RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 42, NUMBER 4 15 AUGUST 1990

Self-organized criticality in sandpiles: Nature of the critical phenomenon

J. M. Carlson
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106

J. T. Chayes
Department of Mathematics, University of California, Los Angeles, California 90024

E. R. Grannan
AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, New Jersey 07974

G. H. Swindle
Department of Mathematics, University of California, Los Angeles, California 90024
(Received 26 April 1990)

We identify the avalanche boundaries in a class of sandpile models, and derive their dynamics, which consist of birth, death, and coalescence events. We then introduce and study a simple "trough" model based on these boundary dynamics. We prove that the trough model has a phase transition, and that at and below the critical point, troughs become extinct in the thermodynamic limit. Numerical results suggest that sandpile models are associated with such a critical point, and that the observed scaling behavior results from finite-size effects.

In nature, many dissipative dynamical systems are seen to exhibit scale-invariant characteristics that are indicative of traditional equilibrium systems at a critical point. Recently, a series of cellular automata, referred to as "sandpiles," were introduced,[1,2] which have generated interest because they seem to exhibit "critical" fluctuations as the result of a dynamical threshold instability, rather than the tuning of a parameter. The term "self-organized criticality" was coined to describe this behavior. It was suggested that events of a wide range of sizes persist in these and related systems because the attractor of the dynamics is minimally stable.[3]

The purpose of this Rapid Communication is to elucidate the nature of the scaling behavior in the sandpile models. We begin by proving that in the sandpile models there exists a set of domain walls, which we call *troughs,* which bound the regions that can experience avalanches. Moreover, we show that the dynamics of these troughs is governed by a simple set of rules involving birth, death, and coalescence events. We then introduce the *trough model,* which is a caricature of the sandpile models based on the dynamics of the domain walls. We prove that the trough model has a phase transition with the density of the troughs as an order parameter, and that, in the thermodynamic limit, the trough density goes to zero at the transition point. Finally, we show that the observed scaling behavior at the critical point is a consequence of finite-size effects.

The relation to the original sandpile models is as follows. First, we claim that sandpiles on a torus also undergo a transition at which the domain-wall density tends to zero. Second, our simulations of sandpile models strongly suggest that the behavior of the domain walls is related to the trough model *at its critical point.* Of course, the precise values of the critical exponents for the trough model differ from those of the specific sandpile model studied here, just as different sandpile models belong to different universality classes. Nevertheless, the qualitative features of the phase transitions are similar.

For this discussion, we will consider the simple one-dimensional "limited local" sandpile models[2] with the following rules. Two positive integers z_c and n are prescribed. The system consists of L consecutive sites: $i=1$ to L. The height $h(i)$ is the number of grains of sand on site i, and the slope is $z(i)=h(i)-h(i+1)$. We assume the system is *closed* at the left boundary ($i=0$), and *open* at the right ($i=L+1$). Sand is dropped one grain at a time onto a randomly chosen site. The system is then examined, and if any slope exceeds the threshold value z_c, n grains fall onto the next site to the right. As a result

$$\begin{aligned} z(i) &\to z(i)-2n\,, \\ z(i-1) &\to z(i-1)+n\,, \\ z(i+1) &\to z(i+1)+n\,. \end{aligned} \tag{1}$$

Equation (1) is somewhat modified at the boundaries, e.g., when $i=L$, $z(i)$ decreases by only n and n grains fall off the pile. Equation (1) is iterated until all of the sites are below threshold, at which time another grain is added.

In the sandpile model described above, we define a *trough* to be any site i for which the slope $z(i)$ is n or more grains below threshold. We now prove that avalanches are terminated at troughs, and derive trough dynamics. To this end we relabel the slopes by $-\infty,\ldots,z_c-n\to t$ (trough), $-\infty,\ldots,z_c\to s$ (stable), $z_c+1-2n,\ldots,z_c-n\to T$ (high trough), $z_c+1-n,\ldots,z_c\to m$ (marginally stable), $z_c+1,\ldots,z_c+n\to F$ (falling). The following theorem describes the resulting configuration after an avalanche [i.e., the iteration of (1)].

Theorem. Suppose that a grain of sand is added so that $z(i)>z_c$. Denote the positions of the first troughs to the left- and right-hand side of i by i_L and i_R, respectively, and define $i_C \equiv i_R - i + i_L$ to be the reflection of i in the interval $[i_L, i_R]$.

Case 1. When $i_R \le L$ the avalanche is confined to the interior of the system, and the final slope configuration after the avalanche is unchanged except for

$$\begin{aligned} z(i_L) &\to z(i_L)+n \quad (t \to m \text{ or } t)\,, \\ z(i) &\to z(i)-n \quad (F \to m)\,, \\ z(i_C) &\to z(i_C)-n \quad (m \to T)\,, \\ z(i_R) &\to z(i_R)+n \quad (t \to m \text{ or } t)\,, \end{aligned} \tag{2}$$

where sites i_L and i_R become marginally stable unless they were deep troughs ($z \le z_c - 2n$) initially. Equation (2) also applies when the avalanche region extends to the left-hand side boundary omitting the change at $z(i_L \equiv 0)$.

Case 2. When the avalanche extends to the right-hand side boundary the above holds, with the exception that $z(i_C)$ and $z(i_R \equiv L+1)$ are unchanged, and $D = n(i - i_L)$ grains of sand fall off the pile.

Remark. The sandpile models have traditionally been simulated in the limit of adding sand at an infinitesimal rate, so that all avalanche durations are less than the time between them. This theorem shows explicitly how the dynamics may alternately be viewed such that sand is added at a fixed finite rate, but the avalanches are instantaneous and long range, since they may extend to a trough arbitrarily far away.

Summary of Proof. Equation (1) implies that the evolution of a site depends only on its value and that of its two neighbors. For Case 1, it can be verified that the only nonstatic configurations which can evolve from the initial configuration, which is of the form $tmm\ldots mmFmm\ldots mmt$, are

$$(1)s\underline{F}s \to \underline{T},\quad (2)s\underline{m}F \to \underline{F},\quad (3)F\underline{T}F \to \underline{F}\,,$$

$$(4)s\underline{t}F \to \underline{s},\ \text{and specifically}\ (4a)s\underline{T}F \to \underline{m}\,,$$

where we have shown what the middle site (indicated by an underline) becomes on the next time step. We establish the results of the theorem by tracing the evolution of the avalanche. The disturbance propagates out, reflects at the troughs at i_L and i_R, adding slope to these sites, and ends by coalescing and depositing a trough at i_C. That sites other than i, i_L, i_R, and i_C are unchanged follows from the fact that sites change by integer multiples of n. ■

This theorem implies that during avalanches troughs are removed and added at precise locations. However, in terms of the configuration before the initiating grain of sand is dropped, two cases arise. In the first case, the left trough exists prior to the addition of the initiating sand, and the trough coalescence described in the theorem results in the net removal of one trough. We call this a *coalescence* event. In the second case, the initiating grain of sand results in the formation of both a falling site and a trough immediately to the left, so that compared to the initial configuration there is no net change in the number of troughs; the right trough simply moves one step to the left. We call this a *slide* event. In addition to causing avalanches, dropped sand can give rise to a trough, which we refer to as a birth event, or if the dropped grain lands on a trough it can remove it, which we describe as a death event.

We proceed to now define the trough model, which is a stochastic process designed to capture the essential behavior of troughs just described in a more tractable two-state system. Slide events and the phenomena of deep troughs, which require more than one death or coalescence event to be removed, are ignored in this formulation. We also ignore certain short-range interactions which arise because potential birth sites for troughs actually involve pairs of sites. None of these simplifications is believed to alter the essential features of the system.

The trough model is defined on the one-dimensional integer lattice. To each site is associated a 0 (vacancy) or a 1 (trough). The rules are the following: (i) *Birth*: vacant sites are filled at rate λ; (ii) *Death*: occupied sites become vacant at rate δ; (iii) *Coalescence*: at each vacant site i at rate 1, the nearest trough to the left- and to the right-hand side of i are removed, and i is filled. Unlike the sandpile model, the trough model has as tuning parameters the birth and death rates λ and δ.

We now consider the infinite system,[4] in which the order parameter is the equilibrium density ρ of troughs. Below we prove that there is a phase transition at $\lambda = 1$.

Theorem. Consider $\delta > 0$. When $\lambda \le 1$ the system goes extinct in the sense that in any translation-invariant equilibrium measure the density is $\rho = 0$. Conversely, when $\lambda > 1$ the system survives and the density is given by

$$\rho = \frac{\lambda - 1}{\lambda + \delta - 1} > 0\,. \tag{3}$$

Remark. Note that the transition becomes discontinuous as $\delta \to 0$, and that for each $\delta > 0$ the order parameter exponent is $\beta = 1$.

Proof. Denote a general configuration by ξ and the configurational probability distribution of the system at time t by P_t. Let $\rho_t = P_t(\xi_t(i) = 1)$, and define l to be the event that a trough is at site i, and the next trough to the right is at site $i + l$. We have

$$\frac{d\rho_t}{dt} = \lambda(1-\rho_t) + (1-\rho_t) - \delta\rho_t - 2\sum_{l=1}^{\infty}(l-1)P_t(l)\,. \tag{4}$$

The following identities are easily verified: $\sum_{l=1}^{\infty} P_t(l) = \rho_t$ and $\sum_{l=1}^{\infty} l P_t(l) = 1$. When $\lambda > 1$, setting $d\rho_t/dt = 0$ in Eq. (4) and using these identities yields the equilibrium density given in Eq. (3). Furthermore, Eq. (4) implies that if the density is ever greater (less) than ρ, it increases (decreases) monotonically to the equilibrium value. It is also clear that if $\lambda \le 1$, then $d\rho/dt < 0$ whenever $\rho > 0$, implying extinction. Thus the critical point is $\lambda_c = 1$, where troughs become extinct. ■

Remark. It is also possible to use the differential equation for $P_t(l)$ [analogous to (4) for ρ_t] with convexity arguments to establish bounds on the moments of $P(l)$ that exist in equilibrium for given values of λ.

Next we direct our attention to the trough model on finite sets, where the single-site density will clearly be positive for any positive birth rate. Nonetheless, the density of troughs $\rho(L)$ in a system of size L exhibits markedly different asymptotics in the different regimes. For convenience we consider the case of open boundary conditions: a coalescence event occurring beyond the left- or right-most trough causes the trough to vanish (corresponding to the open boundary at $i=L$ in the sandpile model). Our numerical simulations verify that the scaling behavior does not depend on whether the boundaries are closed or open. The open boundaries simplify the analysis, because the rate at which a trough is added or removed depends only on the number of troughs in the system.

Theorem. In a finite system of size L with open boundary conditions, for $\delta>0$ the density of troughs $\rho(L)$ is asymptotically

$$\rho(L)\sim\begin{cases}L^{-1} & \text{if } \lambda<1\\ L^{-1/2} & \text{if } \lambda=1\\ C_\lambda & \text{if } \lambda>1\,.\end{cases} \tag{5}$$

Sketch of Proof. Denoting the total number of troughs in the system by n $[\rho(L)=n/L]$, the transition rates are (i) $n\rightarrow n+1$ at rate $(L-n)\lambda$, (ii) $n\rightarrow n-1$ at rate $n\delta+(L-n)$. From these rates it is possible to calculate the equilibrium distribution of the number of troughs. Standard asymptotic methods applied to the resulting expression for the density yield (5). ∎

The theorem implies that when $\lambda<1$ the expected number of troughs is bounded uniformly for all sizes L, so that the characteristic separation is of order L, while for $\lambda>1$ the characteristic distance between troughs remains finite. The only possible regime with nontrivial scaling is $\lambda=1$, where the number of troughs is diverging but the density is vanishing in the limit.

We will now discuss the sandpile models in the context of the above results on the trough model. Numerical results imply that the domain walls in the sandpile models are at a critical point analogous to $\lambda=1$ in the trough model. Figure 1 illustrates that in both cases the density of troughs $\rho(L)$ tends to zero as $L^{-\theta}$ with $0<\theta<1$. Note that θ prescribes the proper scaling for the continuum limit. In Fig. 2 we compare the distribution of amounts of sand falling off the pile with the corresponding quantity for the trough model. The multifractal scaling curves are qualitatively very similar. Since the exponents and the scaling functions are different for the two models, they are not in the same universality class—this is expected since we have neglected some dynamics (e.g., slide events) in constructing the trough model.

The trough model has a tunable parameter λ, which the sandpile models seem to lack. However, if we consider the sandpile models on a finite set with periodic boundary conditions, it is clear from Eq. (1) that total slope is conserved. This allows us to fix the average slope s. It is found that there is a critical value s_c so that if $s>s_c$ the domain walls go extinct, while if $s<s_c$ then they persist in large systems. Furthermore, simulations indicate that as the size of the original sandpile models diverges, the average slope converges to s_c. This convergence of the open systems to the critical point of the closed systems can be explained by noting that, if the mean slope is too great, troughs begin to die (avalanches occur) reducing the slope, while if the slope is too small, the trough density becomes nonzero, and large avalanches become rare causing the slope to increase. (See Ref. 2 for similar arguments involving conservation laws.)

Conclusions. The dynamics of a certain class of sandpile models has been shown to be controlled by special sites, called troughs, which determine the avalanche edges. This sensitivity to troughs may explain the lack of scaling behavior in laboratory experiments on real sand,[5] where we expect inertial effects would allow large cas-

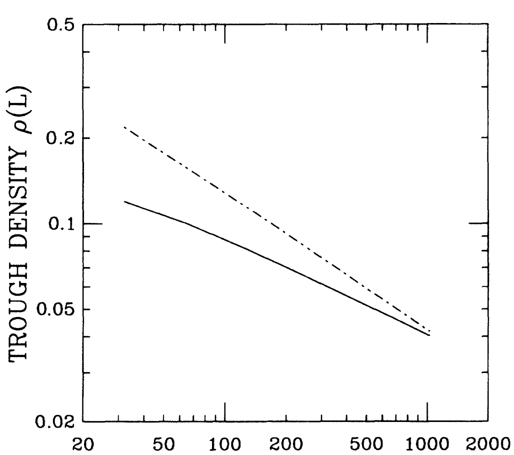

FIG. 1. Trough density vs system size for the sandpile model with $z_c=2$ and $n=2$ (solid line) and the trough model at $\lambda=1$ and $\delta=1$ (dashed line). In each case for large L, $\Delta(L)\propto L^{-\theta}$, where for the trough model $\theta=0.5$, and for the sandpile model $\theta\approx0.34$.

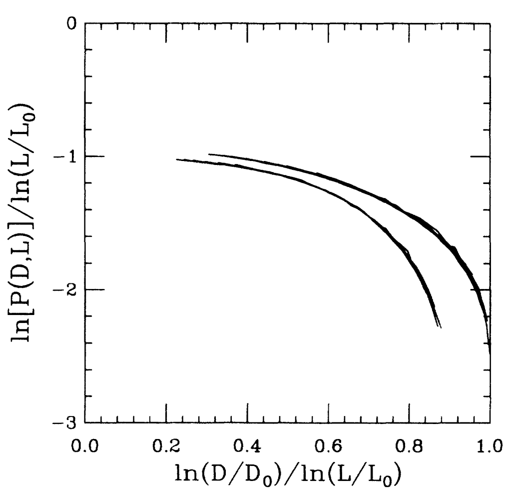

FIG. 2. Multifractal fits to the distributions $F(D,L)$ of amounts of sand $D=n(i-i_L)$ falling off the pile as the result of a single avalanche for system sizes $L=32$–2048. Results for the sandpile model (lower curves) are qualitatively similar to the corresponding results for the trough model at $\lambda=\delta=1$ (upper curves). A standard multifractal fitting form is used (Ref. 2), with $D_0=0.54$ and $L_0=0.29$ for the trough model and $D_0=1.1$ and $L_0=0.30$ for the sandpile model.

cades to overwhelm troughs. Another signature of the lack of inertial effects in the sandpile model may be related to the fact that the net effect of an avalanche on the system is relatively minor—involving the change in slope of, at most, four sites. This is significantly different from the model studied in Ref. 6 where scaling behavior arises as a result of inertial effects and slipping instabilities, resulting in the amplification of spatial irregularities during an event.

We have also established a relation between observations made about the domain walls in a class of one-dimensional sandpile models and the critical behavior of troughs. We suspect that other versions of sandpile models and generalizations to higher dimensions can be identified with variants of the trough model. The dynamic selection of a critical point is vaguely reminiscent of the way in which invasion percolation, a dynamic growth process, selects the critical point of ordinary static site percolation.[7,8] On the other hand, our results are quite different from those obtained in Ref. 9 for a driven diffusion equation, which by construction has no characteristic lengths or times. We conclude that scaling behavior of the distribution of avalanches in the sandpile models results from the observation of a critical system on a finite set, which would, in fact, be extinct on the infinite lattice.

We would like to thank J. Langer, T. M. Liggett, and J. P. Sethna for useful discussions. The work of J.M.C. was supported by Department of Energy Grant No. DE-FG03-84ER45108 and National Science Foundation Grant No. PHY82-17853, supplemented by funds from NASA. The work of J.T.C. was supported by National Science Foundation Grant No. DMS-88-06552. The work of G.H.S. was supported by National Science Foundation Grant No. DMS-89-02152.

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