Wigner symbols, quantum dynamics, and the kicked rotator

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(Received 25 May 1990)

Using the kicked rotator as an example, we show how to apply the formalism of Wigner symbols to the study of quantum dynamics. This approach provides a simple and direct method of comparing quantum and classical dynamics. We investigate in detail the leading quantum effects for small values of $\hbar$ and discuss the time scales at which quantum effects appear.

I. INTRODUCTION

Progress in understanding nonintegrable classical dynamical systems[1] has, in recent years, stimulated considerable interest in the corresponding quantum systems. In particular, much work has been devoted to investigating relationships between the classical and quantum dynamics of such systems. Important subjects of investigation include the qualitative differences between a system's classical and quantum behavior, the time scale at which quantum effects become significant, and how quantum effects can be calculated for small values of $\hbar$.[2]

Insight into these issues may be found by using a formulation of quantum mechanics that resembles, as much as possible, the usual formulation of classical mechanics. A conventional method of doing this is to construct wave packets that are initially localized around a particular position and momentum.[3] The evolution of such a wave packet is then compared to the classical trajectory beginning at the same position and momentum. While this approach has proved useful for many applications, we advocate, in this paper, an alternative approach that we believe can give a simpler and clearer comparison of quantum and classical dynamics.

The central idea is to directly calculate the time dependence of quantum operators, using the phase-space representation associated with the works of Wigner and Weyl.[4,5] For very small $\hbar$, quantum operators represented in this manner (sometimes referred to as Wigner symbols) reduce precisely to their classical counterparts plus corrections, which are typically of $O(\hbar^2)$. Wigner symbols thus provide a quantum generalization for any classical quantity that has an associated quantum operator. The differences between quantum and classical dynamics may be distinguished by, for example, comparing the Wigner symbols for the position and momentum operators with the classical position and momentum trajectories.

The use of Wigner symbols has three principle advantages over the use of wave packets. First, in using Wigner symbols the only parameter added to the classical parameters is $\hbar$. In contrast, with wave packets one must also specify, at least, their initial widths. Second, wave packets usually spread out and eventually are no longer sharply localized around a particular position and momentum, making dubious a comparison with an individual classical trajectory. With Wigner symbols, this spreading problem does not exist. Third, by employing a phase-space representation for quantum operators the difference between quantum and classical mechanics may be separated into two parts. One part arises from the difference in the phase-space distribution functions allowed for quantum and classical systems and is not essentially related to the dynamics. The other part, that which is obtained by calculating Wigner symbols, contains all the dynamical information. Making this separation can clarify the origin of quantum effects.

While the basic formalism of Wigner symbols is well established,[6] it has not, to our knowledge, been previously applied to the calculation of dynamical properties for nonintegrable quantum systems. In this paper, we demonstrate how to do this, using as an example the kicked rotator, a model frequently employed in the investigation of nonintegrable dynamics.[7] This model consists of a particle that is periodically "kicked" and is governed by the Hamiltonian

$$H=\frac{p^2}{2}+K\cos(x)\sum_{n=-\infty}^{+\infty}\delta(t-n)\ , \tag{1.1}$$

where x is the particle's position, p is the particle's momentum, and K is the kicking strength.[8]

For the kicked rotator, we show explicitly how to calculate Wigner symbols, and we study in detail the leading deviation for small $\hbar$ of symbols from their classical limits. We find that the asymptotic time dependence of the leading deviation depends only on general features of the classical trajectory. We also discuss how to estimate the time scale at which quantum effects begin to be important and present a calculation of a so-called quantum crossover time. The object of this paper is to demonstrate the utility of Wigner symbols for studying quantum dynam-

ics, as well as to give new results for the kicked rotator relevant for small values of $\hbar$.

In Sec. II, we review the basic formalism of Wigner symbols. In Sec. III, we give, for the kicked rotator, a practical method for calculating Wigner symbols and present typical numerical results, which we compare to the corresponding classical ones. We then, in Sec. IV, consider in detail the leading quantum correction to a classical trajectory and discuss the time scales at which quantum effects become significant.

II. WIGNER SYMBOL FORMALISM

For a system consisting of a single particle, a dynamical quantity $A \equiv F(x,p)$ (e.g., the particle's energy) can classically be regarded as a function $A(x_0,p_0,t) \equiv F(x(t),p(t))$ of the particle's initial position x_0, initial momentum p_0, and the time t. Similarly, a quantum operator $\hat{A} \equiv F(\hat{x},\hat{p})$ can be regarded as a function $\hat{A}(\hat{x}_0,\hat{p}_0,t) \equiv F(\hat{x}(t),\hat{p}(t))$ of the initial position operator $\hat{x}_0 \equiv \hat{x}(0)$, the initial momentum operator $\hat{p}_0 \equiv \hat{p}(0)$, and time. We assume that $\hat{A}$ is Hermitian, that the function $F(x,p)$ does not have an explicit $\hbar$ dependence, and that $[\hat{x},\hat{p}]=i\hbar$.

Since the position and momentum operators do not commute, the function $\tilde{A}$ is well defined only if an operator ordering is specified. Weyl ordering is obtained from the prescription

$$\hat{A}(t)=\int dx_0 dp_0\, \tilde{A}(x_0,p_0,t) \times \int \frac{d\lambda d\gamma}{(2\pi)^{2d}} e^{i\lambda[\hat{x}(0)-x_0]+i\gamma[\hat{p}(0)-p_0]} , \quad (2.1)$$

where d is the dimension of the system. Defined in this manner, we refer to $\tilde{A}$ as the Wigner symbol for $\hat{A}$.[6] To find $\tilde{A}$, one takes the Wigner transform of $\hat{A}$:

$$\tilde{A}(x_0,p_0,t) = \int d\alpha\, e^{-(i/\hbar)\alpha p_0} \langle x_0+\alpha/2|\, \hat{A}(t)|x_0-\alpha/2\rangle , \quad (2.2)$$

where the state $|x\rangle$ is an eigenstate of the initial position operator. The Wigner symbol of any Hermitian operator is real.

This way of representing a quantum operator is useful because the Wigner symbol $\tilde{A}(x_0,p_0,t)$ is a natural generalization of the classical $A(x_0,p_0,t)$. For small $\hbar$, one can show[9]

$$\tilde{A}(x_0,p_0,t)=A(x_0,p_0,t)+O(\hbar^2) . \quad (2.3)$$

Furthermore, the average of $\hat{A}$ over an initial state $|\psi_0\rangle$ may be expressed as

$$\langle \hat{A}(t)\rangle \equiv \int dx\, dx' \langle \psi_0|x\rangle\langle x|\hat{A}(t)|x'\rangle\langle x'|\psi_0\rangle = \int dx_0 dp_0\, \tilde{A}(x_0,p_0,t)\tilde{\rho}(x_0,p_0) , \quad (2.4)$$

where $\tilde{\rho}$ is the Wigner function of $|\psi_0\rangle$, given by[4]

$$\tilde{\rho}(x_0,p_0)=\int \frac{d\alpha}{(2\pi\hbar)^d} e^{-(i/\hbar)\alpha p_0} \times \langle x_0+\alpha/2|\psi_0\rangle\langle\psi_0|x_0-\alpha/2\rangle . \quad (2.5)$$

Equation (2.4) is very analogous to the classical expression for the average value of A over a distribution of initial conditions.

In Eq. (2.4), the $\hbar$ dependence of $\hat{A}$'s expectation value has been divided into two parts, one due to the Wigner function $\tilde{\rho}$ and one due to the Wigner symbol $\tilde{A}$. The Wigner function, as we have defined it, is independent of time and has no connection to the system's Hamiltonian. It merely represents choice of initial state. Its $\hbar$ dependence arises from the basic rules of quantum mechanics that constrain the possible forms of $\tilde{\rho}$. For example, $\tilde{\rho}$ cannot be chosen to be a δ function in both the position and momentum variables, as this would violate the Heisenberg uncertainty relation. The Wigner symbol, on the other hand, depends essentially on the Hamiltonian and contains dynamical information intrinsic to the system. By comparing Wigner symbols directly with their classical limits, one may distinguish quantum effects particular to a system's dynamics from ones due solely to the general restrictions quantum mechanics places on phase-space distributions.

The phase-space formulation of quantum mechanics developed here is distinct, but complementary to, one which uses time-dependent Wigner functions as the central dynamical objects.[10] Our approach corresponds to the Heisenberg picture of quantum mechanics, while the use of time-dependent Wigner functions corresponds to the Schrödinger picture. In previous studies of quantum dynamics, the Schrödinger viewpoint has predominated. In the following sections, we show how practical calculations can be done within the Heisenberg framework.

III. CALCULATING WIGNER SYMBOLS FOR THE KICKED ROTATOR

The essential features of the kicked rotator's dynamics are contained in a map that gives the position and momentum just before a kick in terms of the position and momentum before the previous kick. This map is found by integrating the equations of motion corresponding to the Hamiltonian (1.1) over one kicking period, leading to the so-called standard map[11]

$$\begin{aligned} x_{n+1} &= x_n + p_{n+1} , \\ p_{n+1} &= p_n + K\sin(x_n) , \end{aligned} \quad (3.1)$$

with x_n being the position and p_n the momentum before the nth kick.

The dependence of a dynamical quantity $A_n \equiv F(x_n,p_n)$ on the initial position and momentum can be obtained from the recurrence relation

$$A_{n+1}(x_0,p_0)=A_n[x_0+p_0+K\sin(x_0),p_0+K\sin(x_0)] , \quad (3.2)$$

which follows from (3.1) and the fact that an $(n+1)$-step evolution from (x_0,p_0) is the same as an n-step evolution from (x_1,p_1). Moreover, by applying (3.2) to the cases with $A_n=x_n$ and $A_n=p_n$, (3.1) can be derived from (3.2). Thus (3.1) and (3.2) provide equivalent representations of the kicked rotator's classical dynamics.

Quantum mechanically, the evolution of an operator over one kicking period is given by[10,12]

$$\hat{A}_{n+1}=\hat{U}^{\dagger}\hat{A}_n\hat{U}\ , \tag{3.3}$$

where $\hat{U}$ is an evolution operator defined as

$$\hat{U}\equiv e^{-i/\hbar(\hat{p}_0^2/2)}e^{i/\hbar K\cos(\hat{x}_0)}\ . \tag{3.4}$$

Using Eq. (2.2) to express (3.3) in terms of Wigner symbols, one finds, after some algebra,

$$\tilde{A}_{n+1}(x_0,p_0)=\sum_{l=-\infty}^{+\infty}J_l[2K\sin(x_0)/\hbar]\times\tilde{A}_n(x_0+p_0+\hbar l/2,p_0+\hbar l/2)\ , \tag{3.5}$$

where $J_l(x)$ is a Bessel function. The recurrence relation (3.5) is a quantum generalization of (3.2), reducing to (3.2) in the $\hbar\to 0$ limit. For large l, the Bessel functions decrease rapidly and only about $4K/\hbar$ terms contribute substantially the sum in (3.5).

The numerical evaluation of (3.5) is especially simple for quantities with the symmetry

$$\tilde{A}_n(x_0,p_0)=\tilde{A}_n(x_0+2\pi,p_0)=\tilde{A}_n(x_0,p_0+2\pi)\ , \tag{3.6}$$

which is preserved by the dynamics. This symmetry can be exploited by choosing $\hbar$ equal to a "resonant" value of $4\pi a/b$, where a and b are positive integers.[7,12] Any other value of $\hbar$ may be approximated to arbitrary accuracy by using sufficiently large a and b.[13]

For these resonant values of $\hbar$, (3.5) becomes

$$\tilde{A}_{n+1}(x_0,p_0)=\sum_{l=0}^{b-1}Q_l(x_0)\tilde{A}_n(x_0+p_0+\hbar l/2,p_0+\hbar l/2)\ , \tag{3.7}$$

with

$$Q_l(x_0)=\frac{1}{b}\sum_{m=0}^{b-l}\cos\left\{\frac{2\pi lm}{b}+\frac{K}{\hbar}\left[\cos\left[x_0+\frac{2\pi m}{b}\right]-\cos\left[x_0-\frac{2\pi m}{b}\right]\right]\right\}\ , \tag{3.8}$$

which now contains a finite number of terms. Further simplication occurs if p_0 is approximated by a number $2\pi j/b$, where j is an integer. The recurrence relation (3.7) is then equivalent to a map of a $b\times b$ matrix and is straightforward to evaluate numerically. The symmetry (3.6) applies in particular to the momentum step $\tilde{p}_{n+1}(x_0,p_0)-\tilde{p}_n(x_0,p_0)$, from which the momentum symbol can be obtained. In fact, the Wigner symbol for any operator that is a polynomial in $\hat{x}$ and $\hat{p}$ can be found by iterating $b\times b$ matrices.

In Fig. 1, we show the time dependence of a typical

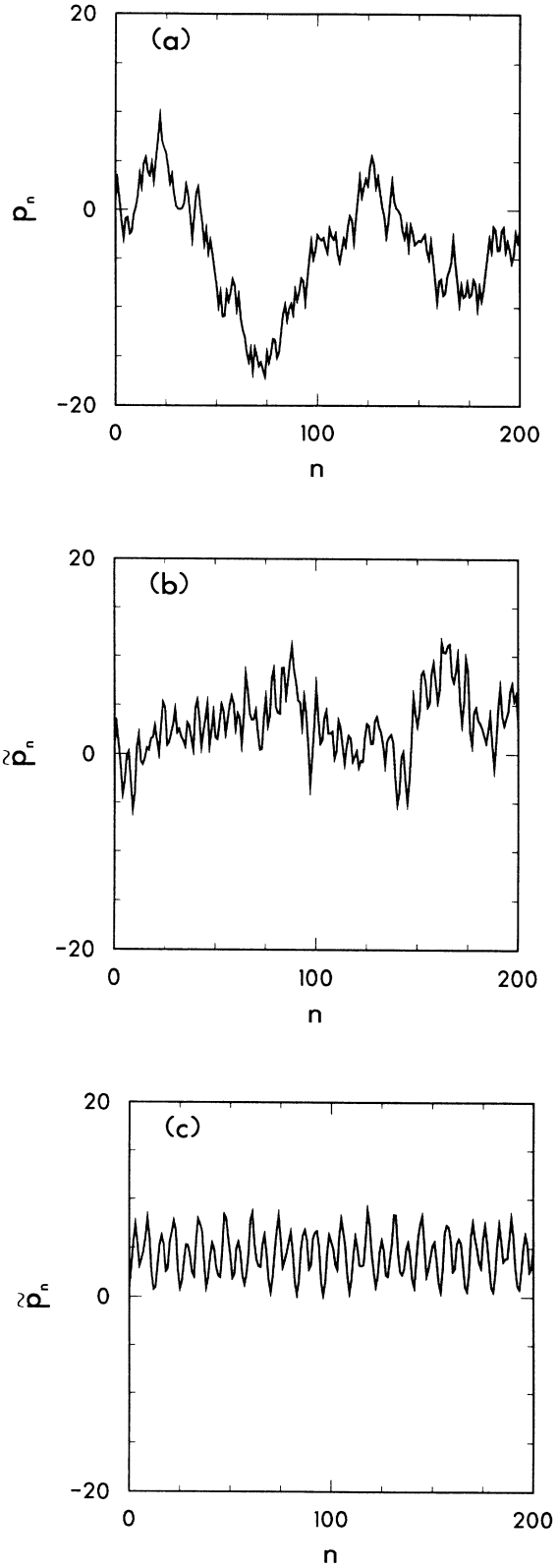

FIG. 1. A classical momentum trajectory and its corresponding momentum symbol for $K=2.5$, $x_0=1.0$, and $p_0=20\pi/67$; (a) is the classical trajectory, (b) is the momentum symbol with $\hbar=8\pi/67\approx 0.38$, and (c) is the symbol with $\hbar=212\pi/67\approx 9.9$. As $\hbar$ is increased the momentum symbol's time dependence evolves from chaotic, in the classical limit, to simple and quasiperiodic, for large values of $\hbar$.

momentum symbol for two values of $\hbar$, as well as the corresponding classical trajectory ($K=2.5$, $x_0=1.0$, $p_0\approx 0.94$). The symbols were calculated by using (3.7) with $b=67$. The classical trajectory, Fig. 1(a), is chaotic. For small $\hbar$, the momentum symbol also appears irregular, as in Fig. 1(b), but as $\hbar$ is increased it becomes more regular as suggested by Fig. 1(c).

A Fourier transform in time of the momentum symbol consists of sharp peaks, indicating quasiperiodic dynamical behavior. For small $\hbar$, many peaks have a substantial weight, while for large $\hbar$ only a few are significant. In the $\hbar\to 0$ limit, the number of important Fourier peaks may diverge, leading to chaotic classical motion. The suppression of the chaotic behavior of the kicked rotator by quantum effects can be understood in terms of a localization argument due to Grempel, Prange, and Fishman.[14]

IV. QUANTUM CORRECTIONS FOR SMALL $\hbar$

We now consider the leading deviation of a Wigner symbol for the kicked rotator from its classical limit. These deviations, generally of $O(\hbar^2)$, represent the first quantum corrections to the classical motion. Here we restrict ourselves to the position and momentum symbols, although the method we use may be extended to other Wigner symbols.

The position and momentum symbols have the expansions

$$\begin{aligned}\tilde{x}_n(x_0,p_0)&=x_n(x_0,p_0)+\hbar^2\alpha_n(x_0,p_0)+O(\hbar^4)\ ,\\ \tilde{p}_n(x_0,p_0)&=p_n(x_0,p_0)+\hbar^2\beta_n(x_0,p_0)+O(\hbar^4)\ ,\end{aligned}\tag{4.1}$$

where α_n and β_n are the coefficients of the leading quantum corrections. These coefficients may be obtained by iterating the map

$$\begin{aligned}\alpha_{n+1}&=\alpha_n+\beta_{n+1}\ ,\\ \beta_{n+1}&=\beta_n-V^{(2)}(x_n)\alpha_n+W_n\ ,\end{aligned}\tag{4.2}$$

with the initial condition $\alpha_0=\beta_0=0$. In (4.2),

$$W_n\equiv\frac{K}{8}\epsilon_{ik}\epsilon_{jl}x_{n,ij}\left[\tfrac{1}{2}\sin(x_n)x_{n,kl}+\tfrac{1}{3}\cos(x_n)x_{n,k}x_{n,l}\right]\ ,\tag{4.3}$$

where we use the notation

$$\begin{aligned}f_{,1}(x_0,p_0)&\equiv\frac{\partial f(x_0,p_0)}{\partial x_0}\ ,\\ f_{,2}(x_0,p_0)&\equiv\frac{\partial f(x_0,p_0)}{\partial p_0}\ ,\\ f_{,ij}(x_0,p_0)&\equiv[f_{,i}(x_0,p_0)]_{ij}\end{aligned}\tag{4.4}$$

for any function $f(x_0,p_0)$. In (4.3), the indices i,j,k,l are summed from 1 to 2, and ϵ_{ij} is an antisymmetric matrix with $\epsilon_{11}=\epsilon_{22}=0$ and $\epsilon_{12}=-\epsilon_{21}=1$. W_n can be calculated from the classical map (3.1). While (4.2) may be derived from (3.5), a more direct derivation is given in the Appendix.

For short times, the quantum corrections obtained from (4.2) are small, and the quantum and classical trajectories nearly coincide. However, as time increases, the quantum corrections tend to grow and eventually become important (see Fig. 2). For small $\hbar$, it is the asymptotic behavior for large time of the corrections which is of interest.[15] Numerical study of (4.2) indicates the asymptotic form of the corrections depends only on the general character of the classical trajectory. Specifically, we find for a generic quasiperiodic classical trajectory that, on the average, $|\alpha_n|$ and $|\beta_n|$ increase proportionally to n^3, while for a chaotic classical trajectory they increase proportionally to $e^{3\sigma n}$, where σ is the largest Liapunov exponent for the classical trajectory. Equations (4.1) then suggest that the time at which the Wigner symbols begin to deviate significantly from the corresponding classical trajectories scales as $\hbar^{-2/3}$ for quasiperiodic trajectories and as $-(2/3\sigma)\ln(\hbar)$ for chaotic trajectories.[16] Clearly, the quantum corrections to a chaotic classical trajectory increase much more rapidly than those for a quasiperiodic trajectory.

Figure 3 gives a comparison of the quantum corrections computed from (4.2) with exact results obtained from Eq. (3.7) for two trajectories with $K=1.5$ and $\hbar=4\pi/250\approx 0.05$. We define the quantum deviation $\Delta_n\equiv(\tilde{p}_n-p_n)/\hbar^2$. For small n, Δ_n is approximately equal to β_n but as n increases the deviation grows and can no longer be obtained from (4.2). The deviation for the chaotic trajectory (squares) initially grows much faster than the deviation for the quasiperiodic trajectory (circles).

As an illustration of how to apply these results, we consider a wave packet that initially has the Gaussian form

$$\langle x|\psi_0\rangle=\left[\frac{\lambda}{\pi\hbar}\right]^{1/4}\exp\left[\frac{ip_0x}{\hbar}-\frac{\lambda}{2\hbar}(x-x_0)^2\right]\ ,\tag{4.5}$$

where λ is a real parameter. From Eq. (2.5), one finds the

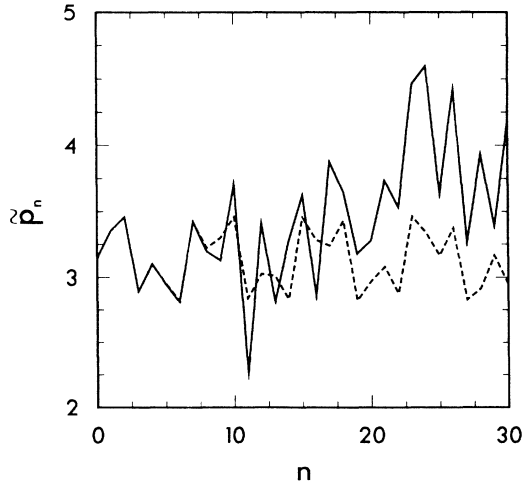

FIG. 2. Momentum symbol ($\hbar=4\pi/250$, solid line) with the corresponding classical trajectory (dashed line) for $K=1.5$, $x_0=3.0$, and $p_0=\pi$. The momentum symbol follows the classical trajectory for several iterations, but then begins to deviate.

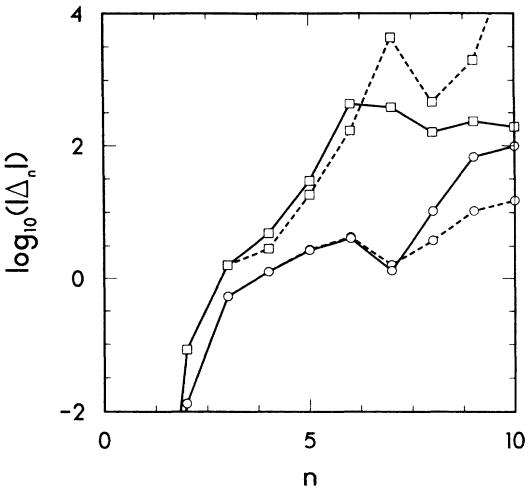

FIG. 3. Logarithm of the deviation $\Delta_n \equiv (\bar{p}_n - p_n)/\hbar^2$ for a quasiperiodic trajectory ($K=1.5$, $x_0=3.0$, and $p_0=\pi$; circles) and a chaotic trajectory ($K=1.5$, $x_0=2.0$, and $p_0=\pi$; squares). The solid lines are exact calculations with $\hbar=4\pi/250$, and the dashed lines are the approximations obtained by using the $\hbar$ expansion coefficient β_n.

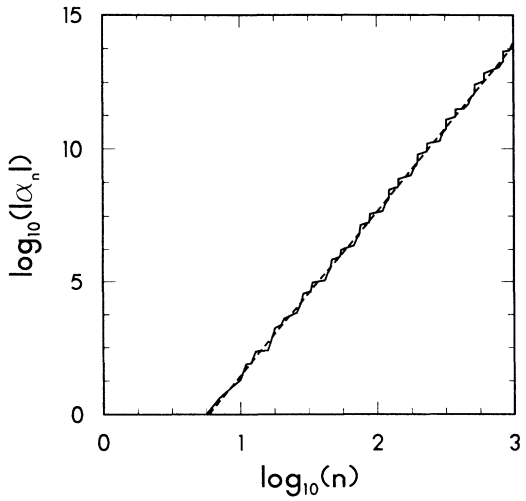

FIG. 4. Logarithm of the quantum correction α_n for the last bounding KAM trajectory ($K=0.971\,635\ldots$, $x_0=\pi$, and $p_0=3.737\,998\ldots$). Only points from the curve's upper envelope are plotted (solid line). The dashed line is a least squares fit of slope 6.2, showing that the correction increases approximately as $n^{6.2}$.

corresponding Wigner function to be

$$\tilde{\rho}(x,p)=\frac{1}{\pi\hbar}\exp\left[-\frac{\lambda}{\hbar}(x-x_0)^2-\frac{1}{\lambda\hbar}(p-p_0)^2\right] . \quad (4.6)$$

The average position of this wave packet as it evolves in time is, as follows from (2.4),

$$\langle\hat{x}(t)\rangle=\int dx'dp'\bar{x}(x',p',t)\tilde{\rho}(x',p') . \quad (4.7)$$

We now ask how this average position differs, for small $\hbar$, from the average position found from the classical evolution of a phase-space distribution that is initially set equal to the Wigner function (4.6). In other words, we compare the wave packet's average position with that obtained from Eq. (4.7) when $\bar{x}$ is replaced by the classical position x. Using (4.1) one finds that the difference between these quantum and classical average positions is simply $\hbar^2\alpha_n(x_0,p_0)$ plus terms of higher order in $\hbar$. Hence the time at which the average position of the Gaussian wave packet begins to deviate substantially from the classical average is the same as the time at which $\bar{x}_n(x_0,p_0)$ begins to deviate substantially from $x_n(x_0,p_0)$. We note that this time cannot be obtained from the usual semiclassical treatment of Gaussian wave packets, since the average position in this approximation is just the classical $x_n(x_0,p_0)$.[17]

As a further example, we apply Wigner symbols to estimate the so-called quantum crossover time for the kicked rotator. The crossover time may be defined, somewhat loosely, as the time at which there appears a significant qualitative difference between the kicked rotator's classical and quantum behavior. For values of K greater than $K_c \equiv 0.9716\ldots$, this is conventionally taken as the time at which the average momentum of a quantum wave packet ceases to exhibit the diffusive behavior observed classically.[7] [If K is less than K_c, classical Kolmogorov-Arnol'd-Moser (KAM) trajectories bound the momentum and this criterion does not apply.[11]]

The classical diffusion rate for $K \gtrsim K_c$ is determined essentially by the dynamics in the vicinity of the cantorus remnant of the last bounding KAM trajectory to disappear (which it does at $K=K_c$). Since this last KAM trajectory plays a central role in determining the kicked rotator's qualitative behavior, it is reasonable to postulate that the time at which quantum corrections to this trajectory become significant should correspond to the quantum crossover time. Numerical calculations obtained from Eq. (4.2) for the leading quantum corrections to the position and momentum symbols indicate that these grow proportionally to n^{ν}, where $\nu=6.2\pm0.1$ (see Fig. 4).[18] Equations (4.1) then imply a crossover time proportional to $\hbar^{-(2/\nu)}$, which is consistent with both numerical simulations and with the result of Fishman, Grempel, and Prange,[19] who use a renormalization argument to find a crossover time scaling as $\hbar^{-(2/\nu')}$, with $\nu'\approx 6.1$.

V. CONCLUDING REMARKS

Both classically and quantum mechanically, there are two complementary perspectives from which one may discuss dynamics. The usual classical approach is to calculate trajectories for individual particles. Alternatively, one may study the evolution of a distribution function of phase-space points, by solving, for example, Liouville's equation. In quantum mechanics, the viewpoint analogous to the classical trajectory approach is the Heisenberg picture, while the Schrödinger picture is analogous to the classical distribution function approach.

Since many of the important concepts pertaining to nonintegrable classical dynamical systems (e.g., the distinction between regular and chaotic dynamics) are most easily understood from the trajectory viewpoint, it seems

natural to employ the Heisenberg picture in comparing quantum and classical dynamics. Nevertheless, this is rarely done, perhaps because of a lack of practical techniques for applying the Heisenberg approach. In this paper, we have sought to develop such techniques, using the model of the kicked rotator.

Our basic method is to calculate directly the time dependence of the Wigner symbols corresponding to quantum observables. When quantum mechanics is formulated in this way, its relationship to classical mechanics is very clear. Each Wigner symbol corresponds to a unique classical quantity, allowing the comparison between quantum and classical dynamics to be made simply and unambiguously. For example, the Wigner symbol corresponding to the derivative of a classical trajectory with respect to its initial position $\partial x(t)/\partial x_0$, which is relevant to the calculation of Liapunov exponents, is simply the derivative of the position symbol $\partial \tilde{x}(t)/\partial x_0$. How to make such a connection in the language of Schrödinger wave functions is less obvious.

In addition to giving a practical algorithm for calculating Wigner symbols for the kicked rotator, we have also studied the behavior of the position and momentum symbols for small values of $\hbar$. We find that the asymptotic behavior of the leading quantum corrections for these Wigner symbols depends only on the general character of the corresponding classical trajectory, and we have shown that these corrections can be used to find the time at which quantum effects become significant. In particular, the quantum crossover time for the kicked rotator with $K \gtrsim K_c$ can be estimated from the deviations to the last bounding KAM trajectory.

While in this paper we have restricted ourselves to the kicked rotator, the Wigner symbol approach can be applied to many other systems. This method is especially useful when a comparison of classical and quantum dynamics is desired. We anticipate applications both to questions concerning the qualitative dynamical behavior of simple quantum systems and to the development of practical quantitative calculational techniques.

ACKNOWLEDGMENTS

We are grateful to S. Fishman, R. E. Prange, and W. Kohn for helpful discussions. This work has been supported in part by National Science Foundation Grant No. DMR87-03434 and U.S. Office of Naval Research Grant No. N00014-84-K-0548. One of us (J.H.J.) also acknowledges financial support of the North Atlantic Treaty Organization.

APPENDIX

Here we present a derivation of map (4.2), which gives the leading quantum corrections for the position and momentum symbols. We begin with some general considerations pertaining to the $\hbar$ expansions of Wigner symbols. In developing such expansions, it is convenient to have a relation between the Wigner symbols for two individual operators $\hat{A}$ and $\hat{B}$, and the Wigner symbol for their product $\hat{A}\hat{B} \equiv \hat{C}$. This is given by the so-called "$*$ product"

$$\tilde{C}(x_0,p_0) = \tilde{A}(x_0,p_0)\exp\left[\frac{i\hbar}{2}\left[\frac{\overleftarrow{\partial}}{\partial x_0}\frac{\overrightarrow{\partial}}{\partial p_0} - \frac{\overleftarrow{\partial}}{\partial p_0}\frac{\overrightarrow{\partial}}{\partial x_0}\right]\right] \times \tilde{B}(x_0,p_0) \equiv \tilde{A}(x_0,p_0) * \tilde{B}(x_0,p_0) , \tag{A1}$$

with the arrows indicating on which sides the derivatives act.[6,20] The $*$ product, like the operator product, is associative but not commutative. Note that the $*$ product reduces to an ordinary product for $\hbar = 0$.

Expanding the exponential in Eq. (A1), we obtain the $\hbar$ expansion

$$\tilde{A}(x_0,p_0) * \tilde{B}(x_0,b_0) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{i\hbar}{2}\right]^n \{\tilde{A},\tilde{B}\}_n , \tag{A2}$$

where

$$\{\tilde{A},\tilde{B}\}_n \equiv \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{\partial^n \tilde{A}}{\partial^{n-m}x_0 \partial^m p_0} \frac{\partial^n \tilde{B}}{\partial^n x_0 \partial^{n-m} p_0} . \tag{A3}$$

$\{\tilde{A},\tilde{B}\}_1$ is simply the conventional Poisson bracket. Evaluating a $*$ product of two Wigner symbols to order n in $\hbar$ requires derivatives up to order n.

We now consider the $\hbar$ expansion for the Wigner symbol of some analytic function of an operator $F(\hat{A})$. The Wigner symbol for $F(\hat{A})$ may be written $F(*\,\tilde{A}(x_0,p_0))$, where $F(*\,\tilde{A}(x_0,p_0))$ is defined in terms of F's Taylor expansion and the rules

$$\begin{aligned} &*\,\tilde{A}(x_0,p_0) \equiv \tilde{A}(x_0,p_0) , \\ &[*\,\tilde{A}(x_0,p_0)]^2 \equiv \tilde{A}(x_0,p_0) * \tilde{A}(x_0,p_0) , \\ &[*\,\tilde{A}(x_0,p_0)]^3 \equiv \tilde{A}(x_0,p_0) * \tilde{A}(x_0,p_0) * \tilde{A}(x_0,p_0) , \end{aligned} \tag{A4}$$

etc. We may expand $F(*\,\tilde{A}(x_0,p_0))$ as

$$\begin{aligned} F(*\,\tilde{A}(x_0,p_0)) &= F(\tilde{A}(x_0,p_0) + [*\,\tilde{A}(x_0,p_0) - \tilde{A}(x_0,p_0)]) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(\tilde{A}(x_0,p_0)) \times [*\,\tilde{A}(x_0,p_0) - \tilde{A}(x_0,p_0)]^n , \end{aligned} \tag{A5}$$

where $F^{(n)}(A)$ is the nth derivative of $F(A)$. Then $n=1$ term vanishes identically. Using (A2), each term of the right-hand side of (A5) can be further expanded in powers of $\hbar$. To $O(\hbar^2)$, one finds that only the first three nontrivial terms (i.e., $n=0,2,3$) contribute, with the higher-order terms being of $O(\hbar^4)$ or smaller. Evaluating these gives

$$F(*\tilde{A}(x_0,p_0))=F(\tilde{A}(x_0,p_0))-\frac{\hbar^2}{8}\epsilon_{ik}\epsilon_{jl}\tilde{A}_{,ij}(x_0,p_0)[\tfrac{1}{2}F^{(2)}(\tilde{A}(x_0,p_0))\tilde{A}_{,kl}(x_0,p_0)$$
$$+\tfrac{1}{3}F^{(3)}(\tilde{A}(x_0,p_0))\tilde{A}_{,k}(x_0,p_0)\tilde{A}_{,l}(x_0,p_0)]+O(\hbar^4)\ , \qquad \text{(A6)}$$

where we have used the notations of (4.4).

The dynamics of the quantized kicked rotator follows from the operator map

$$\hat{x}_{n+1}=\hat{x}_n+\hat{p}_{n+1}\ ,$$
$$\hat{p}_{n+1}=\hat{p}_n+K\sin(\hat{x}_n)\ , \qquad \text{(A7)}$$

which is the quantized form of (3.1) and is equivalent to (3.3). Taking the Wigner transform of this map gives

$$\tilde{x}_{n+1}=\tilde{x}_n+\tilde{p}_{n+1}\ ,$$
$$\tilde{p}_{n+1}=\tilde{p}_n+K\sin(*\tilde{x}_n)\ . \qquad \text{(A8)}$$

Applying (A6) and (4.1) to (A8), and extracting the terms of $O(\hbar^2)$, leads directly to (4.2).

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