Stimulated two-photon free-free transitions in a Coulomb potential: Formalism

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We present a calculation of the cross sections for two-photon free-free transitions of an electron colliding with a Coulomb center of force. The calculation is based on second-order perturbation theory, in the nonrelativistic dipole approximation. The matrix elements for absorption and emission were integrated analytically in momentum space, following a method developed earlier by one of us. This makes use of the Schwinger integral representation for the Coulomb Green's function. The result was expressed in terms of integrals over hypergeometric functions of the Gauss type. Simple limiting forms of these complicated expressions were found in the first Born approximation, and at low and high photon energies. The results derived agree with those obtained by direct calculations done for these limits. Finally, concluding remarks on the analytic part of our work are made. The numerical computation of the two-photon free-free transition cross sections for absorption and emission in various geometries is planned to be presented at a later time.

I. INTRODUCTION

An electron colliding with an atom in a radiation field can absorb or emit photons, thereby undergoing stimulated "free-free" transitions (FFT's). In weak fields, as those of conventional optical sources or of astrophysical nature, only one-photon FFT's have significant probability of occurrence. In intense laser fields, on the other hand, of the kind applied for the heating of plasmas, for example, multiphoton transitions can also occur. We shall be interested in the following in two-photon transitions at moderately high intensities, such as can be covered by the formulas of second-order perturbation theory. (For various aspects of FFT's see the reviews of Refs. 1–3.)

Stimulated FFT's are closely related to bremsstrahlung processes, whereby an electron emits spontaneously one or more photons in a collision, in the absence of any preexisting radiation. In particular, directly related to the two-photon FFT's is the two-photon bremsstrahlung, an experimentally very elusive process which has attracted interest lately.

The theoretical methods for calculating FFT cross sections can be grouped, broadly speaking, into perturbative or nonperturbative with respect to the intensity of the radiation.

Perturbation theory to lowest nonvanishing order yields results valid at sufficiently small radiation intensities and all frequencies ω (except for the limit of vanishing ω). The vast majority of the FFT work done refers to one-photon transitions, the only ones of practical interest in the past. The archetypal result is due to Sommerfeld, who considered the case of a Coulomb center of force.[4] As the one-photon perturbation theory matrix element is simple enough, one could later take into account the complexities of the many-electron structure of the atom.[1]

It is only relatively recently that we have reported the first perturbation theory FFT calculation for more than one photon.[5] It refers to two-photon transitions in a Coulomb potential, and represents the analog of the Sommerfeld one-photon result. We have computed the differential cross section analytically as far as possible, and then resorted to numerical evaluation. Only a special scattering geometry was considered, namely the case when the incident electron momentum is perpendicular to the linearly polarized electric field. This calculation was followed by one for two-photon bremsstrahlung in a Coulomb potential[6] (see also Ref. 7), in view of the experimental interest.

More recently another perturbation theory calculation for two-photon free-free absorption was performed by Chang and Robinson,[8] using a (short-range) model potential for the atomic field. Total absorption cross sections were obtained for He, C, Ne, and Ar. This ends the list of calculations for multiphoton FFT's in perturbation theory. Their scarcity is rather remarkable in view of the numerous efforts done in the related problem of multiphoton ionization from atoms (e.g., see Ref. 9), and may be due to the lack of experimental incentive.

The low-frequency limit of the FFT matrix elements has received considerable attention.[1] The theoretical interest of the problem stems from the realization by Low that the one-photon transition amplitudes are closely related to those for radiationless elastic scattering.[10] His theorem is valid specifically for short-range potentials, and takes a weaker form for a Coulomb or Coulomb-tail potential. In practice this means that the computation of the FFT cross sections at low frequencies (precisely the case of interest for astrophysics) is reduced to the simpler one for elastic scattering. The extension of the Low theorem to two-photon bremsstrahlung in short-range

potentials was obtained and studied by Rosenberg.[11,12] In Refs. 5 and 6 we found that for a Coulomb potential his result again takes a weaker form.

At high laser intensities the formulas of lowest-order perturbation theory become inappropriate because of the growing importance of the higher-order corrections. *Nonperturbative theories* have been developed to encompass this difficulty. While, per definition, valid to all orders in the intensity, they are restricted with respect to other parameters (frequency, electron energy). The complexities of the nonperturbative approach are such that these theories have been developed mainly for one-electron problems. Among them we quote the ones by Bunkin and Fedorov for FFT's in the Born (high-electron energy) approximation,[13] by Kroll and Watson for the low-frequency limit[14] (a case which cannot be covered by lowest-order perturbation theory), and by Gavrila and Kaminski for the high-frequency limit.[15] All these theories give expressions for the multiphoton FFT amplitudes, which, to lowest order in the intensity, agree with the corresponding formulas of perturbation theory when taken in the appropriate limit.

Controlled experimental information on FFT's comes only from *beam experiments,* which are very difficult to do.[2] These were all performed at the infrared frequency of the CO_2 laser (0.117 eV photon energy), on noble gas atoms, and incident electron energy between 10 and 20 eV. The experiments at lower intensity (about 10^4-10^5 W/cm^2) were concerned only with one-photon transitions (Andrick and collaborators, see Refs. 2 and 16, and references therein). At higher intensities (about 10^8 W/cm^2), Weingartshofer and collaborators were the first to demonstrate the existence of multiphoton FFT's.[17] More recently, Wallbank, Holmes, and Weingartshofer,[18] in a continuation of this work, have begun quantitative studies on the magnitude of the cross sections. Special consideration was given to two-photon transitions. The theoretical interpretation of their results is seriously limited, however, by the unknown structure of the laser pulse.[19]

With the present paper we begin to give a complete account of our results on two-photon FFT's. Here we shall describe the analytic formalism. Later we shall present and discuss our numerical results for absorption and emission in a variety of scattering geometries (not considered in Ref. 5). We hope that our work will represent an incentive for more experiments on this process.

Section II of this paper contains the integration of the perturbation theory two-photon matrix elements for the Coulomb potential. We have adopted an integration method developed in a different context by one of us many years ago.[20] The matrix element is finally expressed in terms of rotationally invariant amplitudes, each of which contains integrals over Gauss hypergeometric functions ${}_2F_1$. This is as far as the analytic calculation can go. In Sec. III we discuss this result in certain limits for the parameters photon energy and electron energies. In these limits it reduces to simple forms, easily understood physically. It was thus possible to check that the general result satisfies all limiting forms of the cross sections which could be derived independently. Section IV presents some concluding remarks to the analytic part of our work. The Appendix contains the calculation of an auxiliary integral.

II. MATRIX ELEMENTS

The differential cross section for the scattering of an electron with the absorption (+) or emission (−) of two photons from or into the laser field can be written[21]

$$\frac{d\sigma^{(\pm)}}{d\Omega}=\frac{p_f}{p_i}|f^{(\pm)}(\theta)|^2 , \tag{1}$$

where $f^{(\pm)}(\theta)$ is the scattering matrix element. In the case of absorption, second-order perturbation theory in the dipole approximation yields[21]

$$f^{(+)}(\theta)=\frac{1}{8\pi}I\omega^{-2}\langle u^{\rm in}_{\mathbf{p}_f}|(\mathbf{e}\cdot\mathbf{P})G(\Omega_+)(\mathbf{e}\cdot\mathbf{P})|u^{\rm out}_{\mathbf{p}_i}\rangle . \tag{2a}$$

Similarly, for emission

$$f^{(-)}(\theta)=\frac{1}{8\pi}I\omega^{-2}\langle u^{\rm in}_{\mathbf{p}_f}|(\mathbf{e}^*\cdot\mathbf{P})G(\Omega_-)(\mathbf{e}^*\cdot\mathbf{P})|u^{\rm out}_{\mathbf{p}_i}\rangle . \tag{2b}$$

We have denoted by $\mathbf{p}_i$ and $\mathbf{p}_f$ the momenta corresponding to the initial and final states (energies E_i and E_f), by θ the scattering angle between them, and by $u^{\rm out}_{\mathbf{p}_i}$ and $u^{\rm in}_{\mathbf{p}_f}$ the associated continuum Coulomb wave functions with outgoing or incoming spherical wave behavior, respectively, normalized to unit asymptotic amplitude. $\mathbf{P}$ is the electron momentum operator and $G(\Omega)$ is the Coulomb Green's operator for energy parameter Ω. I is the time-averaged intensity of the laser field, ω is the photon energy, and $\mathbf{e}$ its polarization vector (complex for arbitary polarization, and such that $\mathbf{e}\cdot\mathbf{e}^*=1$).

The two values of Ω appearing in Eqs. (2a) and (2b) are

$$\Omega_\pm=E_i\pm\omega+i\epsilon , \tag{3}$$

where $\epsilon>0$, infinitesimal, and energy conservation requires

$$E_f=E_i\pm2\omega . \tag{4}$$

Further, we take $E_\sigma=p_\sigma^2/2$, $(\sigma=i,f)$.

Our formulas are written in *Z-scaled atomic units,* i.e, lengths are expressed in Z^{-1}a.u., momenta in Za.u., energies in Z^2a.u., and the time-averaged intensity I in Z^6a.u. (the a.u. of intensity is taken to be 3.51×10^{16} W/cm^2).[22]

The two matrix elements Eqs. (2a) and (2b) are represented diagrammatically in Figs. 1(a) and 1(b), respectively, according to rules described in Ref. 23.

In the following $f^{(\pm)}$ will be integrated in momentum space. From Eq. (2a), $f^{(+)}$ can be written as

$$f^{(+)}=\frac{1}{8\pi}I\omega^{-2}\sum_{\alpha,\beta=1}^{3}e_\alpha e_\beta\Pi_{\alpha\beta}(\Omega_+) , \tag{5}$$

where

$$\Pi_{\alpha\beta}(\Omega)=\int\int u^{\rm in*}_{\mathbf{p}_f}(\mathbf{p}_2)p_{2\beta}G(\mathbf{p}_2,\mathbf{p}_1;\Omega)\times p_{1\alpha}u^{\rm out}_{\mathbf{p}_i}(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2 . \tag{6}$$

For the Fourier transform of the continuum wave function with asymptotic outgoing spherical waves we shall use the integral representation[24]

$$u_{\mathbf{p}}^{\mathrm{out}}(\mathbf{q})=\mathcal{N}\oint\left[\frac{\zeta}{\zeta-1}\right]^{n}\times\frac{d\zeta}{\{(\mathbf{q}-\mathbf{p}\zeta)^2+[\eta-ip(1-\zeta)]^2\}^2}\ , \tag{7}$$

where $n=1/ip$, and $\eta>0$, infinitesimal. The contour of integration in Eq. (7) is a loop encircling the points $\zeta=0$ and 1 in the direct sense, but leaving out the pole of the integrand. The principal value of the nth power appearing in the integrand should be taken $(-\pi<\arg\zeta/(\zeta-1)<\pi)$. The normalization constant $\mathcal{N}$ required by Eqs. (6) and (7) is

$$\mathcal{N}=4p(2\pi)^{-3/2}\Gamma(n+1)e^{\pi|n|/2}\ . \tag{8}$$

The incoming spherical waves function can be obtained from Eq. (7) by using $u_{\mathbf{p}}^{\mathrm{in}}(\mathbf{q})=[u_{\mathbf{p}}^{\mathrm{out}}(\mathbf{q})]^*$.

For the Coulomb Green's function in momentum space we shall use the Schwinger integral representation[25]

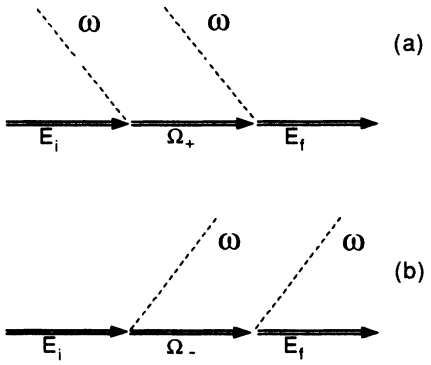

FIG. 1. Furry (bound-state interaction picture) diagrams representing the matrix elements for (a) free-free absorption and (b) emission of two photons (see Ref. 23). Double-line arrows describe the electron propagating in the Coulomb field: the initial (final) ones correspond to the incoming (outgoing) electron; the internal ones represent the Green's function. The dashed lines represent the photons and the their operators; when drawn at an acute (obtuse) angle with the electron arrows, the photon lines describe absorption (emission). The energies of the particles are indicated next to their lines (energy is conserved at vertices). (a) corresponds to Eq. (2a), and (b) to Eq. (2b).

$$G(\mathbf{p}_2,\mathbf{p}_1;\Omega)=F\int_1^{(0+)}\rho^{-\tau}\frac{d}{d\rho}\left[\frac{1-\rho^2}{\rho}\frac{1}{[X^2(\mathbf{p}_1-\mathbf{p}_2)^2+(p_1^2+X^2)(p_2^2+X^2)(1-\rho^2)/4\rho]^2}\right]d\rho\ , \tag{9}$$

where

$$\tau=X^{-1},\quad X^2=-2\Omega,\quad \mathrm{Re}X>0 \tag{10}$$

and

$$F=\frac{X^3}{2\pi^2}\frac{i\exp(i\pi\tau)}{2\sin\pi\tau}\ . \tag{11}$$

The contour of integration in Eq. (9) starts at $\rho=1$ (where one should take $\rho^{-\tau}=1$), encircles the origin $\rho=0$ in the direct sense, and returns to $\rho=1$.

From Eqs. (3) and (4) we have $\mathrm{Re}\Omega_{\pm}>0$, so that, on account of Eq. (10),

$$\tau_{\pm}=i(p_i^2\pm2\omega)^{-1/2}\ , \tag{12}$$

where the square root is taken to be positive. By inserting Eqs. (7) and (9) into Eq. (6), and interchanging the order of integrations we may write

$$\Pi_{\alpha\beta}=\tfrac{1}{4}\mathcal{N}_i\mathcal{N}_f F\oint d\zeta_1\oint d\zeta_2\int_1^{(0+)}d\rho\left[\frac{\zeta_1}{\zeta_1-1}\right]^{n_i}\left[\frac{\zeta_2}{\zeta_2-1}\right]^{n_f}\rho^{-\tau}\times\left[\frac{\partial}{\partial\kappa_{i\alpha}}-\frac{\kappa_{i\alpha}}{\mu_i}\frac{\partial}{\partial\mu_i}\right]\left[\frac{\partial}{\partial\kappa_{f\beta}}-\frac{\kappa_{f\beta}}{\mu_f}\frac{\partial}{\partial\mu_f}\right]\frac{d}{d\rho}\left[\frac{1-\rho^2}{\rho}J\right]\ , \tag{13}$$

where

$$J=\int\int\frac{d\mathbf{p}_1 d\mathbf{p}_2}{[(\mathbf{p}_1-\boldsymbol{\kappa}_i)^2+\mu_i^2][X^2(\mathbf{p}_1-\mathbf{p}_2)^2+(p_1^2+X^2)(p_2^2+X^2)(1-\rho^2)/4\rho]^2[(\mathbf{p}_2-\boldsymbol{\kappa}_f)^2+\mu_f^2]}\ . \tag{14}$$

We have abbreviated

$$\boldsymbol{\kappa}_i=\mathbf{p}_i\zeta_1,\quad \boldsymbol{\kappa}_f=\mathbf{p}_f\zeta_2\ ;\qquad \mu_i=\eta-ip_i(1-\zeta_1),\quad \mu_f=\eta-ip_f(1-\zeta_2)\ . \tag{15}$$

The integral Eq. (14) was encountered and calculated earlier [Gavrila, Ref. 20, Eq. (30)]. We have shown then that

$$\frac{d}{d\rho}\left[\frac{1-\rho^2}{\rho}J\right]=\frac{16\pi^4}{X^2}\frac{1}{\Gamma}\ , \tag{16}$$

with

$$\Gamma=\{[(X+\mu_i)^2+\kappa_i^2]-\rho[(X-\mu_i)^2+\kappa_i^2]\}$$
$$\times\{[(X+\mu_f)^2+\kappa_f^2]-\rho[(X-\mu_f)^2+\kappa_f^2]\}$$
$$+4\rho X^2[(\kappa_i-\kappa_f)^2+(\mu_i-\mu_f)^2]\ . \tag{17}$$

By taking into account Eq. (15) we find

$$\Gamma=s+t_1\xi_1+t_2\xi_2+v\xi_1\xi_2\ , \tag{18}$$

where we have denoted

$$\begin{aligned}
s&=w^2\ ,\\
w&=(p_i+iX)(p_f+iX)-\rho(p_i-iX)(p_f-iX)\ ,\\
t_1&=-2p_iw[(p_f+iX)-\rho(p_f-iX)]\ ,\\
t_2&=-2p_fw[(p_i+iX)-\rho(p_i-iX)]\ ,\\
v&=4p_ip_f\{[(p_i+iX)-\rho(p_i-iX)]\\
&\quad\times[(p_f+iX)-\rho(p_f-iX)]\\
&\quad+4\rho X^2\sin^2(\theta/2)\}\ .
\end{aligned} \tag{19}$$

Further, from Eqs. (16) and (17) we may write[26]

$$\frac{1}{4}\left[\frac{\partial}{\partial\kappa_{i\alpha}}-\frac{\kappa_{i\alpha}}{\mu_i}\frac{\partial}{\partial\mu_i}\right]\left[\frac{\partial}{\partial\kappa_{f\beta}}-\frac{\kappa_{f\beta}}{\mu_f}\frac{\partial}{\partial\mu_f}\right]\frac{d}{d\rho}\left[\frac{1-\rho^2}{\rho}J\right]$$
$$=a\delta_{\alpha\beta}+b\kappa_{i\alpha}\kappa_{i\beta}+c\kappa_{i\alpha}\kappa_{f\beta}+d\kappa_{f\alpha}\kappa_{i\beta}+e\kappa_{f\alpha}\kappa_{f\beta}\ . \tag{20}$$

with

$$\begin{aligned}
a&=2^5\pi^4\rho\Gamma^{-2}\ ,\\
b&=-i2^7\pi^4\rho X\Gamma^{-3}p_i^{-1}(\zeta_1-1)^{-1}(\xi_f\zeta_2+\eta_f)\ ,\\
c&=-2^4\pi^4p_i^{-1}p_f^{-1}(\zeta_1-1)^{-1}(\zeta_2-1)^{-1}\\
&\quad\times[-\Gamma^{-2}(1+\rho^2)\\
&\qquad+2\Gamma^{-3}(\xi_i\zeta_1+\eta_i)(\xi_f\zeta_2+\eta_f)]\ ,\\
d&=2^9\rho^2X^2\Gamma^{-3}\ ,\\
e&=-i2^7\pi^4\rho X\Gamma^{-3}p_f^{-1}(\zeta_2-1)^{-1}(\xi_i\zeta_1+\eta_i)\ .
\end{aligned} \tag{21}$$

These contain the abbreviations

$$\begin{aligned}
\xi_\sigma&=2p_\sigma[(p_\sigma+iX)-\rho^2(p_\sigma-iX)]\ ,\\
\eta_\sigma&=-[(p_\sigma+iX)^2-\rho^2(p_\sigma-iX)^2]\quad(\sigma=i,f)\ .
\end{aligned} \tag{22}$$

By introducing the unit vectors of the electron momenta $\nu_\sigma=\mathbf{p}_\sigma/p_\sigma$ $(\sigma=i,f)$, we can write Eq. (13) as

$$\Pi_{\alpha\beta}=A\delta_{\alpha\beta}+B\nu_{i\alpha}\nu_{i\beta}+C\nu_{i\alpha}\nu_{f\beta}+D\nu_{f\alpha}\nu_{i\beta}+E\nu_{f\alpha}\nu_{f\beta}\ . \tag{23}$$

The amplitudes A, B, etc, appearing here have the form

$$A=2^5\pi^4N\int_1^{(0+)}\rho^{-\tau+1}I(0,0)d\rho\ , \tag{24a}$$

$$B=2^6i\pi^4p_iXN\int_1^{(0+)}\rho^{-\tau+1}\left[\xi_f\frac{\partial}{\partial v}+\eta_f\frac{\partial}{\partial t_1}\right]I(1,0)d\rho\ , \tag{24b}$$

$$C=2^4\pi^4N\int_1^{(0+)}\rho^{-\tau}\left[(1+\rho^2)+\xi_i\xi_f\frac{\partial}{\partial v}+\xi_i\eta_f\frac{\partial}{\partial t_1}+\xi_f\eta_i\frac{\partial}{\partial t_2}+\eta_i\eta_f\frac{\partial}{\partial s}\right]I(1,1)d\rho\ , \tag{24c}$$

$$D=-2^8\pi^4p_ip_fX^2N\int_1^{(0+)}\rho^{-\tau+2}\frac{\partial}{\partial v}I(0,0)d\rho\ , \tag{24d}$$

$$E=2^6i\pi^4p_fXN\int_1^{(0+)}\rho^{-\tau+1}\left[\xi_i\frac{\partial}{\partial v}+\eta_i\frac{\partial}{\partial t_2}\right]I(0,1)d\rho\ , \tag{24e}$$

with the abbreviation

$$N=\mathcal{N}_i\mathcal{N}_fF\ . \tag{25}$$

Equations (24) are expressed in terms of the general integral

$$I(q_i,q_f)=\oint\oint d\zeta_1d\zeta_2\times\left[\frac{\zeta_1}{\zeta_1-1}\right]^{n_i+q_i}\left[\frac{\zeta_2}{\zeta_2-1}\right]^{n_f+q_f}\frac{1}{\Gamma^2}\ , \tag{26}$$

where q_i,q_f are non-negative integers and Γ is defined by Eq. (18). Equations (24) are written with the understanding that after having performed the derivatives of $I(q_i,q_f)$ with respect to s,t_1,t_2,v, these quantities should be replaced by their actual values, given in Eq. (19).

The integral $I(q_1,q_2)$ and its derivatives are calculated in the Appendix, Eqs. (A6)–(A10). By inserting in these the quantities given in Eq. (19), we may finally write the amplitudes of Eqs. (24) in the form

$$A=\gamma_A\int_0^1d\rho\,\rho^{1-\tau}\frac{(1-\rho x_0)^{n_i+n_f-2}}{(1-\rho x_1)^{n_i+1}(1-\rho x_2)^{n_f+1}}\ {}_2F_1(1+n_i,1+n_f;2;z)\ , \tag{27a}$$

$$B=\gamma_B\left[(p_f+Q)^2\int_0^1d\rho\,\rho^{1-\tau}\frac{(1-\rho x_0)^{n_i+n_f-2}(1-\rho^2u_f^2)}{(1-\rho x_1)^{n_i+3}(1-\rho x_2)^{n_f+1}}\ {}_2F_1(n_i+3,n_f+1;2;z)\right.$$
$$\left.+2p_fQ(n_f+1)x_2\int_0^1d\rho\,\rho^{2-\tau}\frac{(1-\rho x_0)^{n_i+n_f-2}}{(1-\rho x_1)^{n_i+2}(1-\rho x_2)^{n_f+2}}\ {}_2F_1(n_i+3,n_f+2;3;z)\right]\ , \tag{27b}$$

$$C=\gamma_C\int_0^1 d\rho\,\rho^{-\tau}\frac{(1-\rho x_0)^{n_i+n_f}}{(1-\rho x_1)^{n_i+2}(1-\rho x_2)^{n_f+2}}[f^{\mathrm{I}}\,{}_2F_1(n_i+2,n_f+2;2;z)+f^{\mathrm{II}}\,{}_2F_1(n_i+3,n_f+3;3;z)$$
$$+f^{\mathrm{III}}\,{}_2F_1(n_i+3,n_f+2;2;z)+f^{\mathrm{IV}}\,{}_2F_1(n_i+2,n_f+3;2;z)]\ , \tag{27c}$$

$$D=\gamma_D\int_0^1 d\rho\,\rho^{2-\tau}\frac{(1-\rho x_0)^{n_i+n_f-2}}{(1-\rho x_1)^{n_i+2}(1-\rho x_2)^{n_f+2}}\,{}_2F_1(n_i+2,n_f+2;3;z)\ , \tag{27d}$$

$$E=\gamma_E\left[(p_i+Q)^2\int_0^1 d\rho\,\rho^{1-\tau}\frac{(1-\rho x_0)^{n_i+n_f-2}(1-\rho^2u_i^2)}{(1-\rho x_1)^{n_i+1}(1-\rho x_2)^{n_f+3}}\,{}_2F_1(n_i+1,n_f+3;2;z)\right.$$
$$\left.+2p_iQ(1+n_i)x_1\int_0^1 d\rho\,\rho^{2-\tau}\frac{(1-\rho x_0)^{n_i+n_f-2}}{(1-\rho x_1)^{n_i+2}(1-\rho x_2)^{n_f+2}}\,{}_2F_1(n_i+2,n_f+3;3;z)\right]\ . \tag{27e}$$

The coefficients appearing here are

$$\gamma_A=-2^7n_in_fp_ip_fQ^3h(3,3,1,1)\ , \tag{28a}$$
$$\gamma_B=-2^8p_i^2p_fn_f(n_i+1)(n_i+2)Q^4h(3,5,3,1)\ , \tag{28b}$$
$$\gamma_C=-2^6Q^3p_ip_f(n_i+1)(n_f+1)h(2,2,2,2)\ , \tag{28c}$$
$$\gamma_D=+2^9p_i^2p_f^2n_in_f(n_i+1)(n_f+1)Q^5h(4,4,2,2)\ , \tag{28d}$$
$$\gamma_E=-2^8p_ip_f^2n_i(n_f+1)(n_f+2)Q^4h(5,3,1,3) \tag{28e}$$

where $h(q_1,q_2,q_3,q_4)$ is the function depending on integers q_1,q_2,q_3,q_4:

$$h(q_1,q_2,q_3,q_4)=i\pi e^{\pi/2p_i}e^{\pi/2p_f}\Gamma(n_i+1)\Gamma(n_f+1)$$
$$\times(p_i+Q)^{n_i-q_1}(p_f+Q)^{n_i-q_2}$$
$$\times(Q-p_i)^{-n_i-q_3}(Q-p_f)^{-n_f-q_4}\ . \tag{29}$$

Equation (27c) also contains the coefficients

$$f^{\mathrm{I}}=1+\rho^2+(n_i+n_f+2)\frac{(1-\rho^2u_i^2)(1-\rho^2u_f^2)}{(1-\rho x_0)^2}\ , \tag{30a}$$

$$f^{\mathrm{II}}=\frac{(n_i+2)(n_f+2)}{2}\left[z\frac{(1-\rho^2u_i^2)(1-\rho^2u_f^2)}{(1-\rho x_0)^2}\right.$$
$$\left.-\frac{(1-u_i^2)(1-u_f^2)\rho^2}{(1-\rho x_0)^2}\right]\ , \tag{30b}$$

$$f^{\mathrm{III}}=-(n_i+2)\frac{(1-\rho^2u_f^2)(1-\rho^2)}{(1-\rho x_0)(1-\rho x_1)}\ , \tag{30c}$$

$$f^{\mathrm{IV}}=-(n_f+2)\frac{(1-\rho^2u_i^2)(1-\rho^2)}{(1-\rho x_0)(1-\rho x_2)}\ . \tag{30d}$$

In Eqs. (27)–(30) we are using the abbreviations

$$x_0=u_iu_f,\quad x_1=u_f/u_i,\quad x_2=u_i/u_f\ ; \tag{31}$$
$$u_\sigma=(p_\sigma-Q)/(p_\sigma+Q)\quad(\sigma=i,f)\ ; \tag{32}$$
$$Q=iX,\quad \mathrm{Im}Q>0\ ; \tag{33}$$
$$z=\frac{16\rho p_ip_fQ^2\sin^2(\theta/2)}{(p_i^2-Q^2)(p_f^2-Q^2)(1-x_1\rho)(1-x_2\rho)}\ . \tag{34}$$

In writing Eqs. (27) we have also taken into account the relation, valid for $\mathrm{Re}\lambda>0$:

$$\int_1^{(0+)}\rho^{\lambda-1}\chi(\rho)d\rho=-\frac{2\sin\pi\lambda}{ie^{-i\pi\lambda}}\int_0^1\rho^{\lambda-1}\chi(\rho)d\rho\ , \tag{35}$$

where $\chi(\rho)$ is analytic in the vicinity of $\rho=0$. Note that the amplitude E of Eq. (27e) can be obtained from B, Eq. (27b), by interchanging the subscripts i and f.

From Eqs. (5), (6), and (23) one may write the absorption matrix element as

$$f^{(+)}=\frac{1}{8\pi}I\omega^{-2}[A^{(+)}\mathbf{e}^2+B^{(+)}(\mathbf{e}\cdot\boldsymbol{\nu}_i)^2$$
$$+(C^{(+)}+D^{(+)})(\mathbf{e}\cdot\boldsymbol{\nu}_i)(\mathbf{e}\cdot\boldsymbol{\nu}_f)$$
$$+E^{(+)}(\mathbf{e}\cdot\boldsymbol{\nu}_f)^2]\ , \tag{36}$$

where the amplitudes $A^{(+)}$, $B^{(+)}$, etc., are to be calculated with Ω_+ of Eq. (3). By a similar calculation starting from Eq. (2b), the emission matrix element becomes

$$f^{(-)}=\frac{1}{8\pi}I\omega^{-2}[A^{(-)}\mathbf{e}^{*2}+B^{(-)}(\mathbf{e}^*\cdot\boldsymbol{\nu}_i)^2$$
$$+(C^{(-)}+D^{(-)})(\mathbf{e}^*\cdot\boldsymbol{\nu}_i)(\mathbf{e}^*\cdot\boldsymbol{\nu}_f)$$
$$+E^{(-)}(\mathbf{e}^*\cdot\boldsymbol{\nu}_f)^2]\ , \tag{37}$$

where $A^{(-)}$, $B^{(-)}$, etc., are the same amplitudes as in Eqs. (27)–(34), but now taken for Ω_- of Eq. (3). Obviously, E_f and E_i are connected differently in Eqs. (36) and (37); see Eq. (4).

The coefficients Eqs. (28a)–(28e) contain complex powers of the form

$$T_\sigma=[(Q+p_\sigma)/(Q-p_\sigma)]^{n_\sigma}\quad(\sigma=i,f) \tag{38}$$

for which the principal value branch must be taken. They have a different form for absorption and emission. For *absorption*, $p_i<Q<p_f$, and for $\sigma=i$ one finds immediately

$$T_i^{(+)}=\exp\{n_i\ln[(Q+p_i)/(Q-p_i)]\} . \quad (39a)$$

However, for $\sigma=f$ an ambiguity appears in the choice of the imaginary part ($\pm i\pi$) of the complex logarithm involved. The ambiguity should be eliminated by referring to Eqs. (4), (10), and (33), to get

$$T_f^{(+)}=e^{-\pi|n_f|}\exp\{n_f\ln[(p_f+Q)/(p_f-Q)]\} . \quad (39b)$$

The case of *emission,* for which $p_f<Q<p_i$, can be handled similarly. One finds

$$T_i^{(-)}=e^{-\pi|n_i|}\exp\{n_i\ln[(p_i+Q)/(p_i-Q)]\} , \quad (40a)$$

$$T_f^{(-)}=\exp\{n_f\ln[(Q+p_f)/(Q-p_f)]\} . \quad (40b)$$

At this point we would like to mention that Eqs. (36) and (37) satisfy the time reversal condition

$$f^{(+)}(\mathbf{p}_i\to\mathbf{p}_f;\mathbf{e})=f^{(-)}(-\mathbf{p}_f\to-\mathbf{p}_i;\mathbf{e}^*) , \quad (41)$$

where we have indicated the initial and final momenta of the transitions, and the polarization vectors. To check Eq. (41) we first note that the transformation implied by Eq. (41),

$$\mathbf{p}_i\to-\mathbf{p}_f,\quad \mathbf{p}_f\to-\mathbf{p}_i,\quad \mathbf{e}\to\mathbf{e}^* , \quad (42)$$

changes the scalar products of Eq. (36) into those of Eq. (37). However, this transformation relates the amplitude $A^{(+)}$ to $A^{(-)}$, $B^{(+)}$ to $E^{(-)}$, $E^{(+)}$ to $B^{(-)}$, and $C^{(+)}+D^{(+)}$ to $C^{(-)}+D^{(-)}$. To prove the invariance of Eq. (41) we need show that the related amplitudes are actually equal. Now, the amplitudes $A,\ldots,E$ depend on the magnitudes of the momenta $|\mathbf{p}_i|,|\mathbf{p}_f|$ and their angle θ [contained in z of Eq. (34)]. Therefore the transformation Eq. (42) affects on one hand the amplitudes by interchanging the subscripts i and f. In view of Eqs. (27)–(34) it is easy to see that, *at given* τ, when performing this interchange, A is invariant; B goes over into E, and vice versa; C and D are invariant. Thus, *at given* τ,

$$\begin{aligned} &A(f,i)=A(i,f),\quad B(f,i)=E(i,f) ,\\ &C(f,i)=C(i,f)\quad D(f,i)=D(i,f) , \end{aligned} \quad (43a)$$

On the other hand, the amplitude $f^{(+)}$ of Eq. (2a) is associated to the value $\tau_+=i(p_i^2+2\omega)^{-1/2}$ of Eq. (12), whereas $f^{(-)}$ of Eq. (2b) is associated to the value $\tau_-=i(p_i^2-2\omega)^{-1/2}$. By performing the transformation Eq. (42), τ_- becomes $i(p_f^2-2\omega)^{-1/2}$ and, since $p_f^2=p_i^2+4\omega$, is equal to τ_+. We are, therefore, precisely in the case of Eq. (43a), and we may write

$$\begin{aligned} &A^{(+)}(f,i)=A^{(-)}(i,f),\quad B^{(+)}(f,i)=E^{(-)}(i,f) ,\\ &C^{(+)}(f,i)=C^{(-)}(i,f),\quad D^{(+)}(f,i)=D^{(-)}(i,f) . \end{aligned} \quad (43b)$$

This completes the proof of Eq. (41).

We have pushed the analytic calculation as far as possible. At this point a numerical computation of Eqs. (36) and (37) had to be carried out. This will be described in a forthcoming paper. In the following we shall discuss some limiting cases of our analytical results.

III. ANALYTICAL LIMITS

The unwieldly form of our general result Eqs. (27)–(37) does not allow direct insight into the physics of the process. However, our formulas simplify substantially in the limiting cases for the parameters ω,E_i,E_f we shall now consider, and a number of salient features will become apparent. The simpler formulas obtained can be compared with the ones derived independently for these limits, which allows a check of our general analytic results Eqs. (27)–(37), and also of its numerical computation (to be presented later). Finally, they will serve as a convenient reference for the discussion of the computed angular distributions. Some of the limits apply to both absorption and emission, others to one of them only, as specified.

A. High initial and final electron energies (case of absorption and emission)

The conditions $E_i\gg1$ and $E_f\gg1$ (for emission the latter has to be stated separately), or, equivalently, $|n_i|\ll1$ and $|n_f|\ll1$, characterize the *first Born approximation,* if second-order terms in n_i and/or n_f are ignored. At the same time, in view of Eqs. (12) and (4), τ should be neglected with respect to 1 because for absorption $|\tau_+|<|n_i|$, and for emission $|\tau_-|<|n_f|$. On the other hand, *no assumption on* ω will be made.

Inspection of the formulas for the amplitudes Eqs. (27)–(34) shows that A and D are proportional to the product n_in_f, and thus do not contribute to the first Born approximation. The amplitudes B and E are of first order in n_i or n_f, and do contribute. This allows us to set n_i and n_f equal to zero in the rest of their expressions. The Gauss functions in the integrands then have integer parameters and reduce to simple rational functions of the variable z [or ρ, see Eq. (34)].[27] The complex powers in the integrands become integer powers. Thus Eqs. (27b) and (27e) will contain integrals over rather complicated rational functions of ρ. Their calculation is elementary, albeit very arduous. The result is

$$B_B=2\pi p_i^2/\omega^2\Delta^2,\quad E_B=2\pi p_f^2/\omega^2\Delta^2 , \quad (44)$$

containing the momentum transfer $\mathbf{\Delta}=\mathbf{p}_f-\mathbf{p}_i$ (subscript B stands for Born).

The amplitude C appears to be of *zeroth* order with respect to n_i and n_f [see Eq. (28c)]. Since in ordinary (non-Z-scaled) atomic units n_i and n_f are proportional to Z, this would imply that C, and consequently the matrix elements $f^{(\pm)}$, would be nonvanishing in the absence of the potential ($Z=0$). That this cannot happen has been checked by calculating C for $|n_i|=|n_f|=0$, and showing that it actually vanishes. To *first* order one finds, similarly as for B and E,

$$C_B=-4\pi p_ip_f/\omega^2\Delta^2 . \quad (45)$$

The matrix element for absorption Eq. (36) thus reduces in first Born approximation to the simple form[28]

$$f_B^{(+)}=\frac{1}{4\omega^4}\frac{(\mathbf{e}\cdot\mathbf{\Delta})^2}{\Delta^2} . \quad (46)$$

Note that for arbitrary elliptic polarization $f_B^{(+)}$ is complex because of $\mathbf{e}$. $f_B^{(-)}$ can be obtained by replacing $\mathbf{e}$ by $\mathbf{e}^*$ in Eq. (46).

This simple result can also be obtained by a direct calculation, in which not only the interaction of the electron with the radiation field is treated as a perturbation, but also its interaction with the Coulomb potential. One may proceed in two equivalent ways. First, one may replace the initial and final wave functions, as well as the Green's function entering Eq. (2a) or (2b), by their Born expansions through first order in the potential.[29] A term independent of the potential emerges, but this is proportional to a δ function expressing momentum conservation. Since this cannot be satisfied concomitantly with energy conservation (a free electron cannot absorb or emit photons), the δ function vanishes. The matrix element will, therefore, contain the first and higher powers of the potential in its expression, i.e., to lowest order will be proportional to Z in ordinary atomic units. The result obtained fully agrees with Eq. (46),[30,31] which represents a stringent test of our general formulas.[32] Alternatively, one may calculate the matrix elements by applying directly the Feynman rules of perturbation theory, for the interaction of free electrons with a radiation field and a potential[23] as represented diagrammatically in Fig. 2, to obtain the same result.

The cross section corresponding to both absorption and emission can be written as

$$\frac{d\sigma_B^{(\pm)}}{d\Omega}=\frac{1}{16}\frac{p_f}{p_i}\frac{I^2}{\omega^8}\frac{|(\mathbf{e}\cdot\boldsymbol{\Delta})|^4}{\Delta^4}\ . \tag{47}$$

In contrast to the general cross section, Eqs. (1), (36), and (37), which depends on the individual orientation of the vectors $\mathbf{p}_i$ and $\mathbf{p}_f$ with respect to $\mathbf{e}$, Eq. (47) depends only on the angle between the momentum transfer $\boldsymbol{\Delta}$ and $\mathbf{e}$. The dependence of Eq. (47) on the direction of $\boldsymbol{\Delta}$ is smooth. Indeed, for linear polarization ($\mathbf{e}$ real), for example, the quantity $(\mathbf{e}\cdot\boldsymbol{\Delta})^4/\Delta^4=(\mathbf{e}\cdot\boldsymbol{\delta})^4$, where $\boldsymbol{\delta}$ is the unit vector of $\boldsymbol{\Delta}$, is bracketed between 0 (for δ orthogonal to $\mathbf{e}$) and 1 (for δ parallel to $\mathbf{e}$). In particular, there is a smooth variation, and the cross section stays finite, when $\boldsymbol{v}_i$ approaches $\boldsymbol{v}_f$ (forward scattering, $\Delta\rightarrow|p_i-p_f|$), or $-\boldsymbol{v}_f$ (backward scattering, $\Delta\rightarrow p_i+p_f$), whatever the common direction of $\boldsymbol{v}_i$, $\boldsymbol{v}_f$, and $\boldsymbol{\Delta}$ may be, with the exception of the case when all three end up being perpendicular to $\mathbf{e}$. Indeed, in the latter case the cross section may drop quite rapidly for forward and backward scattering.[33] Note that the first Born cross section Eq. (47), breaks down as an approximation when it vanishes because then second-order Born corrections (as those contained in A) should be taken into account and will yield a nonvanishing, although small contribution to Eq. (1).

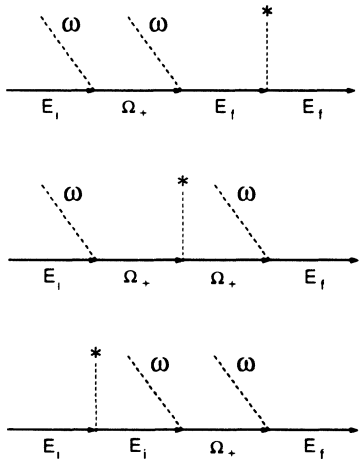

FIG. 2. Feynman diagrams for the first Born approximation of the two-photon free-free absorption (see Ref. 23). Single-line arrows represent the free electron propagation, dotted lines represent the photons, and dashed lines ending in a star the Coulomb potential. Other conventions as for Fig. 1. Similar diagrams can be drawn for two-photon emission.

B. Low ratio of photon energy to electron energy (case of absorption and emission)

We shall now assume that $\omega/E_i\ll1$ (and therefore also $\omega/E_f\ll1$), although ω *need not be small* with respect to 1. Since we are interested only in the leading term of the matrix elements in this limit, whenever possible we shall work to lowest order in the ratio ω/E_i. In this case $p_i\cong p\cong Q$, and $n_i\cong n_f$; their common values will be denoted by p and n, respectively. Also $\tau_\pm\cong-n$. From Eqs. (31) and (34) we find $x_0\cong-\omega^2/4p^4$, $x_i\cong x_f\cong-1$, and

$$z\cong z_0\rho(1+\rho)^{-2},\quad z_0=-4p^4\omega^{-2}\sin^2\frac{\theta}{2} \tag{48}$$

for both absorption and emission.

With these simplifications at hand we can consider the limiting form of the amplitudes $A-E$. The coefficients $\gamma_A-\gamma_E$, Eqs. (28), and $f^{\mathrm{I}}-f^{\mathrm{IV}}$, Eqs. (30) present no problems, but the integrals in Eqs. (27a)–(27e) require special attention. They become of the type

$$\int_0^1\rho^{n+a}(1+\rho)^{-2n-b}{}_2F_1(n+r,n+s,t;z)d\rho\ , \tag{49}$$

where a,b,r,s,t are positive integers, except for a which may be also zero (n should not be confused with an integer). The variable z of Eq. (48) has a peculiar dependence on ρ, θ, and ω, which requires a separate consideration of the forward ($\theta=0$) and nonforward ($\theta\neq0$) scattering.

For *nonforward scattering* when $\omega\rightarrow0$, we have $|z_0|\rightarrow\infty$ and $|z|\rightarrow\infty$ for all ρ, except for $\rho=0$. Thus, when $\rho\neq0$, one may use the asymptotic form of the Gauss functions.[34] If $r<s$ this is

$${}_2F_1(n+r,n+s;t;z)\underset{|z|\rightarrow\infty}{\rightarrow}\frac{\Gamma(t)\Gamma(s-r)}{\Gamma(t-r-n)\Gamma(n+s)}(-z)^{-n-r}\ , \tag{50}$$

and if $r=s$

$$ {}_2F_1(n+r,n+r;t;z) \underset{|z|\to\infty}{\to} \frac{\Gamma(t)}{\Gamma(n+r)\Gamma(t-r-n)}(-z)^{-n-r}\ln(-z) . \tag{51} $$

In the vicinity of the origin, $\rho=0$, more specifically on the small interval $\rho \le |z_0^{-1}| = O(\omega^2)$, we have $|z|<1$ and one may use the series expansion of the Gauss functions (at $\rho=0$ these reduce to 1). The other two factors in the integrand of Eq. (49) then reduce to ρ^{n+a}. When $a>0$ (which is the case for the amplitudes A,B,D,E), the contribution of this interval to the integral is therefore of order z_0^{n+a-1}. It turns out that in all cases this is of higher order than that of the rest of the integration interval in Eq. (49), and hence negligible. We are then left with integrals extended over the interval $(|z_0|^{-1},1)$, containing the asymptotic forms of the Gauss functions. By inserting Eqs. (50) and (51) into Eq. (49) the complex power $\rho^n(1+\rho)^{-2n}$ is canceled, and integrals with simple rational or logarithmic integrands emerge. To lowest order in $|z_0^{-1}|$, these can be extended over the whole interval (0,1). We find (subscript SP stands for "soft photon")[35]

$$ B_{\rm SP}=E_{\rm SP}=\frac{\pi}{2\omega^2}\frac{\Gamma(1+n)}{\Gamma(1-n)}\left[\sin^2\frac{\theta}{2}\right]^{-n-1} , \tag{52} $$

$$ A_{\rm SP}=O\left[\ln\frac{\omega}{E}\right], \quad D_{\rm SP}=O\left[\ln\frac{\omega}{E}\right] . \tag{53} $$

Thus $A_{\rm SP}$ and $D_{\rm SP}$ are negligible with respect to $B_{\rm SP}$ and $E_{\rm SP}$ to lowest order in ω/E.

Some of the terms of C have to be treated differently, because also the case $a=0$ may now occur in Eq. (49). As $r>0$, Eqs. (50) and (51) show that the integrand vanishes in the limit $\omega\to 0$, except in the vicinity of the origin ($\rho \le |z_0|^{-1}$), where it is constant. On this interval, to lowest order in $|z_0|^{-1}$, one may neglect ρ with respect to 1, and the variable z then becomes $z\cong z_0\rho$. By changing also the integration variable from ρ to $x\cong -z_0\rho$, it can be shown that the dominant behavior of the integral Eq. (49) is given by

$$ (-z_0)^{-n-1}\int_0^\infty x^n {}_2F_1(n+r,n+s;t;-x)dx = (-z_0)^{-n-1}(r-2)!(s-2)!(t-1)! \times \frac{\Gamma(n+1)}{\Gamma(n+r)\Gamma(n+s)\Gamma(t-n-1)} , \tag{54} $$

where we have used a tabulated formula.[36] Summing up we get

$$ C_{\rm SP}=-2B_{\rm SP} , \tag{55} $$

with $B_{\rm SP}$ given by Eq. (52).

By inserting Eqs. (52), (53), and (55) into the matrix element Eq. (36), we finally obtain to lowest order in ω/E

$$ f_{\rm SP}^{(+)}=\frac{1}{8\omega^4}I(\mathbf{e}\cdot\Delta)^2 f_c(\theta) , \tag{56} $$

where

$$ f_c(\theta)=\frac{1}{2p^2}\frac{\Gamma(1+n)}{\Gamma(1-n)}\left[\sin^2\frac{\theta}{2}\right]^{-1-n} \tag{57} $$

is the Coulomb elastic scattering amplitude (on the energy shell). To obtain $f^{(-)}$ one needs only replace $\mathbf{e}$ by $\mathbf{e}^*$ in Eq. (56). The corresponding cross sections for *nonforward scattering* are

$$ \frac{d\sigma_{\rm SP}^{(\pm)}}{d\Omega}=\frac{1}{64}\frac{I^2}{\omega^8}|(\mathbf{e}\cdot\Delta)|^4\frac{d\sigma_c}{d\Omega} , \tag{58} $$

where $(d\sigma_c/d\Omega)=|f_c|^2=4\Delta^{-4}$ is the Rutherford cross section.

Characteristic of Eqs. (56) and (58) is their rapid increase as $\omega\to 0$, which is related to the infrared divergence of QED matrix elements involving soft photons.[37] For $\mathbf{e}\cdot\Delta=0$ both equations vanish. This simply means that the ω/E corrections (powers and/or logarithms) to Eqs. (56) and (58) are then taking over. A remarkable feature of Eq. (58) is that it does not depend on the energy of the electron (recall that $E_i\cong E_f$).

It does not appear to be possible to extract a simple formula for the amplitude $f_{\rm SP}$ in *forward, or nearly forward, scattering*. For strictly forward scattering, $\theta=0$, the variable z of Eq. (34) vanishes and the Gauss functions in Eqs. (27) and (49) reduce to 1. To lowest order, the integral Eq. (49) no longer depends on ω but the coefficients do depend on it. The amplitudes B,C,D,E thus obtained are of order ω^{-4} whereas A is of order ω^{-2}. Thus the contributions B,C,D,E would appear to be dominant, but it turns out that when added together in $f_{\rm SP}$, their sum vanishes to order ω^{-4}, which indicates that $f_{\rm SP}$ is in forward scattering of order ω^{-3} or less. (We could not check its actual order of magnitude because the calculation becomes prohibitively difficult.) For nearly forward scattering ($\theta^2\ll 1$), z and z_0 of Eq. (48) are very sensitive to the ratio $(\theta/\omega)^2$. This can vary from 0 (case of strictly forward scattering) to ∞ (case of nonforward scattering), when θ is increased from zero and ω is small. It is conceivable that the variation of z for nearly forward scattering may give rise to a rapid change in the cross section.

Formulas similar to Eq. (56) have been derived for *one-photon* FFT's (spontaneous and stimulated) where they have been termed as the *soft-photon* (low-frequency) *approximation*. The basic result there is the formula of Low, connecting the first *two* terms of the expansion of the FFT matrix element in powers of ω with the elastic scattering amplitude (on the energy shell) and its derivative with respect to the energy.[1,10] The theorem refers specifically to the case of a short-range potential. It has been realized meanwhile that for a Coulomb potential the connection holds only for the *first* term of the expansion in ω, and only for nonforward scattering.[38]

Our Eq. (56) shows that we are dealing here with a situation similar to the one-photon case: to lowest order in ω/E the matrix element splits into a radiation-dependent factor $I(\mathbf{e}\cdot\Delta)^2/8\omega^4$, times the elastic scattering amplitude. Recall that this is valid only for $\theta\neq 0$. Thus Eq. (56) represents the extension (as far as it goes) of the soft-photon theorem to the two-photon FFT's in a Coulomb

potential. This result agrees (apart for kinematical factors) with the dominant term of the general $1/\omega$ expansion derived by Rosenberg for the matrix element of two-photon bremsstrahlung in a short-range potential (which, however, holds also for $\theta=0$); see Ref. 11, Eq. (4.6).

C. High ratio of photon energy to initial electron energy (case of absorption only)

This limit is characterized by $\omega/E_i \gg 1$, which cannot be considered for emission. Thus, from Eq. (4), we have

$$\frac{p_i^2}{p_f^2}=\frac{p_i^2}{4\omega}\left[1+\frac{p_i^2}{4\omega}\right]^{-1}=O\left[\left[\frac{\omega}{E_i}\right]^{-1}\right], \tag{59}$$

and hence p_i should be neglected with respect to p_f in this limit, even though p_i may be quite large itself. In the beginning we shall make *no separate assumption on* ω; therefore n_i, and the lowest order forms in ω/E_i of n_f and τ: $n_f \cong -i/(4\omega)^{1/2}$, $\tau \cong i/(2\omega)^{1/2}$, can be arbitrary. The x_0, x_1, x_2 of Eq. (31) tend to $\bar{x}, \bar{x}, 1/\bar{x}$, respectively, where $\bar{x}$ is the constant $\bar{x}=-(1-2^{1/2})^2$. Further, the variable z of Eq. (34) tends to zero, irrespective of θ, so that, to lowest order, the ${}_2F_1$ functions in the integrands of Eq. (27) reduce to 1. The integrals then become of the general form

$$\int_0^1 \rho^{r-\tau}(1-\rho\bar{x})^{n_f-s}[1-(\rho/\bar{x})]^{-n_f-t}d\rho\ , \tag{60a}$$

where r,s,t are nonnegative integers. Note that the n_i dependence has dropped out of these integrals and that Eq. (60a) does not depend on the scattering angle θ. The integral Eq. (60a) can be expressed in terms of the Appell hypergeometric function of two variables $F_1(a;b_1,b_2;c;x_1,x_2)$.[39] The result is

$$\frac{1}{r-\tau}F_1(r-\tau;s-n_f,t+n_f;r+1-\tau;\bar{x},(1/\bar{x}))\ . \tag{60b}$$

By also taking into account the coefficients Eqs. (28)–(30), it is simple to derive the expressions of the amplitudes $A,\ldots,E$, but we shall not give them here.

Their form simplifies considerably if, in addition to the condition $(\omega/E_i)\gg 1$, we assume the *high-frequency condition* $\omega \gg 1$. Then $|n_f|, |\tau| \ll 1$ (but n_i may still be arbitrary), and, by setting them equal to zero, the definite integral Eq. (60a) can immediately be performed. It turns out that in this limit the amplitude E dominates, being of order ω^{-2}, whereas the other amplitudes are of order $\omega^{-5/2}$ or higher. The absorption matrix element Eq. (36) becomes then in the high-frequency limit (subscript HF stands for "high frequency")

$$f_{\rm HF}^{(+)}=(I/4\omega^4)e^{\pi/2p_i}\Gamma(1+n_i)(\mathbf{e}\cdot\boldsymbol{\nu}_f)^2\ . \tag{61a}$$

This is the result obtained from the formula of second-order perturbation theory (with respect to the field intensity I) Eq. (2), if also the preceding high-frequency conditions are imposed. Alternatively, Eq. (61a) may be derived from the high-frequency perturbation theory developed by Gavrila and Kaminsky,[15] which expresses the exact two-photon FFT matrix element $f_{\rm ex}^{(+)}$ as an expansion in $1/\omega$, with coefficients which depend on the intensity I. In order to compare this with Eq. (61a), we need pick out the lowest order term in $1/\omega$ of this expansion, and then take its lowest-order approximation with respect to I. This gives (in Z-scaled a.u.), when also p_i is assumed to be small with respect to p_f [as was done for Eq. (61a)],

$$f_{\rm HF}^{(+)}=-(\pi^2/2)\alpha_0^2(\mathbf{e}\cdot\mathbf{p}_f)^2\mathcal{V}(p_f)u_{\mathbf{p}_i}^{\rm out}(0)\ , \tag{61b}$$

where $\alpha_0=I^{1/2}\omega^{-2}$, $\mathcal{V}(p)=-(2\pi^2p^2)^{-1}$ is the Fourier transform of the Coulomb potential (considered as the limit of a screened potential), and $u_{\mathbf{p}_i}^{\rm out}(0)$ is the value at the origin of the coordinate space continuum wave function $u_{\mathbf{p}_i}^{\rm out}(\mathbf{r})$, entering Eqs. (2a) and (2b). As $u_{\mathbf{p}_i}^{\rm out}(0)=e^{\pi/2p_i}\Gamma(1+n_i)$, Eqs. (61a) and (61b) are indeed identical, which is another check on the present calculation.

By calculating the cross section Eq. (1) with the help of Eq. (61a) or (61b) we find

$$\frac{d\sigma_{\rm HF}^{(+)}}{d\Omega}=\frac{I^2}{8p_i\omega^{15/2}}\frac{(2\pi/p_i)}{1-e^{-(2\pi/p_i)}}(\mathbf{e}\cdot\boldsymbol{\nu}_f)^4\ , \tag{62}$$

where we have used

$$|e^{\pi/2p}\Gamma(1+n)|^2=\frac{2\pi|n|}{1-e^{-2\pi|n|}}\ . \tag{63}$$

If E_i also is assumed to be large ($E_i\gg 1$, $|n_i|\ll 1$), the conditions under which Eq. (61a) was derived [$(\omega/E_i)\gg 1$, $\omega\gg 1$] overlap with those for the Born approximation ($E_i\gg 1$, $E_f\gg 1$). One may inquire then if Eqs. (61a) and (46) become identical. With $|n_i|\ll 1$, Eq. (61a) reduces to

$$f_{\rm HF}^{(+)}\cong(I/4\omega^4)(\mathbf{e}\cdot\boldsymbol{\nu}_f)^2\ . \tag{64}$$

When comparing Eq. (64) with Eq. (46) we should recall, however, that, on the basis of Eq. (59), we have neglected p_i with respect to p_f. If this is done in Eq. (46), it then indeed reduces to Eq. (64), as it should.

D. High ratio of photon energy to final electron energy (case of emission only)

For emission only we may consider the limit $\omega/E_f\gg 1$, with E_i arbitrary. By adding also the high-frequency condition $\omega\gg 1$, this case can be related to Eq. (61a) if use is made of the time reversal formula Eq. (41). By applying the transformation Eq. (42) to Eq. (61a) we find

$$f_{\rm HF}^{(-)}=(I/4\omega^4)e^{\pi/2p_f}\Gamma(1+n_f)(\mathbf{e}^*\cdot\boldsymbol{\nu}_i)^2\ . \tag{65}$$

A particular case of this limit is the one in which the final electron is left at rest ($E_f\to 0$, $E_i\cong 2\omega$). This new limit cannot be performed directly on the matrix element Eq. (65) because $\Gamma(1+n_f)$ has an indefinitely oscillating phase for $|n_f|\to\infty$. However, by taking into account Eq. (63), it is apparent that the limit exists for the cross section Eq. (1). We find, to lowest order in $(1/n_f)$,

$$\left[\frac{d\sigma_{\rm HF}^{(-)}}{d\Omega}\right]_0\cong\frac{\pi}{16}\frac{I^2}{\omega^{17/2}}|(\mathbf{e}^*\cdot\boldsymbol{\nu}_i)|^4\ . \tag{66}$$

IV. CONCLUDING REMARKS

The analytic evaluation of the general second-order perturbation theory matrix elements for two-photon FFT's has yielded the exact, albeit intricate formulas Eq. (36) and (37), the quantities involved being defined in Eqs. (27)–(34). However, in the four limiting cases considered in Sec. III, simple analytic expressions emerge.

Thus, in Sec. III A it was shown that, in first Born approximation ($E_i, E_f \gg 1$), the cross sections reduce to Eq. (47). In Sec. III B it was shown that, at sufficiently low frequency [$(\omega/E_i),(\omega/E_f) \ll 1$], they reduce to Eq. (58). Although obtained under different conditions, Eqs. (47) and (58) are quite similar in form. As the two conditions may overlap [when E_i, $E_f \gg 1$ and $(\omega/E_i),(\omega/E_f) \ll 1$], it is reassuring to see that then the two formulas actually coincide. By simple inspection one realizes that Eq. (47) can be formally applied to both cases of Sec. III A and Sec. III B: in the first case it is applicable for all θ, whereas in the second only for $\theta \neq 0$. The simple cross section Eq. (47) appears to play a rather central role since we have been able to identify it as an underlying structure of our numerical results, even in cases when the aforementioned conditions are not well satisfied.

The other two limiting cases considered in Secs. III C and III D refer to high photon energy compared with the initial and final electron energy, for the cases of absorption and emission, respectively, and end up with Eqs. (61) and (65). Since also the condition $\omega \gg 1$ was invoked in their derivation, they are at present mainly of theoretical interest. A numerical computation of the absorption and emission differential cross sections, Eqs. (1) and (2), based on our formulas of Sec. II, is now in progress.

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APPENDIX: CALCULATION OF INTEGRAL $I(q_i,q_f)$

We shall now calculate the integral of Eq. (26):

$$I(q_i,q_f)=\oint d\xi_1 \oint d\xi_2 \left[\frac{\xi_1}{\xi_1-1}\right]^{n_i+q_i}\left[\frac{\xi_2}{\xi_2-1}\right]^{n_f+q_f} \times \frac{1}{(s+t_1\xi_1+t_2\xi_2+v\xi_1\xi_2)^2} . \quad \text{(A1)}$$

Here q_i and q_f are non-negative integers, and the integral is to be calculated along the closed contours defined in Sec. II. Principal branches of the complex powers entering Eq. (A1) have to be taken (i.e., $-\pi < \arg\xi/(\xi-1) < \pi$).

The integral over ξ_2 can immediately be performed by the residue theorem to give

$$I(q_i,q_f)=2\pi i(n_f+q_f)\frac{s^{n_f+q_f-1}}{(s+t_2)^{n_f+q_f+1}} \times \oint d\xi_1 \left[\frac{\xi_1}{\xi_1-1}\right]^{n_i+q_i}\left[1+\xi_1\frac{t_1}{s}\right]^{n_f+q_f-1} \times \left[1+\xi_1\frac{t_1+v}{t_2+s}\right]^{-n_f-q_f-1} . \quad \text{(A2)}$$

Since the integration contour in Eq. (A2) is a loop around the points $\xi_1=0$ and 1, it may be tightened such as to encircle $\xi_1=0$ infinitesimally. In the vicinity of this point the integrand is bounded (for $q_i=0$) or vanishes (for $q_i>0$). Therefore the infinitesimal arc around $\xi_1=0$ does not give a contribution to the integral and we may write

$$I(q_i,q_f)=2\pi i(n_f+q_f)\frac{s^{n_f+q_f-1}}{(s+t_2)^{n_f+q_f+1}}e^{i\pi(n_i+q_i)} \times \int_0^{(1+)} \xi_1^{n_i+q_i}(1-\xi_1)^{-n_i-q_i} \times \left[1+\frac{t_1}{s}\xi_1\right]^{n_f+q_f-1} \times \left[1+\frac{t_1+v}{t_2+s}\xi_1\right]^{-n_f-q_f-1} d\xi_1 . \quad \text{(A3)}$$

Now the contour starts at $\xi_1=0$, encircles $\xi_1=1$ in the direct sense, and then returns to $\xi_1=0$.

We may compare Eq. (A3) with the integral representation of the Appell hypergeometric function of two variables F_1:

$$F_1(a;b_1,b_2;c;x_1,x_2) = \frac{ie^{-i\pi(c-a)}}{2\sin\pi(c-a)}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \times \int_0^{(1+)} \xi^{a-1}(1-\xi)^{c-a-1}(1-x_1\xi)^{-b_1} \times (1-x_2\xi)^{-b_2} d\xi , \quad \text{(A4)}$$

valid for $\mathrm{Re}\,a>0$. Thus Eq. (A3) becomes

$$I(q_i,q_f)=4\pi^2(n_i+q_i)(n_f+q_f)\frac{s^{n_f+q_f-1}}{(s+t_2)^{n_f+q_f+1}} F_1\left[n_i+q_i+1;1-n_f-q_f,1+n_f+q_f;2;-\frac{t_1}{s},-\frac{t_1+v}{t_2+s}\right] . \quad \text{(A5)}$$

This can be simplified by using one of the transformation formulas for $F_1(a;b_1,b_2;c;x_1,x_2)$,[40] and noting that the transformed $F_1(a';b_1',b_2';c';x_1',x_2')$ has in our case $b_1'=0$, and consequently reduces to an ordinary Gauss function ${}_2F_1$. We finally get the expression

$$I(q_i,q_f)=G(q_i,q_f)\ {}_2F_1(n_i+q_i+1,n_f+q_f+1,2;z)\ , \tag{A6}$$

where

$$G(q_i,q_f)=-4\pi^2(n_i+q_i)(n_f+q_f)\frac{s^{n_i+q_i+n_f+q_f}}{(s+t_1)^{n_i+q_i+1}(s+t_2)^{n_f+q_f+1}}\ , \tag{A7}$$

$$z=\frac{t_1t_2-su}{(s+t_1)(s+t_2)}\ . \tag{A8}$$

In Eqs. (27)–(31) of Sec. II we also need the derivatives of $I(q_i,q_f)$ with respect to t_1, t_2, and u. By using the formula for the derivative of a Gauss function,[41] we find

$$\frac{\partial I}{\partial t_1}=\frac{n_i+q_i+1}{s+t_1}G(q_i,q_f)\left[\frac{t_2}{s+t_2}\frac{n_f+q_f+1}{2}\,{}_2F_1(n_i+q_i+2,n_f+q_f+2,3;z)\right.$$
$$\left.-{}_2F_1(n_i+q_i+2,n_f+q_f+1,2;z)\right]\ . \tag{A9}$$

The expression for $\partial I/\partial t_2$ can be obtained from Eq. (A9) by interchanging both t_1 and t_2, and the subscripts i and f. [Note that $G(q_i,q_f)$ remains invariant.] Finally,

$$\frac{\partial I}{\partial v}=-\tfrac{1}{2}(n_i+q_i+1)(n_f+q_f+1)s(s+t_1)^{-1}(s+t_2)^{-1}G(q_i,q_f)\ {}_2F_1(n_i+q_i+2,n_f+q_f+2,3;z)\ . \tag{A10}$$

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[31]In usual (non-Z-scaled) atomic units, $f_B^{(+)}$ of Eq. (46) is indeed proportional to Z.

[32]Equation (46) also agrees with the low-intensity limit of the first Born approximation result for the two-photon FFT matrix element obtained (for arbitary intensities) by Bunkin and Fedorov, Ref. 13; see also M. Gavrila, A. Maquet, and V. Véniard, Ref. 5, Sec. 4.1, in particular Eq. (4.4).

[33]This is precisely what happens for forward and backward scattering, $\mathbf{v}_i \cong \pm \mathbf{v}_f$, in the scattering geometry chosen in Ref. 5, where $\mathbf{v}_i \perp \mathbf{e}$. An illustrative case is that of Fig. 1(b) given there which, although not truly within the validity of the Born approximation (as the condition $E_i \gg 1$ is not really satisfied), indicates nevertheless this trend.

[34]See, A. Erdelyi *et al.*, Ref. 27, Sec. 2.10, Eqs. (7) and (8).

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