff errors of the stable table entries in the lower right-hand corner (for $n \geq 3$, $k > 100$, say). The numbers in square brackets are estimates of how much the last digits are affected by error propagation. The last column gives the relative accuracy of the numerical estimates for γ^*_n as compared with the rational numbers, which are analytically exact for $n = 1, 2$ and, in any case, extremely accurate for $n = 3$.

Finally, in order to test the relations (71) and (72) for general p, we have performed similar numerical analyses of higher-order corrections also for $p = 3, 4, 5$. Since we are not aware of any published γ_n for general p, we have generated (see Appendix) and analyzed the oscillator perturbation series as well. Our results in Table VII nicely confirm that $\gamma^*_n = \gamma_n$ for $n < p - 1$. Furthermore, γ^*_{p-1} and γ_{p-1} are indeed related by Eq. (72), as can be read off from the last two lines for γ^*_n at each p.

V. DISCUSSION

The large-order formula (49) shows that the leading factorial growth behaviors $\propto \Gamma(pk + D/2)$ of the coefficients E_k (Coulomb) and ϵ_k (anharmonic oscillator) are identical. In the latter case this behavior has been explained as follows. In the path-integral or instanton approach[7,14] (the field-theoretical generalization of the WKB method) the starting point is the partition function $Z = \int \mathcal{D}\vec{x} \exp(-\mathcal{A}[\vec{x}])$ with the euclidean action

$$\mathcal{A}[\vec{x}] = \int d\tau \left(\tfrac{1}{2}\dot{\vec{x}}^2 + \frac{\omega^2}{2}\vec{x}^2 + \lambda(\vec{x}^2)^{p+1}\right). \qquad (77)$$

A saddle-point approximation yields, for small $\lambda < 0$, the leading exponential behavior of Im ϵ. A simple rescaling, $\vec{x} = \lambda^{-1/2p}\vec{y}$, reveals its dependence on λ,

$$\mathrm{Im}\,\epsilon \propto e^{-A/|\lambda|^{1/p}}, \qquad (78)$$

where the proportionality constant $A \equiv a^{-1/p}$ is the extremal action (in the rescaled variables) at the saddle point. Recalling the dispersion relation (39), this implies that the perturbation coefficients ϵ_k grow like $(-a)^k\Gamma(pk + \beta)$. The remaining constant β can be deduced from symmetry considerations. The saddle-point solution breaks "spontaneously" τ-translational and internal O(D)-rotational invariance. This gives rise to $1 + (D - 1) = D$ so-called zero-frequency fluctuations (Nambu-Goldstone modes) which leave the action invariant. Their proper treatment leads to prefactors $\propto (1/|\lambda|^{1/p})^{1/2}$ in Im ϵ. Since there is one such factor for each zero mode, the total prefactor for the ground-state energy is $\propto (1/|\lambda|^{1/p})^{D/2}$. (For excited states there are additional factors of this type.) Using (39), this fixes $\beta = D/2$.

Thus, via the above derived equivalence with the Coulomb systems, also the leading large-order behavior of the coefficients E_k can be understood without detailed calculations. It goes without saying that, although all calculations have been performed explicitly only for the ground-state energies, the considerations in this paper apply also to excited states without additional problems.

Finally, let us briefly discuss the "true" Stark and Zeeman Hamiltonians which result from the following replacements in the potential term of (3):

$$r \longrightarrow z \text{ (Stark)}, \qquad r^2 \longrightarrow x^2 + y^2 \text{ (Zeeman)}. \qquad (79)$$

Using the Kustaanheimo-Stiefel transformation,[9,10] it is

easy to show[12] that the equivalent four-dimensional oscillator systems have anharmonic terms

$$(\vec{x}^2)^2 \longrightarrow (x_1^2+x_2^2)^2-(x_3^2+x_4^2)^2 \text{ (Stark)},$$
$$(\vec{x}^2)^3 \longrightarrow 4\vec{x}^2(x_1^2+x_2^2)(x_3^2+x_4^2) \text{ (Zeeman)}, \tag{80}$$

respectively. In the first case, this reproduces the known energy formula[1,11]

$$\epsilon(\sqrt{-E/2},\lambda)+\epsilon(\sqrt{-E/2},-\lambda)=1, \tag{81}$$

relating the Stark resonances to two decoupled two-dimensional anharmonic oscillators of the $|\vec{x}|^4$ type. The second oscillator with the coupling $-\lambda$ is unstable and produces the decay-rates of the Stark resonances. In previous work,[1,11] this conclusion was derived from a direct separation of the Stark Hamiltonian when expressed in squared parabolic coordinates. In any case, the scaling and symmetry arguments given above show that $p=1$ and $\beta=2/2=1$. This implies for the perturbation expansion of the real part of the Stark ground-state resonance the large-order behavior

$$E_k \propto \Gamma(k+1)=k!, \tag{82}$$

in agreement with detailed computations.[1,11]

In the case of the Zeeman Hamiltonian we stay even closer to the problems studied in this paper.[12] The only difference is the fact that the equivalent four-dimensional anharmonic oscillator (of the $|\vec{x}|^6$ type) is no longer rotationally symmetric. There remains, however, a residual O(2)×O(2) symmetry which is responsible for the $1+1=2$ rotational zero modes. Adding the contribution of the τ-translational zero mode, we obtain $\beta=\frac{3}{2}$ and consequently, since $p=2$, the leading large-order behavior

$$E_k \propto \Gamma(2k+\tfrac{3}{2}), \tag{83}$$

in agreement with direct WKB analyses.[1,2]

VI. CONCLUSION

The intimate relationship between the Coulomb potential in *three* dimensions and the harmonic oscillator in *four* dimensions has been generalized to include interaction terms. This allowed for an efficient calculation of the large-order behavior of Coulomb systems in external potentials in terms of the known large-order results for anharmonic oscillators.

Apart from the theoretical interest in this very appealing structure, it is conceivable that the extended equivalence opens the way to new efficient numerical computation schemes, especially for the Zeeman Hamiltonian. The present application has only made use of the relation between the energy spectra. By exploiting the relations between the amplitudes it will be possible to connect the entire quantum mechanics of the two systems. In this way, a major part of the rich theoretical information available for the anharmonic oscillator systems (such as rigorous inequalities) will be transferable to the Coulomb systems.

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APPENDIX: RECURSIONS FOR GROUND-STATE PERTURBATION COEFFICIENTS OF COULOMB PROBLEM

In this appendix we derive recursions for a direct and simple calculation of the ground-state perturbation coefficients of the Coulomb problem. These recursions were used to compute the numbers in Tables III and V.

We start with $D^C=3$ dimensions and the radial Schrödinger equation for $\ell^C=0$,

$$-\tfrac{1}{2}\psi''-\frac{1}{r}\psi'-\frac{1}{r}\psi+gr^p\psi=E\psi. \tag{A1}$$

Separating out the unperturbed ground-state wave function, $\psi_0=e^{-r}$,

$$\psi=\psi_0\varphi=e^{-r}\varphi, \tag{A2}$$

Eq. (A1) becomes

$$-\tfrac{1}{2}\varphi''+\left(1-\frac{1}{r}\right)\varphi+gr^p\varphi=(E+\tfrac{1}{2})\varphi. \tag{A3}$$

Inserting the perturbation expansions $[b_0(r)\equiv 1\equiv b_{0,0}]$

$$\varphi(r)=\sum_{k=0}b_k(r)g^k,\quad E+\tfrac{1}{2}=\sum_{k=1}E_kg^k, \tag{A4}$$

and comparing equal powers in g, we find

$$-\tfrac{1}{2}b_k''(r)+\left(1-\frac{1}{r}\right)b_k'(r)+r^pb_{k-1}(r)=\sum_{\ell=0}^{k-1}E_{k-\ell}b_\ell(r). \tag{A5}$$

This can be solved by the ansatz

$$b_k(r)=\sum_{j=1}b_{k,j}r^j, \tag{A6}$$

leading to the recursion

$$b_{k,j}=\frac{1}{j}\left(\frac{(j+1)(j+2)}{2}b_{k,j+1}-b_{k-1,j-(p+1)}-3\sum_{\ell=1}^{k-1}b_{\ell,j-1}b_{k-\ell,2}\right), \tag{A7}$$

$$E_k=-3b_{k,2},$$

which has to be worked through with increasing $k = 1,2,3,\ldots$, and, for each k, decreasing $j = k(p+1),\ldots,2$. Initially, $b_{k,j} = 0$, $k \neq 0$, $j \neq 0$, and $b_{0,0} = 1$. From (A5) and (A6) it is easy to see that all $b_{k,1}$ remain zero.

Repeating this analysis for general dimensions D^C ($-(1/r)\psi' \longrightarrow -[(D^C-1)/2r]\psi'$ in (A1), $\psi_0 \longrightarrow e^{-2r/(D^C-1)}$ in (A2), $E+\frac{1}{2} \longrightarrow E+2/(D^C-1)^2$ in (A3)), one finds a similar structure. The generalization of Eq. (A7) which is valid for all dimensions D^C reads

$$b_{k,j} = \frac{D^C-1}{2j}\left(\frac{(j+1)(j+D^C-1)}{2} b_{k,j+1} - b_{k-1,j-(p+1)} - D^C \sum_{\ell=1}^{k-1} b_{\ell,j-1} b_{k-\ell,2} \right), \tag{A8}$$

$$E_k = -D^C b_{k,2},$$

$$k = 1,2,3,\ldots,\quad j = k(p+1),\ldots,2,\quad b_{0,0} = 1.$$

The scheme used in Eqs. (A2)–(A8) is completely analogous to that for anharmonic oscillators.[3] In the latter case, starting from the radial ground-state ($\ell = 0$) Schrödinger equation in D dimensions,

$$-\tfrac{1}{2}\phi'' - \frac{D-1}{2r}\phi' + \frac{r^2}{2}\phi + \lambda r^{2(p+1)}\phi = \epsilon\phi, \tag{A9}$$

the analog of Eq. (A8) reads ($\epsilon = D/2 + \sum_{k=1} \epsilon_k \lambda^k$)

$$b_{k,j} = \frac{1}{2j}\left((j+1)(2j+D) b_{k,j+1} - b_{k-1,j-(p+1)} - D \sum_{\ell=1}^{k-1} b_{\ell,j} b_{k-\ell,1} \right), \tag{A10}$$

$$\epsilon_k = -D b_{k,1},$$

$$k = 1,2,3,\ldots,\quad j = k(p+1),\ldots,1,\quad b_{0,0} = 1.$$

These recursions were used to generate the numbers in Table II.

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