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Mobility of singularities in the dissipative Ginzburg-Landau equation

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The velocity of a vortex solution of the real and complex dissipative Ginzburg-Landau equation in a weak external field is obtained by combining the method of matched asymptotic expansions with the numerical solution in the core region. The velocity of two interacting vortices as a function of their separation is estimated using the quasistationary approximation of the phase field.

Dynamics of topological singularities (defects) is a major outstanding problem of the theory of nonequilibrium patterns. Defects are necessarily present in realistic convective or reaction-diffusional patterns, and play a crucial role in the overall organization of patterns in extended systems and their long-time evolution. The most efficient (and, at sufficiently high aspect ratios, the only practical) tool of description of large-scale dynamics is given by phase equations governing smooth modulations of a basic short-scale structure.[1] Phase equations have a universal form dependent on the symmetry of the underlying systems. The same symmetry properties determine the character of singularities.[2] At the same time, in the immediate vicinity of singularities, the phase equations break down, thus forcing one to revert to nonuniversal short-scale dynamics. So far, due to the inability to incorporate realistic description of defects into large-scale phase-dynamical computations, applications of phase equations has been very limited, and computations of distorted patterns had to employ short-scale grids and be restricted to model equations and moderate box sizes.[3]

A rational approach to large-scale computations calls for the use of phase equations everywhere except the vicinity of defects. The latter, being localized objects ("particles") with a well-defined structure, should respond in a prescribed way to local values of a smooth phase field. A relationship between the force exerted by this field and the defect velocity (if one is found) would determine the equation of motion of singularities. In its turn, their displacement would affect the phase field governed by long-scale equations. The crucial step towards the realization of this approach is to define the effective particle-field interaction by matching the short-scale solution near the core of the defect with the outer smooth phase field.

We shall work with the simplest pattern-forming model—the two-dimensional (2D) dissipative Ginzburg-Landau equation (DGLE)

$$\dot{u} = \nabla^2 u + u - |u|^2 u \,, \tag{1}$$

which can be rewritten using $u = \rho e^{i\theta}$ as a pair of real equations

$$\dot{\rho} = \nabla^2 \rho + (1 - |\nabla\theta|^2 - \rho^2)\rho \,, \tag{2}$$

$$\dot{\theta} = \nabla^2\theta + 2\rho^{-1}\nabla\rho \cdot \nabla\theta \,. \tag{3}$$

In the context of reaction-diffusion systems, this equation can be derived by bifurcation expansion in the vicinity of a cusp singularity. Studies of defects in Rayleigh-Bénard convection patterns[4] involved a much more complicated fourth-order Newell-Whitehead-Segel amplitude equation. The DGLE model turned out, however, to be applicable near the onset of convection in liquid crystals under conditions when a definite orientation of convection rolls is induced by boundary conditions.[5]

A stationary defect with the topological charge n corresponds to a circulary symmetric vortex solution of Eqs. (2) and (3) that is given,[6] in polar coordinates (r,ϕ), by $\theta = n\phi$ and the amplitude $\rho = \rho_0(r)$ that verifies

$$\rho_0'' + r^{-1}\rho_0' + (1 - n^2 r^{-2} - \rho_0^2)\rho_0 = 0 \,. \tag{4}$$

We shall further restrict to $|n| = 1$, since defects with higher charges are unstable. It is also sufficient to consider explicitly the solution rotating in the positive sense $(n=1)$, its counterpart being just the complex conjugate. The solution of Eq. (4) with $|n| = 1$ is a monotonic function with the asymptotics $\rho_0 = ar$ (where a is a constant determined numerically) near the origin, and $\rho_0 = 1 - \frac{1}{2} r^{-2} + O(r^{-4})$ at $r \to \infty$.

One could be tempted to look for a perturbed solution, corresponding to the defect slowly propagating with a constant speed $\epsilon\mathbf{v}$ under the action of a weak externally imposed phase gradient $\epsilon\mathbf{A}$, by rewriting Eq. (1) in a gauged form

$$\epsilon\mathbf{v}\cdot\nabla u + |\nabla + i\epsilon\mathbf{A}|^2 u + u - |u|^2 u = 0 \,, \tag{5}$$

and expanding (5) in ϵ around the quiescent vortex solution $u_0 = \rho_0(r)e^{i\phi}$. The two components of the gradient $\nabla u_0 = (\mathbf{\hat{r}}\rho_0' + i\mathbf{\hat{\ell}}r^{-1})e^{i\phi}$ (where $\mathbf{\hat{r}}$, $\mathbf{\hat{\ell}}$ are, respectively, the radial and circumferential unit vectors) are eigenfunctions of the linearized Eq. (1) with the zero eigenvalue, that correspond to the symmetry of (1) to planar translations. One could attempt, therefore, to compute $\mathbf{v}$ by applying the solvability condition $\mathrm{Re}\int \nabla\bar{u}\psi\, d^2\mathbf{x} = 0$, where the overbar marks the complex conjugate, and $\psi = (\mathbf{v} + 2i\mathbf{A})\cdot\nabla u_0$ is the inhomogeneity in the first-order expansion of (5). The attempt fails, since the integral diverges logarithmically at large distances. The same problem arises when one adopts an approach[4] that utilizes the existence of a potential $\mathcal{L}$ generating the evolution equation $\dot{u} = -\delta\mathcal{L}/\delta\bar{u}$. We note in parentheses that the difficulty disappears when (1) is replaced by the Ginzburg-Landau equation in its original (conservative) form,[7] with $i\dot{u}$ rather than $\dot{u}$ in the left-hand side; the velocity can then be just gauged away by setting $\mathbf{A} = -\frac{1}{2}\mathbf{v}$, and the vortex remains symmetric as it moves along a constant phase gradient.

Using a nonintegrable eigenfunction to derive a solvability condition is, of course, a mathematically dubious procedure. One can argue, however, that the logarithmic divergence should be viewed with caution, since the zero-order phase gradient decays to $O(\epsilon)$, i.e., becomes comparable with the first-order correction at $r=O(\epsilon^{-1})$. Introducing a long-scale cutoff at this distance and identifying the expansion parameter ϵ with $v=|\mathbf{v}|$ gives immediately the mobility relationship $|\mathbf{A}| \propto v \ln v$; it is also easy to see that the vortex migrates in the direction normal to the imposed phase gradient. Bodenschatz, Pesch, and Kramer[5] have arrived at this result with more precision and ingenuity by using in the solvability condition the zero-order solution only up to an intermediate distance $1 \ll r_0 \ll v^{-1}$, and replacing it onwards by the asymptotic solution with $\rho=1$ and θ verifying the phase equation, written in the coordinate frame comoving with a vortex steadily propagating along the y axis as

$$v\theta_y+\theta_{xx}+\theta_{yy}=0\,. \tag{6}$$

This procedure ensures convergence, but still lacks mathematical justification. The solvability condition does not operate here in a proper context of a regular perturbation scheme, but attempts to incorporate, together with the zero-order solution, also a long-distance tail of the very first-order solution whose existence it is supposed to ensure. We shall seek therefore an alternative approach based on a more reliable method of matched asymptotic expansions.[8] This method, though being somewhat more laborous for the particular problem in question, does not hinge upon the Hermiticity of the linearized equation that makes the conjugate eigenfunction readily available, or on the existence of a potential.

We start with Eqs. (2) and (3), assuming, without loss of generality, that the defect propagates along the y axis, and rewriting them in the comoving frame as

$$v\rho_y+\nabla^2\rho+(1-|\nabla\theta|^2-\rho^2)\rho=0\,, \tag{7}$$

$$v\theta_y+\nabla^2\theta+2\rho^{-1}\nabla\rho\cdot\nabla\theta=0\,. \tag{8}$$

Let

$$\rho=\rho_0(r)+v\psi(r)\sin\phi,\quad \theta=\phi+v\chi(r)\cos\phi\,. \tag{9}$$

Then the first-order equations read

$$\psi''+r^{-1}\psi'+(1-2r^{-2}-3\rho_0^2)\psi+2r^{-2}\rho_0\chi+\rho_0'=0\,. \tag{10}$$

$$\chi''+r^{-1}\chi'-r^{-2}\chi+2\rho_0^{-1}(\rho_0'\chi'+r^{-2}\psi)+r^{-1}=0\,. \tag{11}$$

Note that we use ungauged equations; the weak external phase gradient driving the vortex with the prescribed velocity will be further determined by matching conditions. The cosine function in the expression for θ indicates that the computed gradient will be directed along the x axis; trying first-order functions with the sine and cosine switched between ρ and θ would just yield equations lacking a forcing term.

The asymptotic expressions for the phase function χ at $r\to\infty$ (the outer limit of the inner solution) following from Eq. (11) is

$$\chi=br-\tfrac{1}{2}r\ln r+\delta\,, \tag{12}$$

$$\delta=-\tfrac{1}{2}r^{-1}\ln^2 r+(2b-1)r^{-1}\ln r+cr^{-2}+\cdots\,, \tag{13}$$

where b,c are indefinite constants that have to be fitted to the numerically computed solution. We integrated Eqs. (10) and (11) numerically with the initial conditions $\chi(0)=\psi(0)=\psi'(0)=0$, that are required to remove the singularity at $r\to 0$. The numerical integration cannot actually commence at the origin where Eqs. (10) and (11) are singular. Instead, the power expansion has to be used to advance to an appropriate starting point; this expansion gives, in particular, $-\psi''(0)=\rho_0'(0)=a$. The value of $\chi'(0)$ has to be adjusted iteratively in such a way as to approach the required asymptotic function (12). There is a unique value of $\chi'(0)$ insuring the convergence, and it has to be fine tuned with an increased precision when one wishes to extend the integration interval to larger r. The solution eventually diverges due to numerical instabilities.

Though only two leading terms in (12) are required for matching, more have been added for practical reasons stemming from the fact that, in contrast to the classical method, the matching is assisted by solving the inner equations (10) and (11) numerically. The asymptotic series is expressed, in fact, in powers of $r^{-2}\ln r$, and, say, at $r\sim 10$, the error still runs in a few percentage points if only the two leading terms are used. Rather than extending the integration to longer intervals, one can keep the limits short, and use the full expression of (12) and (13) with two numerically fitted constants. The numerical solution (solid line) and the matching asymptotic solutions (dashed lines) for the phase and amplitude functions are shown in Fig. 1. The value of the numerical constant obtained by matching is $b=0.309$.

This constant has to be further connected to the extrinsic phase gradient by solving the outer phase Eq. (6), which is rewritten using extended variables $X=vx$, $Y=vy$ as

$$\theta_Y+\theta_{XX}+\theta_{YY}=0\,, \tag{14}$$

and taking the inner limit of the outer solution at $R=vr\to 0$.

In the presence of defects, the phase is not defined globally as a continuous univalued function. It is convenient, therefore, to replace it in the outer region by the dual function $\Phi(X,Y)$ that satisfies[9]

$$\Phi_Y+\Phi_{XX}+\Phi_{YY}=2\pi\delta(X)\delta(Y)\,, \tag{15}$$

$$\Phi_Y+\Phi=-\theta_X,\quad \Phi_X=\theta_Y\,. \tag{16}$$

Using Eq. (16) in the contour integral $\oint\nabla\theta\cdot dl$ and applying the Gaussian theorem we see that this integral equals 2π as required, by virtue of Eq. (15). It is easily checked that Eq. (14) is satisfied automatically due to (16), while the integrability of Eq. (16) is insured by Eq. (15). The solution of Eq. (15) in the infinite region is

$$\Phi=-\exp(-\tfrac{1}{2}R\sin\phi)K_0(\tfrac{1}{2}R)\,, \tag{17}$$

where K_0 is a modified Bessel function. The components

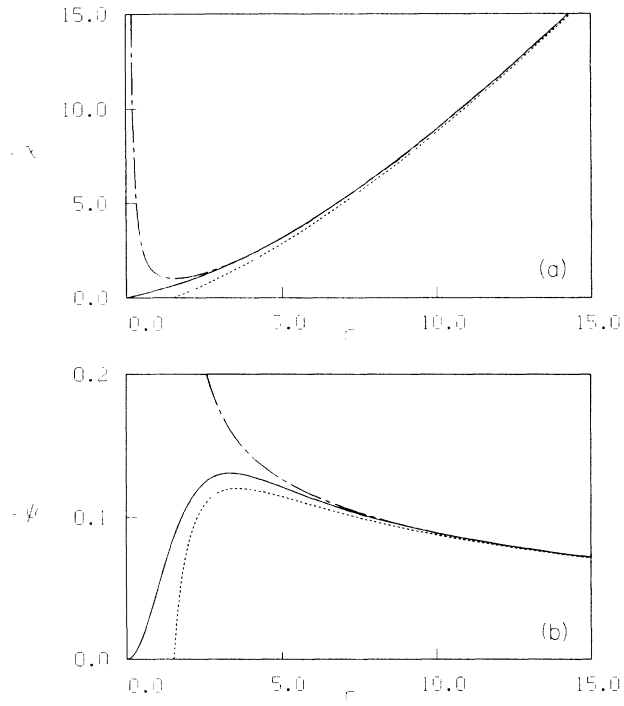

FIG. 1. The numerical solution for (a) the phase and (b) amplitude first-order functions (solid line), and matching asymptotes using Eq. (12) with one matching constant (---) and Eqs. (12) and (13) with two matching constants (-·-·-).

of the wave vector read from (16) and (17) are

$$\theta_X = A + \tfrac{1}{2}\exp(-\tfrac{1}{2}R\sin\phi)[K_0(\tfrac{1}{2}R) - \sin\phi K_1(\tfrac{1}{2}R)] ,$$
$$\theta_Y = \tfrac{1}{2}\exp(-\tfrac{1}{2}R\sin\phi)\cos\phi K_1(\tfrac{1}{2}R) , \tag{18}$$

where an arbitrary constant phase gradient A directed along the X axis (representing the external phase field) has been added. Using the asymptotics of Bessel functions at $R \rightarrow 0$, $K_0(z) = -\ln(\frac{1}{2}z) - C$ (where $C = 0.577\ldots$ is the Euler constant), $K_1(z) = z^{-1}$, and reverting to the inner variables yields

$$\theta_x = -r^{-1}\sin\phi + \tfrac{1}{2}v[2A - \ln(\tfrac{1}{4}vr) - C + \sin^2\phi] ,$$
$$\theta_y = r^{-1}\cos\phi - \tfrac{1}{2}v\cos\phi\sin\phi . \tag{19}$$

These expressions, representing the inner limit of the outer solution, have to be matched with the outer limit of the inner solution read from Eqs. (9) and (13). The inner expression for θ_y is identical to the above, and the expression for θ_x reads

$$\theta_x = -r^{-1}\sin\phi + \tfrac{1}{2}v(2b - \ln r - \cos^2\phi) . \tag{20}$$

Matching both expressions yields

$$A = \tfrac{1}{2}\ln(v/v_0) , \tag{21}$$

where

$$v_0 = 4\exp(1 - C - 2b) = 3.29 .$$

The result coincides with that computed by Bodenschatz *et al.*[5] using the integral solvability condition. Note that the analytical form of the mobility relationship is universal, and only the number v_0 depends on the numerical integration in the core region. If the algebraic nonlinearity in Eq. (1) is replaced by another smooth function vanishing at $|u| = 0$ and 1, nothing changes except this numerical value.

The above computation can be extended in a straightforward way to the complex Ginzburg-Landau equation (CGLE)

$$\dot{u} = (1 + i\eta)\nabla^2 u + (1 + i\mu)u - (1 + i\nu)|u|^2 u , \tag{22}$$

provided it is restricted to the case of a vanishing group velocity $\nu = \eta$. Using $u = \rho e^{i\theta}$ and transforming to the comoving frame, we rewrite Eq. (22) in the form [cf. Eqs. (7) and (8)]

$$\hat{v}(\rho_y + \eta\rho\theta_y) + \nabla^2\rho + (1 - |\nabla\theta|^2 - \rho^2)\rho = 0 , \tag{23}$$
$$\hat{v}(\theta_y - \eta\rho^{-1}\rho_y) + \nabla^2\theta + 2\rho^{-1}\nabla\rho\cdot\nabla\theta = 0 , \tag{24}$$

where $\hat{v} = v/(1 + \eta^2)$. The zero-order solution remains unchanged; furthermore, the first-order terms proportional to η in (23) and (24) can be removed by the gauge transformation $u \rightarrow ue^{-iBy}$, or by $\theta \rightarrow \theta + By$ with $B = \frac{1}{2}\eta\hat{v}$. This means that in the complex case the motion across the phase gradient [governed, as in the real case, by the mobility relationship (21), modified by changing v to $\hat{v}$] is complemented by the motion along the phase gradient.

Suppose that the external phase gradient at the core of the defect is due to a fixed defect with the charge $n = \pm 1$ placed at a distance $r = O(\epsilon^{-1})$ in the positive y direction. Then the driving gradient is directed along the x axis and equals $\hat{A} = vA = -nr^{-1}\sin\phi = nr^{-1} = O(\epsilon)$. According to the mobility relationship (21), the induced velocity is defined implicitly by $v\ln(v_0/v) = -2n/r$. The mobile defect (supposed to be positively charged) is repelled by the like charge ($n = 1$), and is attracted to the opposite charge ($n = -1$). In the case of the CGLE, the velocity along the y axis is reduced by the factor $1 + \eta^2$, and a circumferential velocity component inversely proportional to r is added.

The realistic case when both defects are moving presents a far more difficult problem, since the phase field induced by each defect at the core of another one depends

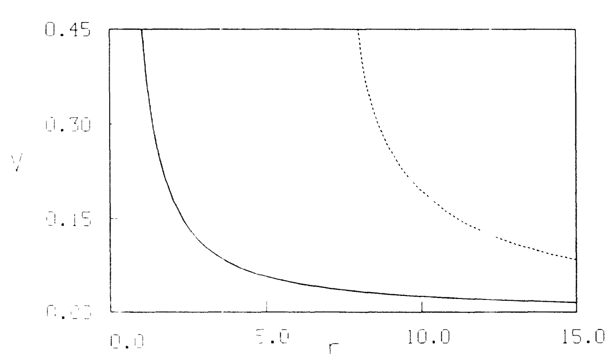

FIG. 2. Quasistationary velocities for attracting (solid line) and repelling (dashed line) vortices.

on its velocity and, strictly speaking, on its entire past history of accelerations. A marked asymmetry of the phase field (19), that decays at large distances exponentially ahead and algebraically behind the migrating vortex, accounts for the asymmetry in interaction of defects of like and opposite charges. A rough estimate can be given by assuming that the quasistationary phase field corresponding to the instantaneous velocity is observed at each location. The migration velocity as the function of the instantaneous separation between the vortex cores is obtained then by computing with the help of Eq. (19) the value of the phase gradient due to one of the vortices at the core of another one, and using it in the mobility relationship (21). For oppositely charged defects moving towards each other the phase gradient generated by one of the defects at the location of the other defect is $\theta_x(r, \frac{1}{2}\pi)$, while for like-charged ones, moving apart, it is $\theta_x(r, -\frac{1}{2}\pi)$. Quasistationary velocities in both cases are shown in Fig. 2. A stable quasi-stationary solution exists only beyond a certain separation; at shorter distances velocities are large, and the entire approximation scheme breaks down, so that results should be discarded when the curves in Fig. 2 start to rise sharply. As expected, the velocities of mutually repelling vortices are much larger, at a comparable distance, than of attracting ones.

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