First-passage-time distribution in a random random walk

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(Received 24 July 1989; revised manuscript received 25 January 1990)

A detailed study of the statistical properties of first-passage times in a one-dimensional random environment is presented. It is found that the spectrum of relaxation times of the master equation describing the hopping dynamics in such a system possesses a gap, which separates the leading relaxation time from the next one by an amount that is exponentially large in the square root of the size of the system. The leading relaxation time for a *given* (typical) system equals τ, the mean first-passage time for *that* realization. One consequence of the existence of this gap is that for times that are larger than a fraction of the mean first-passage time τ, the probability of the first-passage time to have a value t is approximated, to a high degree of accuracy, by $(1/\tau)\exp(-t/\tau)$. The appearance of the above-mentioned gap is, surprisingly, related to the existence of a dominant *single* effective trap (or "potential well"), the next leading trap being of no importance for large systems. A detailed study of the properties of the mean first-passage time and its fluctuations among the members of the ensemble of realizations is presented (the probability distribution of τ is shown to possess anomalous long-time behavior). The realization-averaged mean first-passage time is shown to be dominated by rare realizations and is contrasted with typical values of that quantity. The mean first-passage-time distribution is shown to be related to a new characterization of random walks that we coin the *extent*. The latter quantity is investigated in detail.

I. INTRODUCTION

It is by now well established that transport phenomena in random environments are qualitatively very different from those in regular ones.[1-4] Even minute amounts of randomness may be of major consequences for all transport properties. It is pointless to summarize the wealth of already existing knowledge in this rapidly developing field. Here, we shall concentrate on the properties of a prototypical model for one-dimensional random transport, known as the Sinai model.[5-7] The Sinai model consists of a one-dimensional random walk where the nearest-neighbor hopping probabilities are independent identically distributed random variables. Thus the model describes diffusion in systems with quenched disorder. Among the numerous physical and chemical examples for which this kind of model is relevant, we wish to mention diffusive electronic transport in amorphous media,[8] transport in porous media,[9] turbulent passive scalar diffusion,[10] molecular charge transfer in large molecules, and charge separation in photosynthetic systems.[11,12] The model has also been shown to possess a $(1/f)$-type noise.[13]

Pioneering work on the Sinai model has shown that the diffusion rate in such a system is drastically slower than that in the corresponding regular system. Sinai's own work[6] and subsequent investigations[14-16] have found that the mean square distance $\langle x^2 \rangle$ travelled by a random walker in such a system is proportional to $\ln^4 t$, t being the time. Moreover, Sinai[6] has shown that the position of the walker, starting at $i=0$, after n steps, denoted by $y(n)$, assumes values of order $\ln^2 n$ for large n. Asymptotically in n, the probability distribution of $y(n)/\ln^2(n)$ becomes concentrated in an arbitrarily small neighborhood of a value which depends on the realization of the hopping probabilities. The model was generalized to higher dimensions and studied by using the replica method and the renormalization-group technique.[17-18] Using these techniques it was found that for dimensions $d \geq 2$ the diffusion is regular whereas for $1 \leq d < 2$ it is not. The current understanding is that in one dimension the walker is trapped in deep wells in which the exit time increases exponentially with the depth of the well. In higher dimensions the walker has sufficient "phase space" to avoid these wells, whereas in one dimension the walker must cross each of them as he advances.

It is interesting to note the difference between the Sinai model and two other kinds of random hopping models. One is a random-bond-hopping model[19-20] with finite mean waiting time. In that case, the diffusion rate is given by $\langle x^2 \rangle \simeq 2Dt + O(\sqrt{t})$ where D is the effective diffusion constant. Another form of randomness is that of the waiting times[21-22] at sites which produce anomalous diffusion. An example of this behavior can be found in the diffusion on a random comb in which the length of the teeth L is distributed according to $P(L) \propto L^{-(1+\gamma)}$($L \geq 1$ and $\gamma > 0$). It is found (Refs. 21 and 22) that when the mean waiting time does not exist (corresponding to $\gamma < 1$) then $\langle x^2 \rangle \propto t^{(1+\gamma)/2}$ (where $\langle x^2 \rangle$ is measured along the backbone). When $\gamma \geq 1$, i.e., a mean waiting time exists, the diffusion is regular: $\langle x^2 \rangle \propto t$. When a bias is applied in the direction of the teeth of a random comb defined by the above distribution of teeth's lengths, one obtains[23] $\langle x^2 \rangle \propto (\ln t)^{2\gamma}$. The physical difference between the latter model and the Sinai model is rather obvious: while random waiting times at a site can be interpreted as motion of a particle having a space-dependent random mass, the Sinai dynamics is that of the motion of a particle in a random potential. It has been realized[6,13,24] that the potential itself can be regarded as performing a "random walk" in space. Thus when a segment of length x is considered, one expects a typical potential barrier whose size is of the order of $\sqrt{x}$. The logarithm of the time required to traverse an opposing field of an extent of $\sqrt{x}$ is typically proportional to $\sqrt{x}$; thus one might expect $\langle x^2 \rangle \propto \ln^4 t$. As mentioned before, this argument is borne out by rigorous derivations. Another quantity of interest in any model for random walk is the mean first-passage time. It has been shown by the present authors[25] that in a finite segment of length L, the typical mean first passage time τ satisfies $\ln\tau \propto \sqrt{L}$. The average of τ over the ensemble of realizations of the random hopping rates $\langle \tau \rangle$ satisfies $\ln\langle \tau \rangle \propto L$, which shows that some rare realizations have very long mean first-passage times, affecting the average.

In the present paper, the latter result is generalized in two ways. First, the distribution of first-passage times in a long segment is found for typical (i.e., almost all) realizations. The distribution of mean first-passage times among the various members of the ensemble of realizations is also computed. The investigation of these properties involves the study of a new quantity characterizing random walks, which we coin the *extent*: it is the maximal positive distance travelled by a random walker in a given time. We relate this quantity to the properties of the effective potential encountered by the random walker in the Sinai model.

The present article is organized as follows. Section II presents the formulation of the model as well as the main results obtained in the paper. Sections III and IV can be skipped by readers uninterested in the technical details. Section III is devoted to an analysis of the properties of the mean first-passage time. Its probability distribution in the ensemble of realizations and its average and typical values are computed. In Sec. IV, the probability distribution of first-passage times in a typical realization is computed. This distribution is shown to be simply related to the value of the mean first-passage time. The properties of the characteristic polynomial of the master equation describing the Sinai walk are elucidated, the existence of a gap in its spectrum is established and the size of the gap is computed. Section V offers a brief summary. Technical details involving lengthy calculations are presented in the appendices.

II. FORMULATION OF THE PROBLEM AND STATEMENT OF THE MAIN RESULTS

Consider a one-dimensional lattice of length L: $0 \le j \le L-1$ (j being integers). At each site j there is a probability p_j to hop to site $j+1$ per (discrete) time unit and a probability $q_j = 1-p_j$ to hop to site $j-1$. The set $\{p_j\}$ is one of independent random variables satisfying $0 < p_j < 1$. A probability distribution for the values of $\{p_j\}$ is defined so that $\log(p_j/q_j)$ has zero mean and a finite variance σ^2. The system is assumed to have reflecting and absorbing boundaries at sites 0 and L, respectively. For simplicity, we choose $p_0 = 1$ (else one may introduce a waiting time probability at 0). At site L: $q_L = 0$. Consider a given realization of the random environment (i.e., a specific set $\{p_j\}$). Assume a random walker starts at time 0 from the origin (site 0). Define $\hat{P}_i(n)$ to be the probability to find the random walker after n time units at site i. Clearly, the $\hat{P}$'s satisfy the following master equation:

$$\hat{P}_i(n) = p_{i-1}\hat{P}_{i-1}(n-1) + q_{i+1}\hat{p}_{i+1}(n-1) \tag{2.1}$$

with $P_{-1} \equiv 0$ for convenience. The generating function $D(z)$ (z can be complex) corresponding to any probability distribution function (PDF), $\hat{D}(n)$ is defined as

$$D(z) = \sum_{n=0}^{\infty} \hat{D}(n) z^n . \tag{2.2}$$

Multiplying both sides of Eq. (2.1) by z^n and summing over n from zero to infinity, the following master equation for the generating functions of the P's is obtained:

$$\begin{aligned} P_i(z) = {} & p_{i-1} z P_{i-1}(z) \\ & + q_{i+1} z P_{i+1}(z) + \delta_{i,0}, \quad 0 \le i \le L-1 . \end{aligned} \tag{2.3}$$

The probability distribution function for the first-passage time $G_L(n)$ is the probability that a walker starting at 0 arrives at L, for the first time, after n steps. This quantity is related to $\hat{P}_i$ in an obvious manner:[26]

$$\hat{G}_L(n+1) = \sum_{i=0}^{L-1} \hat{P}_i(n) - \sum_{i=0}^{L-1} \hat{P}_i(n+1) . \tag{2.4}$$

It follows from Eqs. (2.2) and (2.4) that $G_L(z)$ is given by

$$G_L(z) = 1 + (z-1) \sum_{i=0}^{L-1} P_{i(z)} . \tag{2.5}$$

In Appendix A we present a summary of the relevant properties of generating functions and calculate some PDF's which are useful in the sequel. There, it is shown that $G_L(z)$ is analytic inside a disc of radius $1+\epsilon$ where ϵ is the distance of the nearest singularity of $G_L(z)$ to the unit disk. The only singularities of $G_L(z)$ are L simple poles located on the real axis (see Appendix A). The poles are the zeros of an Lth-order polynomial (the characteristic values of the master equation) whose coefficients are functions of the set $\{p_n\}$ of hopping probabilities. Higher-order poles can exist in principle but the realizations leading to such situations have vanishing probability (in the nonrandom case, i.e., all p_i = const, the poles are distinct). In addition, it can be shown that the residues corresponding to these poles are real (see Appendix A). For definiteness, we consider the case of even L (when L is odd, the formulas below are modified as shown in Appendix A). The main results remain unchanged. One has

$$G_L(z) = \sum_{i=1}^{L} \frac{A_i}{z - z_i} + K \tag{2.6}$$

with $|z_i| > 1$, the $\{z_i\}$ being real, $\{A_i\}$ are the corresponding residues, and K is a constant. In the latter sum, to every pole z_i there corresponds a pole $-z_i$ accounting for the fact that $\hat{G}_L(n)$ vanishes for odd n (since L is assumed to be even). Transforming back to "time" space (i.e., inverting Eq. (2.2) with D replaced by G_L (see Appendix A) one obtains, for even n,

$$\hat{G}_L(n) = -2 \sum_{i=1}^{L/2} \frac{A_i}{(z_i)^{n=1}} , \tag{2.7}$$

where the sum extends over the positive z_i's. It is assumed that the z_i's are ordered so that $|z_1| \le |z_2| \le \cdots$. It is shown (Sec. IV) below that for long segments ($L \gg 1$) and for "times" satisfying $n > \tau \exp(-\beta\sqrt{L})$, β being $O(1)$, the sum in Eq. (2.7) is dominated by its first term. Thus, for large times [much smaller though than the mean first-passage time (MFT)], $\hat{G}_L(n)$ decays at a single (exponential) rate $\tau_1 \equiv 1/\ln(|z_1|)$. The value of τ_1 is computed (Sec. IV) and found to equal τ, the mean first-passage time corresponding to the given realization up to a correction which is exponentially small is $\sqrt{L}$: $\tau_1 \simeq \tau[1-\exp(-\gamma\sqrt{L})]$ where $\gamma = O(1)$. Consequently, for a given typical realization, the distribution of first-passage times is well approximated by $(2/\tau)\exp(-n/\tau)$ [since L is assumed even $\hat{G}_L(n)=0$ for odd n, explaining the factor of 2]. Figure 1 presents results of a simulation demonstrating this result. The physical source of this result is the fact, proven below (Sec. IV and Appendix C) that, in a random environment there is typically only one leading "trap." In other words, a random potential in a system of length L has one effective well of leading depth (which may contain smaller wells inside), the next deepest wells being insignificant for transport processes.

The probability distribution of the MFT itself, among the members of the ensemble of realizations is also computed (Sec. III). It is found to be related to a property of one-dimensional walks, which we coin the *extent* (see Appendix B); the latter entity is defined as the maximal (net) distance travelled by a random walker in a given direction. We find that [cf. Eq. (3.10)] asymptotically in L (the number of sites), the probability for the logarithm of the mean first-passage time τ to be smaller or equal to α is (i.e., $L \to \infty$ and keeping the ratio $\alpha/\sqrt{L}$ constant)

$$\text{Prob}(\ln\tau \le \alpha; L) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-)^m}{2m+1} e^{-[L\sigma(2m+1)^2\pi^2/8\alpha^2]} . \tag{2.8}$$

When $\alpha \gg \sqrt{L}$ the latter can be summed and the result transformed to yield the probability density for τ:

$$\text{Prob}(\tau) \simeq \frac{4}{\sqrt{\pi L \sigma^2}} \exp\left[-\frac{1}{2}\frac{\ln^2\tau}{L\sigma^2}\right]\frac{1}{\tau} .$$

Notice the slow decay (essentially proportional to $1/\tau$) of this probability density and the nonstandard exponential factor which ensures that Prob(τ) is normalizable.

III. THE DISTRIBUTION OF MEAN FIRST-PASSAGE TIMES

In this section we study the properties of the mean first-passage time, denoted by τ, for the problem at hand. This quantity as well as higher moments of first-passage times are obtained from the nth derivative of $G_L(z)$ [cf. Eqs. (2.5)] at $z=1$. Obviously,

$$G_L(1) = \sum_{n=0}^{\infty} \hat{G}_L(n) = 1 , \tag{3.1}$$

since the walker arrives at site L with probability unity. We also define the following moments:

$$G_L^{(k)}(1) \equiv \left.\frac{d^k G_L}{dz^k}\right|_{z=1} .$$

In particular $G_L^{(1)}(1) \equiv \tau$ is the mean first-passage time, and $G_L^{(2)}(L) = E(n^2) - \tau$ is the mean-square fluctuation of the first-passage times. The symbol E denotes the expectation value (for a given realization of the set $\{p_i\}$). Averages over an ensemble of realizations are denoted by $\langle\ \rangle$. It follows from Eq. (2.5) that $G_L^{(1)}(1) \equiv \tau$ equals

$$\tau = \sum_{i=0}^{L-1} S_i ,$$

where $S_i \equiv P_i(1)$. We define

$$r_j = \frac{q_j}{p_j} .$$

It can be checked that the solution of Eq. (2.3) for $z=1$ is

$$S_i = \frac{1}{p_i} \sum_{k=i}^{L-1} \prod_{j=i+1}^{k} r_j \tag{3.2}$$

with the convention that $\prod_{j=i+1}^{k} r_j = 1$ when $k < j+1$. Using Eqs. (3.2) we find that the MFT (Refs. 11 and 27)

$$\tau = \sum_{i=0}^{L-1} S_i = \sum_{i=0}^{L-1} \frac{1}{p_i} \sum_{k=i}^{L-1} \prod_{j=i+1}^{k} r_j . \tag{3.3}$$

Equation (3.3) contains a sum of $L(L+1)/2$ products. Assume $p_i > c > 0$, c being a constant, for all $0 \le i \le L-1$. It follows from Eq. (3.3) that

$$\max_{0\le i\le k\le L-1} \prod_{j=i+1}^{k} r_j < \tau < \frac{1}{c}\frac{L(L+1)}{2} \max_{0\le i\le k\le L-1} \prod_{j=i+1}^{k} r_j . \tag{3.4}$$

The lower bound for τ is obtained by keeping the largest product in the sum, in Eq. (3.3), whereas the upper bound is obtained by replacing each of the products in Eq. (3.3) by the largest one. Recall that, by assumption, the random variable $\xi_j = \log r_j$ has zero mean and a variance of σ^2. We define $X_0 = 0$ and $X_j = \sum_{i=1}^{j} \xi_i$ for $1 \le j \le L$. The variables X_j are coordinates of the path of an unbiased random walk, whose steps are ξ_j and $\langle \xi_j^2 \rangle = \sigma^2$. Notice that here the space coordinates $0 \le i \le L$ play a role equivalent to that of time in a conventional random walk. Taking the logarithm of inequality (3.4) yields

$$\max_{0\le i\le k\le L-1} \sum_{j=i+1}^{k} \xi_j < \ln\tau < \max_{0\le i\le k\le L-1} \sum_{j=i+1}^{k} \xi_j + \ln[L(L+1)/2c] . \tag{3.5}$$

Thus, $\ln\tau$ is bounded both from above and below by sums related to an unbiased random walk. It is convenient to define the set S^L (coordinates of a given path, consisting of L steps)

$$S^L=\{X_j;0\le j\le L\} \tag{3.6}$$

and the quantity $\rho(S^L;L)$:

$$\rho(S^L;L)=\max_{1\le i\le k\le L}\left|\sum_{j=i}^{k}\xi_j\right| , \tag{3.7}$$

which is the maximum (over all points) of the distance between any point on a given path and any previous point. We shall call the quantity ρ the *extent* of the path. This quantity is to be *distinguished* from the span[34] (or range) $r(S^L;L)$ defined as

$$r(S^L;L)=\max(x;x\in S^L)-\min(x;x\in S^L) . \tag{3.8}$$

The essential difference between r and ρ is that the extent is an oriented quantity whereas r is not [see Fig. (2)]. They coincide for half of the paths. For the other half, the span corresponds to a "negative extent" (i.e., the extent for the corresponding backward going path starting from L). Surprisingly, it turns out that r and ρ have *different* statistical properties. Their distributions differ (even asymptotically) and so do the corresponding moments. A detailed analysis of the properties of the extent is given in Appendix B. We define

$$P(\alpha;L)=\text{Prob}[\rho(S^L;L)\le\alpha] . \tag{3.9}$$

The limit distribution ($L\to\infty,\alpha/\sqrt{L}$ const) is found to be

$$P(\alpha;L)=\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\exp\left[-\frac{L(2m+1)^2\pi^2\sigma^2}{8\alpha^2}\right] . \tag{3.10}$$

It is intriguing that Eq. (3.10) coincides with the Erdös-Kac[28] formula for the maximum *displacement* in a random walk. It follows from Eq. (3.9) that $E(\rho^L)=(\pi/2)\sigma L^{1/2}$ and $E((\rho^L)^2)\simeq 2G\sigma^2 L$, where G is Catalan's constant ($G=0.916\,966+$). This is to be contrasted with the average of the span which equals $E(r^L)=\sqrt{8\pi}\sigma L^{1/2}$.

As seen from Eq. (3.10), the typical values of α are all of order $\sqrt{L}$: $\alpha=\beta/\sqrt{L}$, β typically being $O(1)$. Using Eqs. (3.5) and (3.7) it follows that asymptotically in L, $\ln\tau\simeq\rho(S^L;L)$, a quantity whose distribution is given by Eq. (3.10). Hence, typically,

$$\tau=e^{\beta\sqrt{L}} , \tag{3.11}$$

where β is an $O(1)$ parameter whose expectation value is $\sqrt{\pi/2}$ and the parameter β is *realization dependent.* Averaging τ over all realizations (denoted by $\langle\ \rangle$) one obtains[27]

$$\langle\tau\rangle=\frac{2x}{x-1}\frac{x^L-1}{1-x}-\frac{x+1}{x-1}L ,$$

where $x=\langle g/p\rangle$. The latter equation shows that the logarithm of the realization averaged mean first-passage time $\ln\langle\tau\rangle$ is proportional to L whereas its typical value (i.e., with probability 1) is proportional to $\sqrt{L}$. This means that rare values of τ do influence τ in a significant way. To see how this comes about, we investigate the tail of the distribution of MFT's. Let $\epsilon\equiv L\pi^2\sigma^2/8\alpha^2$, then Eq. (3.10) reads

$$I(\epsilon)=\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}e^{-\epsilon(2m+1)^2} , \tag{3.12}$$

where $I(\epsilon)\equiv P(\alpha,L)$, for later convenience. We define the Laplace transform

$$I(s)\equiv\int_0^\infty e^{-s\epsilon}I(\epsilon)d\epsilon . \tag{3.13}$$

Using straightforward manipulations one finds

$$I(s)=I_1(s)+I_2(s)+I_3(s) , \tag{3.14}$$

where

$$I_1(s)=\frac{1}{s} , \tag{3.15a}$$

$$I_2(s)=-\frac{2}{\pi s}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1+i\sqrt{s}} , \tag{3.15b}$$

$$I_s(s)=-\frac{2}{\pi s}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1-i\sqrt{s}} . \tag{3.15c}$$

The sum in $I_2(s)$ can be performed by noting that

$$I_2(s)=\frac{-2}{\pi s}\sum_{m=0}^{\infty}(-1)^m\int_0^\infty e^{-\alpha(2m+1+i\sqrt{s})}d\alpha \tag{3.16}$$

or

$$I_2(s)=-\frac{1}{\pi s}\int_0^\infty\frac{e^{-i\alpha\sqrt{s}}}{\cosh\alpha}d\alpha .$$

Hence

$$I_2(s)+I_3(s)=-\frac{1}{\pi s}\text{Re}\int_{-\infty}^{\infty}\frac{e^{i\alpha\sqrt{s}}}{\cosh\alpha}d\alpha . \tag{3.17}$$

Using complex integration analysis (i.e., considering a contour that includes the real axis and an upper-half circle) one finds the value of the integral in Eq. (3.17), from which by adding $I_1(s)=1/s$,

$$I(s)=\frac{1}{s}-\frac{2}{s}\sum_{m=0}^{\infty}(-1)^m\int_0^\infty e^{-\alpha\pi(m+1/2)\sqrt{s}}d\alpha . \tag{3.18}$$

The limit of small ϵ corresponds to large values of s. In that limit, one can find the inverse Laplace transform of Eq. (3.18) using the method of steepest descent. The result is

$$I(\epsilon)\simeq 1-\frac{4\sqrt{2\epsilon}}{\pi^{3/2}}\sum_{m=0}^{\infty}\frac{1}{m+\frac{1}{2}}\exp\left[-\frac{\pi^2(n+1/2)^2}{4\epsilon}\right] . \tag{3.19}$$

Since

$$I(\epsilon)=\text{Prob}\left[\ln\tau<\alpha;\ \epsilon=\frac{L\pi^2\sigma^2}{8\alpha^2}\right]$$

[cf. Eq. (3.10)] one can transform Eq. (3.19) to obtain the probability density for the value of τ, $P(\tau)$. The result is

$$P(\tau)\simeq\frac{4}{\sqrt{\pi L\sigma^2}}\exp\left[-\frac{1}{2}\frac{\ln^2\tau}{L\sigma^2}\right]\frac{1}{\tau} . \tag{3.20}$$

The above result shows that the tail of the distribution decays very slowly, which is the reason for the difference between the average and the typical value of τ. Notice that the (nonstandard) exponential factor insures integrability of the distribution function.

IV. THE DISTRIBUTION OF FIRST-PASSAGE TIMES FOR A TYPICAL REALIZATION

A. The characteristic polynomial

In this section we study the generating function of the first-passage times $G_L(z)$. The master equation [see Eq. (2.3)] can be written as

$$\mathbf{P}=\underline{M}(z)\mathbf{P}+\mathbf{e}_0 \tag{4.1}$$

where $\mathbf{P}$ is an L vector whose ith component is $P_i(z)$ and $\mathbf{e}_0$ is the unit vector with components $\delta_{i,0}$ and

$$\underline{I}-\underline{M}=\begin{bmatrix} 1 & -zq_1 & 0 & 0 & & \cdots & 0\\ -zp_0 & 1 & -zq_2 & 0 & & \cdots & 0\\ 0 & -zp_1 & 1 & -zq_3 & & \cdots & 0\\ & \vdots & & & & & \\ 0 & 0 & & & -zp_{L-2} & & 1 \end{bmatrix} . \tag{4.2}$$

The matrix $\underline{T}\equiv\underline{I}-\underline{M}$ is $L\times L$ and tridiagonal with the following nonzero elements:

$$t_{i,i}=1,\quad t_{i,i-1}=-zp_{i-1},\quad t_{i,i+1}=-zq_{i+1}$$

(row and column indices range from 0 to $L-1$).

Since from Eq. (4.1) $\mathbf{P}=\underline{T}^{-1}\mathbf{e}_0$, it follows that the poles of $P_i(z)$ in the z plane are also zeros of the determinant of $\underline{T}$. By Eq. (2.5) these are also the poles of $G_L(z)$. Next we compute the determinant of $\underline{T}$ which is a polynomial of degree L in z. For definiteness, we consider the case of even L (see Appendix A for the case of odd L). In this case, $\hat{G}_L(n)$ must vanish for odd n. It is easy to see that as a result $G_L(z)$ is an even function of z. Recall that the poles of $G_L(z)$ are real and satisfy $|z|>1$ (cf. Appendix A). Hence a pole at z_p (real and $z_p\geq 1$) must be accompanied by a pole at $-z_p$. Let the variable μ be defined by

$$\frac{1}{z}=-(1+\mu) . \tag{4.3}$$

Next, we perform elementary transformations on T. Dividing the ith column of $\underline{T}$ for each and every i by $-zp_i$ $(i=0,\ldots,L-1)$ we obtain a new (tridiagonal) matrix, denoted by $\underline{W}$. Thus

$$\det|\underline{T}|=\det|\underline{W}|\left[z^L\prod_{j=0}^{L-1}p_j\right] . \tag{4.4}$$

Since we know that $G_L(z)$ is regular at $z=0$, the zeros of $\det|\underline{T}|$ are those of $\det|\underline{W}|$. Using the relation $1/p_j=1+r_j$ (recall that $r_j\equiv q_j/p_j$), the nonvanishing elements of $\underline{W}$ are $(0\leq\alpha\leq L-1)$:

$$w_{\alpha,\alpha}=1+r_\alpha+\frac{\mu}{p_\alpha},\quad w_{\alpha,\alpha-1}=1,\quad w_{\alpha,\alpha+1}=r_{\alpha+1} .$$

We define the set of tridiagonal matrices $\{\underline{U}^{i,j}\}$ $(0\leq i\leq j\leq L-1)$ whose elements are

$$u^{i,j}_{\alpha,\alpha}=1+r_\alpha+\frac{\mu}{p_\alpha},\quad u^{i,j}_{\alpha,\alpha-1}=1,\quad u^{i,j}_{\alpha,\alpha+1}=r_{\alpha+1} ,$$

where the row and column indices range from i to j. Obviously, $\underline{W}=\underline{U}^{0,L-1}$. The determinants $D(i,k,\mu)$ are defined as

$$D(i,k,\mu)\equiv\det|\underline{U}^{i,k}| . \tag{4.5}$$

It follows that $D(0,L-1,\mu)\equiv\det|\underline{W}|$. When $\mu=0$, it is convenient to define $D(i,k,0)\equiv E(i,k)$. The poles of G_L (or the zeros of $\det|\underline{W}|$) are the solutions of $D(0,L-1,\mu)=0$. It is straightforward to see that the D's satisfy the following recursion relation:

$$D(i,k,\mu)=\left[1+r_i+\frac{\mu}{p_i}\right]D(i+1,k,\mu)-r_{i+1}D(i+2,k,\mu)$$

with

$$D(k,k,\mu)=1+r_k+\mu/p_k . \tag{4.6}$$

Solving the recursion Eq. (4.6) backwards in i, for $\mu=0$, we find for $D(i,k,0)$ $[\equiv E(i,k)]$

$$E(i,k)=1+r_i+r_ir_{i+1}+\cdots+r_i\cdots r_k \tag{4.7}$$

(which is also easy to prove by direct induction). Thus $E(k,k)=1+r_k$ and it is convenient to define $E(k+1,k)\equiv 1$.

The determinant $D(0,L-1,\mu)$ is a polynomial of degree L in μ:

$$D(0,L-1,\mu)=f_0+f_1\mu+f_2\mu^2+\cdots+f_L\mu^L , \tag{4.8}$$

where the f's are functions of the set $\{p_i\}$. From Eq. (4.8), $f_0=D(0,L-1,0)=E(0,L-1)$. It follows then that $f_0=1$ since $E(0,k)=1$ for all k (because $p_0=1$, by assumption, implying $r_0=0$). The other f_i's are calculated below.

First, the constant f_1 is equal to the sum of the minors of elements $w_{\alpha,\alpha}$ $(0 \le \alpha \le L-1)$ in which μ has been set equal to zero. Thus

$$f_1 = \frac{1}{p_0}E(1,L-1) + \frac{1}{p_1}E(0,0)E(2,L-1) + \frac{1}{p_2}E(0,1)E(3,L-1) + \cdots + \frac{1}{p_i}E(0,i-1)E(i+1,L-1) + \cdots + \frac{1}{p_{L-1}}E(0,L-2) .$$

Using the fact that $E(0,k)=1$,

$$f_1 = \frac{1}{p_0}E(1,L-1) + \frac{1}{p_1}E(2,L-1) + \cdots + \frac{1}{p_i}E(i+1,L-1) + \frac{1}{p_{L-1}} . \tag{4.9}$$

Using Eq. (4.7) in Eq. (4.9), we obtain the r.h.s of Eq. (3.3). Hence $f_1 = \tau$.

The next term f_2 is found by deleting rows and columns i and j of the matrix $\underline{W}$ and summing the appropriate minors over i and j:

$$f_2 = \sum_{j=1}^{L-1} \sum_{i=0}^{j-1} \frac{1}{p_i p_j} E(i+1,j-1)E(j+1,L-1) . \tag{4.10}$$

Note that in the r.h.s. of the above, the factor $E(0,i-1)$ has been omitted since it is unity. Also, the convention $E(i,i-1)=1$ takes care of the case $i=j-1$. The sum over i in Eq. (4.10) is identical to the sum defining f_1 in Eq. (4.9) where L is to be replaced by j. Defining $\tau_{0,j}$ as the MFT for walk from site 0 to site j (under the same conditions as the walk considered so far), we find

$$f_2 = \sum_{j=1}^{L-1} \tau_{0,j} \frac{1}{p_j} E(j+1,L-1) . \tag{4.11}$$

The general structure of f_n can now be written (and proven by induction):

$$f_n = \sum_{j_n=n-1}^{L-1} \cdots \sum_{j_1=0}^{j_2-1} \frac{1}{p_{j_1}p_{j_2}\cdots p_{j_n}} E(j_1+1,j_2-1)E(j_2+1,j_3-1)\cdots E(j_n+1,L-1) .$$

Using:

$$\tau_{0,j_2} = \sum_{j_1=0}^{j_2-1} \frac{1}{p_{j_1}} E(j_1+1,j_2-1) ,$$

the sum over j_1 can be performed to yield

$$f_n = \sum_{j_n=n-1}^{L-1} \cdots \sum_{j_2=1}^{j_3-1} \frac{1}{p_{j_2}\cdots p_{j_n}} \tau(0,j_2)E(j_2+1,j_3-1)\cdots E(j_n+1,L-1) . \tag{4.12}$$

Also, note that

$$f_L = \prod_{j=1}^{L-1} \frac{1}{p_j} .$$

It is convenient to define the rescaled variables $x \equiv \mu\tau$. In terms of this variable, the equation $D(0,L-1,\mu)=0$ reads [see Eq. (4.8)]

$$1 + x + \frac{f_2}{\tau^2}x^2 + \cdots + \left[\frac{f_n}{\tau^n}\right]x^n + \cdots = 0 . \tag{4.13}$$

In Sec. IV B an analysis of the values of the coefficients f_n as given by Eq. (4.12) is presented.

B. Analysis of the coefficients f_n

In order to find the solutions of Eq. (4.13), we investigate the values of the coefficients f_n given by Eq. (4.12). It turns out that it is convenient to define the following "small parameter" for large L:

$$\epsilon = e^{-\sqrt{L}} . \tag{4.14}$$

Consider a segment $[r,s]$ belonging to $[0,L]$. When the hopping probability at the leftmost site p_r is replaced by unity, one may consider the distribution of first-passage times from site r to site s, in analogy with the one from 0 to L. That is, the walker starts at r and is not allowed to return to $r-1$. Clearly, the MFT for such a walk is [cf. Eq. (3.3)]

$$\tau_{r,s} = E(r+1,s-1) + \frac{1}{p_{r+1}}E(r+2,s-1) + \cdots + \frac{1}{p_{s-1}} . \tag{4.15}$$

In particular $\tau_{j,L} > E(j+1,L-1)$ since all terms in (4.15) are positive. Clearly, since $p_j > c$ (by assumption) one has

$$\frac{1}{p_j}E(j+1,L-1) < \frac{1}{c}E(j+1,L-1) < \frac{1}{c}\tau(j,L) .$$

The latter inequality combined with Eq. (4.11) yields

$$\frac{f_2}{\tau^2} < \frac{1}{c\tau^2} \sum_{j=1}^{L-1} \tau_{0,j}\tau_{j,L} . \tag{4.16}$$

Denote the index of the maximal term in the above sum by M. It follows that

$$\frac{f_2}{\tau^2} < \frac{L}{c\tau^2}\tau_{0,M}\tau_{M,L} \ . \tag{4.17}$$

Henceforth, the extent of $[r,s]$ will be denoted $\mathrm{Ext}(r,s)$. According to the results of Sec. III [see Eq. (3.5)], $\ln\tau_{r,s}$ (the logarithm of the MFT for the segment $[r,s]$) satisfies the following inequality:

$$\ln\tau_{r,s} < \mathrm{Ext}(r,s) + \ln\left[\frac{1}{2c}(s-r)(s-r+1)\right] . \tag{4.18}$$

We define

$$\zeta(r,s) = \exp[\mathrm{Ext}(r,s)] \ , \tag{4.19}$$

that is [cf. Eq. (3.7)]

$$\zeta(r,s) = \max_{r\le i\le k\le s} \prod_{j=i}^{k} r_j \ . \tag{4.20}$$

Using Eq. (4.18) and definition (4.19), to bound $\tau_{0,M}$ and $\tau_{M,L}$ in Eq. (4.17) and replacing factors like $L-M$ by $L+1$, one obtains the following bound:

$$\frac{f_2}{\tau_2} < \frac{(L+1)^5}{4c^3}\frac{1}{\tau^2}\zeta(0,M)\zeta(M,L) \ . \tag{4.21}$$

Since $\tau > \mathrm{Ext}(0,L)$, it follows from the last inequality that

$$\frac{f_2}{\tau^2} < \frac{(L+1)^5}{4c^3}\frac{\zeta(0,M)\zeta(M,L)}{\zeta(0,L)^2} \ . \tag{4.22}$$

The r.h.s. of Eq. (4.22) is investigated in detail in Appendix C. It is proven there that for a typical large system ($L \gg 1$) there is an $O(1)$ number γ such that the above ratio of ζ's is $O(\epsilon^\gamma)$. Thus

$$\frac{f_2}{\tau_2} \le O(\epsilon^\gamma) \ . \tag{4.23}$$

The coefficient f_3 can be read off Eq. (4.12) for $n=3$:

$$f_3 = \sum_{j_1=1}^{L-2}\sum_{j_2=j_1+1}^{L-1} \tau_{0,j_1}\frac{1}{p_{j_1}p_{j_2}}E(j_1+1,j_2-1) \times E(j_2+1,L-1)$$

or by rearranging terms and dividing by τ^3:

$$\frac{f_3}{\tau^3} = \frac{1}{\tau^2}\sum_{j_1=1}^{L-1}\tau_{0,j_1}\frac{1}{p_{j_1}}E(j_1+1,L-1)K(j_1,j_2) \ , \tag{4.24}$$

where $K(j_1,j_2)$ is defined as

$$K(j_1,j_2) \equiv \sum_{j_2=j_1+1}^{L-1}\frac{E(j_1+1,j_2-1)(1/p_{j_2})E(j_2+1,L-1)}{\tau E(j_1+1,L-1)} \tag{4.25}$$

and $K(L-1,j_2)\equiv 0$. Thus, f_3/τ^3 as given by Eq. (4.24) is represented by $1/\tau^2$ times the sum appearing in the r.h.s. of Eq. (4.11) with every term weighted by $K(j_1,j_2)$. It is shown next that $K(j_1,j_2)<1$ implying $f_3/\tau^3 < f_2/\tau^2$.

First, we note that since all the terms contained in $E(j_1+1,j_2-1)$ are also contained in $E(j_1+1,L-1)$ we have

$$K(j_1,j_2) < \frac{1}{\tau}\sum_{j_2=j_1+1}^{L-1}\frac{1}{p_{j_z}}E(j_z+1,L-1) \ .$$

Replacing j_1+1 by zero increases the r.h.s. of the above inequality,

$$K(j_1,j_2) < \frac{1}{\tau}\sum_{j_2=0}^{L-1}\frac{1}{p_{j_z}}E(j_z+1,L-1) \ .$$

It follows that Eq. (4.9) and the fact that $f_1=\tau$ that the r.h.s. of the last inequality is unity. Thus $K(j_1,j_2)<1$ and $f_3/\tau^3 < f_2/\tau^3$. This result can be easily generalized to arbitrary n: the sum defining f_{n+1} is to be rearranged as a sum over terms belonging to f_n weighted by the same K, as defined in Eq. (4.25). Hence

$$\frac{f_{n+1}}{\tau^{n+1}} < \frac{f_n}{\tau^n} \ . \tag{4.26}$$

C. The zeros of the characteristic polynomial

It follows from Eqs. (4.23) and (4.26) that all coefficients $\{f_n/\tau^n; n\ge 2\}$ are bounded by positive powers of ϵ. Thus the characteristic polynomial Eq. (4.13) has the form

$$1 + x + \epsilon^{\kappa_2}x^2 + \cdots = 0 \ , \tag{4.27}$$

where $0<\kappa_2\le\kappa_3\cdots$ are $O(1)$ numbers.

When $\epsilon=0$, there is only one solution $x=-1$ to Eq. (4.27). This means that for finite, yet very small ϵ, there is a solution of Eq. (4.27) of the form

$$x_1 = -1 + a_1\epsilon^{\delta_1} \ , \tag{4.28}$$

where a_1 is $O(1)$. The other zeros $x_j (2\le j\le L)$ tend to infinity as some positive power of ϵ [Ref. (29)]:

$$x_j = a_j\epsilon^{-\delta_j} \ , \tag{4.29}$$

where the a_j's are $O(1)$ numbers. Let us consider the negative poles z_j of $G_L(z)$ only (as mentioned before each z_j has a corresponding pole $-z_j$). Since $x\equiv\mu\tau$ and $1/z = -(1+\mu)$ [cf. Eq. (4.3)], the pole corresponding to x_1 [Eq. (4.28)] is given by

$$z_1 = -\frac{1}{1-(1/\tau)(1-a_1\epsilon^{\delta_1})} \simeq -1-(1/\tau)(1-a_1\epsilon^{\delta_1}) \tag{4.30}$$

and those corresponding to x_j [Eq. (4.29)], are given by

$$z_j = -\frac{1}{1+\frac{1}{\tau} a_j \epsilon^{-\delta_j}} . \tag{4.31}$$

Next, we prove that the z_j's are separated by an exponentially large *gap* from z_1. Using definition (4.14) and Eq. (4.11), it follows that $\tau = \epsilon^{-\beta}$. Hence, from Eq. (4.31)

$$z_j = -\frac{1}{1+a_j \epsilon^{\beta-\delta_j}} . \tag{4.32}$$

Note that β cannot be smaller than δ_j, otherwise the denominator in Eq. (4.32) would be larger than 1, thereby yielding a pole inside the unit disc which is forbidden by the analyticity of $G_L(z)$ in that domain. Thus $\beta \geq \delta_j$. If $\beta > \delta_j$ then

$$z_j \simeq -1 + a_j \frac{\epsilon^{-\delta_j}}{\tau} \tag{4.33}$$

(note that $a_j < 0$ by analyticity of $G_L(z)$ inside the unit disc). In this case, the sought exponential gap in the relaxation time spectrum $\{\ln|z_j|\}$ is obtained at once: $\ln|z_2|/\ln z_1| \simeq \epsilon^{-\delta_2}$. The case $\beta = \delta_j$ is examined next. Here one has

$$z_j = -\frac{1}{1+a_j} .$$

The condition $z_j < -1$ implies $-1 < a_j < 0$. Now, for z_j to be close to -1, a_j must be close to zero. Hence $|z_j + 1| >> 1/\tau$, else a_j is a positive power of ϵ, contrary to assumption.

It remains to show that the residues A_l [see Eq. (2.6)] are well behaved, that is, they are not too large so as to compensate for the "gap" in the poles. In Appendix A, an explicit formula yielding the residues as functions of the poles is derived. It is found, there, that $|A_i / A_1| < e^{kL}$, where k is an $O(1)$ number. For odd n, $\hat{G}_L(n) = 0$, whereas for even n

$$\hat{G}_L(n) \simeq -2A_1 \left[\frac{1}{z_1}\right]^{n+1} - 2A_2 \left[\frac{1}{z_2}\right]^{n+1} + \cdots , \tag{4.34}$$

since L is assumed even, here (see Appendix A). Denote by R_j the absolute value of the ratio of the jth term to the first term in Eq. (4.34). Thus

$$R_j = \left|\frac{A_j}{A_1}\right| \left|\frac{z_1}{z_j}\right|^{n+1} .$$

Using Eqs. (4.30), (4.33), and the upper bound for $|A_j / A_1|$ one finds that

$$R_j < e^{kL} \left[\frac{1+\frac{1}{\tau}}{1+\frac{1}{\tau}|a_j|\epsilon^{-\delta_j}}\right]^{n+1} .$$

Hence

$$R_j < \exp(\tilde{\Delta}) , \tag{4.35}$$

where

$$\tilde{\Delta} = kL + (n+1)[e^{-\beta\sqrt{L}} - |a_j| e^{\delta_j - \beta\sqrt{L}}] .$$

It follows from Eq. (4.35) that the condition for R_j to be far smaller than unity is $\tilde{\Delta} \ll -1$. Since $\delta_j > 0$ and $\tau = e^{\beta\sqrt{L}}$, this condition is equivalent to

$$n >> \frac{kL}{|a_j|} \tau e^{-\delta_j \sqrt{L}} \tag{4.36}$$

for all j. Since k and $|a_j|$ are $O(1)$, it follows from Eq. (4.36) that the first term in Eq. (4.34) dominates for

$$n >> \tau e^{-\delta_j \sqrt{L}} ,$$

that is, for an exponentially small (in $\sqrt{L}$) fraction of the mean first-passage time. This result is demonstrated in Fig. 1(a).

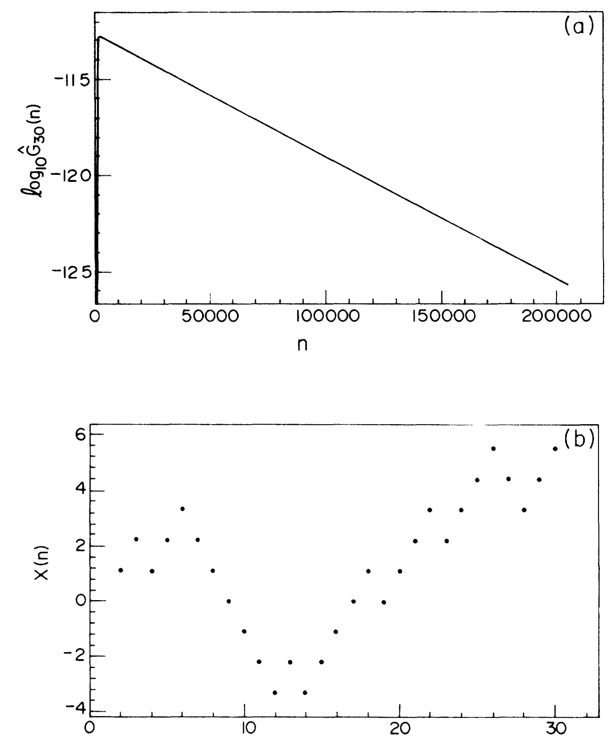

FIG. 1. (a) The logarithm of the first-passage time distribution as a function of the number of steps for $L = 30$. The MFT computed by means of formula (3.3) in the text equals 1.563×10^5 for the specific configuration [see (b)]. The inverse of the slope equals the MFT within numerical errors. Note that the exponential behavior begins at $n \simeq 1000$, well before the MFT. (b) The realization for which the first-passage time distribution is computed. The y axis represents the value of the random potential $X(n) = \sum_{i=1}^{n} \ln(p_i / q_i)$ and the x axis is the number of steps. The distribution of the hopping probabilities is taken in this simulation to be binomial $\text{Prob}(p_j = \frac{1}{4}) = \text{Prob}(p_j = \frac{3}{4}) = \frac{1}{2}$. Note that the span and the extent coincide in this case and are situated between sites 11 and 25.

V. SUMMARY

It has been shown that the MFT in the Sinai model is typically dominated by *one* largest potential barrier. The latter is identified with the exponential of a quantity called the extent. The probability for having a first-passage time t is $(1/\tau)e^{-t/\tau}$ for times t that are larger than a small fraction of τ. Here τ is the MFT of the (typical) realization considered. It is thus different for different realizations. The spectrum of relaxation times of the master equation corresponding to hopping in a finite segment has an exponentially large gap, which is the reason for the simple exponential decay of the distribution of first-passage times. This demonstrates, as has been shown directly, that, surprisingly, the next to the leading "potential barrier" in a one-dimensional random potential is, typically, exponentially small with respect to the leading one. The distribution of mean first-passage times in the ensemble of realizations of the environment is shown to have a long $(1/\tau)$ tail, cut off by an exponential term in $(\ln\tau)^2$, which explains the strong difference between the average value of τ (i.e., $\ln\langle\tau\rangle\propto L$) and its typical value ($\ln\tau\propto\sqrt{L}$).

APPENDIX A: THE METHOD OF GENERATING FUNCTIONS

1. Outline of the method

In the present appendix we outline some relevant features of the method of generating functions. A detailed analysis can be found in Refs. 30 and 31. Consider a discrete network of sites with hopping probabilities defined among them per unit discrete time. The following probability distribution functions are called elementary.

(i) $\hat{T}_{AB}(n)$: the probability to leave site A on the first step and reach B, for the first time, in n steps, without ever returning to A.

(ii) $\hat{Q}_{AB}(n)$: the probability to leave A on the first step and return to A for the first time without ever reaching B, after n steps.

It turns out that many other PDF's of interest may be expressed in terms of these elementary PDF's. The first-passage probability distribution of going from A to B, $\hat{G}_{AB}(n)$, is defined similarly to $\hat{T}_{AB}(n)$, except that returns to A are allowed. $\hat{P}_A(n)$ is defined as the probability to be found at A after n steps. Note that $n=0$ is included in the definition. The generating function $D(z)$ associated with any PDF, $\hat{D}(n)$, is defined as

$$D(z)=\sum_{n=0}^{\infty}\hat{D}(n)z^n \tag{A1}$$

with z complex. The total probability of paths belonging to the class described by a PDF $\hat{D}(n)$ is $D(z=1)$. Moments of n can be obtained from the nth derivative at $z=1$, e.g., $E(n)=D'(1)$.

Since the sum of probabilities of any process does not exceed unity, Eq. (A1) is an absolutely convergent series on the unit disc. Thus, the singularities of a generating function always lie outside the unit disc. The PDF corresponding to a given generating function can be found using Cauchy's theorem. Thus

$$\hat{D}_{AB}(n)=\frac{1}{2\pi i}\int_\Gamma\frac{dz}{z^{n+1}}D_{AB}(z) \tag{A2}$$

where Γ is a contour enclosing the origin. It turns out that most generating functions of interest have simple poles only. In this case, by considering the (infinite) domain outside the unit disc, the value of the integral, Eq. (A2), is given as a sum over residues (the infinite circle $|z|=\infty$ does not contribute in most cases of interest). Moreover, generating functions for processes in finite systems possess a finite number of poles.

As an example, we solve the simple problem of finding $\hat{T}_{0,N}(n)$ for a one-dimensional lattice of length $N+1$ with a nearest-neighbor hopping probability equal to $\frac{1}{2}$. $T_{0,N}$ and $Q_{0,N}$ are denoted by t_N and q_N, respectively. The following relations can be easily verified (see Refs. 30 and 31):

$$q_{N+1}=\left(\frac{z}{2}\right)^2\frac{1}{1-q_N}\ , \tag{A3}$$

$$t_N=\frac{z}{2}\left(1-\frac{q_N}{q_{N+1}}\right)^{1/2}\ . \tag{A4}$$

We define the quantity

$$\Lambda=\frac{1-\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\ .$$

It follows that

$$q_N=\frac{\Lambda}{\Lambda+1}\frac{1-\Lambda^{N-1}}{1-\Lambda^N}$$

and

$$t_N=\frac{z}{2}\Lambda^{(N-1)/2}\frac{1-\Lambda}{1-\Lambda^N}\ . \tag{A5}$$

It remains to invert $t_N(z)$ to find $\hat{t}_N(n)$. First, t_N has no branch point at $\Lambda=1$ (a rotation in z plane around $z=1$ implies $\Lambda\to-\Lambda$ which leaves t_N invariant). The poles of t_N (as a function of Λ) are the Nth roots of unity (but $\Lambda=1$ is not a pole):

$$\Lambda=e^{-2i\pi p/N},\quad p=1,\ldots,N-1\ .$$

A summation over the residues yields the sought result [cf. Ref. 31)]

$$\hat{t}_N(n)=\frac{1}{2N}\sum_{p=1}^{N-1}(-1)^{p+1}\cos^n\left(\frac{\pi p}{N}\right)\tan^2\left(\frac{\pi p}{N}\right)\ . \tag{A6}$$

Approximating the cosine by an exponential, we obtain for large N

$$\hat{t}_N(n)\simeq\frac{\pi^2}{N^3}e^{-(\pi/N)^2 n}\ , \tag{A7}$$

which is a valid approximation for $n\geq N^2$. Note that the leading terms are $p=1$ and $p=N-1$. When N is even (odd) $\hat{t}_N(n)$ vanishes for n odd (even) as it should.

2. The first-passage time distribution of the Sinai model: analytic structure

In this section the analytic structure of G_L [cf. Eq. (2.5)] is analyzed for the one-dimensional random walk (many of the properties are valid for more complicated networks, as well). Consider Eq. (2.5) in the text, where Eq. (2.3) applies. The singularities of $G_L(z)$ are those of $\{P_i(z)\}$. The linear set of L equations (2.3) can be solved by Cramer's rule, the poles of $P_i(z)$ being the zeros of the $L\times L$ determinant of the matrix describing Eq. (2.3) [cf. Eq. (4.3)]. Thus, $G_L(z)$ must have at most L poles outside the unit disc and no branch point. $P_i(z)$ is a rational function of z with numerator and denominator of degree $L-1$ and L, respectively. Thus, when z tends to infinity G_L tends to a constant K. Therefore G_L is of the form

$$G_L(z)=\sum_{i=1}^{L}\frac{A_i}{z-z_i}+K\ . \tag{A8}$$

Note that $\hat{G}_L(n)$ vanishes for even (odd) n when L is odd (even). We consider the case of even L first. Since $G_L(z)$ is a function of z^2, to each pole z_i, associated with a residue A_i, there must be a corresponding pole $(-z_i)$, associated with a residue $(-A_i)$, as can be easily deduced from Eq. (A8). There are precisely $L/2$ poles with $z_i>0$; let their indices be $i=1,2,\ldots,L/2$. It follows that

$$G_L(z)=\sum_{i=1}^{L/2}\frac{2z_iA_i}{z^2-z_i^2}+K\ . \tag{A9a}$$

Hence [since $\hat{G}_L(n)$ is the coefficient z^n in the expansion of $G_L(z)$]

$$\hat{G}_L(n)=-2\sum_{i=1}^{L/2}\frac{A_i}{z_i^{n+1}},\quad n\text{ even}$$
$$\hat{G}_L(n)=0,\quad n\text{ odd}\ . \tag{A9b}$$

Consider now the case of odd L. Let $G_{1,L}$ be defined as the generating function corresponding to the first-passage time distribution from point 1 to point L, in which case the walker is allowed to visit site 0. Clearly, Eq. (2.5) is also valid for $G_{1,L}(z)$ and on the r.h.s. of Eq. (2.3), $\delta_{i,0}$ is replaced by $\delta_{i,1}$. The number of steps in this process is even and $G_{1,L}$ is a function of z^2. By the same reasoning that led to Eq. (A8)

$$G_{1,L}(z)=\sum_{i=1}^{L}\frac{B_i}{z-z_i'}+K'\ , \tag{A10}$$

with poles at z_i' and $-z_i'$. Noting that $G_L(z)=zG_{1,L}$ we have

$$G_L(z)=\sum_{i=1}^{L'}\frac{2z_iB_i}{z^2-z_i'^2}+K'z\ , \tag{A10a}$$

where $L'=(L-1)/2$ (that is, $L-1$ finite poles plus one pole at infinity). It is thus obvious that the analysis of the case of odd L is simply related to that of even L. Therefore, it is sufficient to analyze in detail only the latter case, which is what is done below.

a. Proof that the poles and residues are real

In Sec. IV it is shown that the master equation (2.5) can be written as Eq. (4.1), where $\mathbf{P}$ is an L vector with components $P_i(z)$ and $\hat{e}_0$ is the unit vector with components $\delta_{i,0}$ and

$$\underline{I}-\underline{M}=\begin{bmatrix} 1 & -zq_1 & 0 & 0 & & & \cdots & 0\\ -zp_0 & 1 & -zq_2 & 0 & & & \cdots & 0\\ 0 & -zp_1 & 1 & -zq_3 & & & \cdots & 0\\ & \vdots & & & & -zq_{L-1} & & \\ 0 & 0 & & & -zp_{L-2} & & 1 & \end{bmatrix}\ . \tag{A11}$$

Thus the matrix $\underline{T}\equiv\underline{I}-\underline{M}$ is a $(L\times L)$ tridiagonal matrix whose nonvanishing elements are

$$t_{i,i}=1,\quad t_{i,i-1}=-zp_{i-1},\quad t_{i,i+1}=-zq_{i+1}$$

(row and column indices range from 0 to $L-1$). The poles of $G_L(z)$ are the solutions of

$$\det|\underline{T}|=0\ . \tag{A12}$$

Dividing all columns of $\underline{T}$ by $-z$ and defining $\lambda\equiv 1/z$, one obtains a matrix $\tilde{\underline{T}}$ whose nonvanishing elements are

$$\tilde{t}_{i,i}=0,\quad \tilde{t}_{i,i-1}=p_{i-1},\quad \tilde{t}_{i,i+1}=q_{i+1}\ ,$$

i.e., a positive tridiagonal matrix (obviously $\det|\underline{T}|=0$ when $\det|\tilde{\underline{T}}|=0$). The matrix $\underline{T}$ belongs to a class of matrices for which it known that the spectrum is real.[32] The reality of the eigenvalues was realized in similar cases.[33] Hence, all the poles of $G_L(z)$ are real. Consider now Eq. (A8) for $z\in[-1,1]$. Clearly, $G_L(z)$ is real, since $\hat{G}_L(n)\geq 0$ are real (and positive). Consequently, the residues (i.e., $\{A_i\}$) must be real.

b. Relation between poles and residues of G_L

In this section we find expressions for the residues of $G_L(z)$ in terms of its poles. Clearly, $\hat{G}_L(n)=0$ for $n<L$. Thus the l.h.s. of Eq. (A9b) vanishes for every $n<L$ (it vanishes trivially for odd n) yielding the following linear system of $L/2$ equations for the $L/2$ residues A_i's:

$$\sum_{i=1}^{L/2}x_iu_i^n=K\delta_{n,0},\quad 0\leq n\leq L/2-1\ , \tag{A13}$$

where $x_i\equiv 2A_i/z_i$ and $u_i\equiv 1/(z_i)^2$ have been defined for convenience. The matrix of this system is of the Vandermonde type and can thus be solved explicitly using Cramer's rule:

$$x_i = K\frac{V_i}{V} , \tag{A14}$$

where V_i is the minor of matrix element $(1,i)$ and V is the Vandermonde determinant of the system:

$$V_i = (-)^{i+1} \prod_{j<k;j,k\neq i} (u_j - u_k) , \tag{A15}$$

$$V = \prod_{j<k} (u_j - u_k) . \tag{A16}$$

Using Eq. (A14) and the definition of x_i, we find

$$\frac{A_i}{A_1} = \frac{z_1}{z_i}\frac{V_i}{V_1} . \tag{A17}$$

Next, a closed-form representation for this ratio is obtained in terms of the characteristic polynomial of the master equation. The determinant of the matrix $\underline{T} \equiv \underline{I} - \underline{M}$ is a polynomial, which we denote by $Q(z)$, it is of degree L in z and its zeros are the poles z_i. Hence

$$Q(z) = \sum_{n=0}^{L} a_n z^n = a_L \prod_{j=1}^{L} (z - z_j) . \tag{A18}$$

We define the function $h(1/z^2)$,

$$h\left[\frac{1}{z^2}\right] \equiv \frac{Q(z)}{a_L z^L \prod_{j=1}^{L} z_j} , \tag{A19}$$

or

$$h\left[\frac{1}{z^2}\right] = \prod_{j=1}^{L}\left[\frac{1}{z} - \frac{1}{z_j}\right] = \prod_{j=1}^{L/2}\left[\frac{1}{z^2} - \frac{1}{z_j^2}\right]$$

by symmetry of the poles. It is easy to see that the derivative of $h(1/z^2)$ with respect to $1/z^2$, calculated at $1/z_1^2$ equals V_i [cf. Eq. (A15)] (recall that $u_i \equiv 1/z_i^2$):

$$V_i = \left.\frac{d}{d\left[\frac{1}{z^2}\right]} h\left[\frac{1}{z^2}\right]\right|_{z=z_i} . \tag{A20}$$

Using Eq. (4.4)

$$Q(z) = \det|\underline{T}| = \det|\underline{W}| \left[z^L \prod_{j=0}^{L-1} p_j\right]$$

and Eq. (4.8)

$$\det|\underline{W}| = D(0, L-1, \mu) = 1 + f_1\mu + f_2\mu^2 + \cdots + f_L\mu^L .$$

Thus

$$h\left[\frac{1}{z^2}\right] = \frac{\prod_{j=0}^{L-1} p_j}{a_L \prod_{j=1}^{L} z_j}(f_0 + f_1\mu + f_2\mu^2 + \cdots f_L\mu^L) . \tag{A21}$$

Using now the definition of μ [Eq. (4.3)],

$$\frac{1}{z} = -(1+\mu) ,$$

we have

$$\frac{d}{d(1/z^2)} = \frac{1}{2(1+\mu)}\frac{d}{d\mu} .$$

Finally, using Eqs. (A17), (A20), and (A21),

$$\frac{A_i}{A_1} = \frac{f_1 + 2f_2\mu_i + \cdots + Lf_L\mu_i^{L-1}}{f_1 + 2f_2\mu_1 + \cdots + Lf_L\mu_1^{L-1}} . \tag{A22}$$

By our convention [cf. Eq. (2.7)] $\min_i \ln|z_i| \equiv 1/\tau$. Thus $|z_i| > 1 + 1/\tau$, which, by the definition of μ_i, Eq. (4.3), implies

$$-2 + \frac{1}{\tau} \le \mu_i \le -\frac{1}{\tau}, \quad \mu_1 = -\frac{1}{\tau} .$$

Since the f_i's are sums of products [of no more than L ratios of hopping probabilities (see Sec. IV B)], it follows that an $O(1)$ constant k_0 exists such that $|f_i| < e^{k_0 L}$. This fact, in conjunction with the bound for the μ_i's shows that the numerator in Eq. (A22) is bounded in absolute value by $e^{k_1 L}$, k_1 being a constant (which is close to k_0). Next, the denominator can be rewritten, using the fact that $\mu_1 = -(1/\tau)$ and $f_1 = \tau$ as

$$\tau\left[1 - 2\frac{f_2}{\tau^2} + 3\frac{f_3}{\tau_3} \cdots + L\frac{f_L}{\tau^L}\right] .$$

It has been proven before (cf. Sec. IV B) that f_j/τ^j are small quantities, i.e., positive powers of $e^{-\sqrt{L}}$. Thus, the denominator is bounded from below by, say, $\frac{1}{2}\tau = \frac{1}{2}e^{\beta\sqrt{L}}$. Consequently,

$$\left|\frac{A_i}{A_1}\right| < \tfrac{1}{2}e^{k_1 L - \beta\sqrt{L}} < e^{kL} ,$$

where k is an $O(1)$ number (when L is large enough, one may choose $k = k_1$). This concludes the proof of Eq. (4.36) in the text.

APPENDIX B: THE PROBABILITY DISTRIBUTION OF THE EXTENT

1. Calculation of the PDF of the extent

The following language will be used for convenience. A given set S^n [cf. Eq. (3.6)] is called a path. First, consider the rescaled variable ξ_i/σ to be denoted ξ_i: thus $\langle \xi_i^2 \rangle = 1$ [see the paragraph preceding Eq. (3.4)]. The variable ξ_i is called the length of the ith step (in the path). We first treat the simple case $\text{Prob}(\xi_i = 1) = \text{Prob}(\xi_i = -1) = \frac{1}{2}$.

It will be shown below that for large L, this assumption leads to the same result as the general case would, due to a universality property[35] (the "invariance principle"). In this case, there are exactly 2^n different paths. Let $G^n(\alpha)$ be the set of all paths for which $\rho(S^n; n) \le \alpha$ [cf. Eq. (3.7)], for $\alpha \in \mathbb{Z}$. Obviously, $G^n(\alpha) = \phi$ for $\alpha < -1$. Also, $G^n(\alpha)$

contains all 2^n paths when $\alpha \geq n$.

We define

$$M_k = \min_{1 \leq j \leq k-1}\{X_j\}, \quad 1 \leq k \leq n \ . \tag{B1}$$

Define $G_k^n(\alpha)$ to be the subset of $G^n(\alpha)$ which satisfies $X_n = M_n + \alpha - k$ ($k \in \mathbb{Z}_+$). Let $\hat{N}_k(\alpha;n), \hat{N}(\alpha;n)$ be the number of paths belonging to $G_k^n(\alpha)$ and $G^n(\alpha)$, respectively. Obviously, $\hat{N}_{\alpha+2+m}(\alpha;n)=0$ for all $m \geq 0$ since otherwise $X_n = M_n - (2+m)$, which is impossible (i.e., the value of X_n cannot be lower than that of previous X_{n-1} by more than 1, by construction). The quantities $\hat{N}(\alpha;n)$ and $\hat{N}_k(\alpha;n)$ satisfy

$$\hat{N}(\alpha;n+1) = \hat{N}_0(\alpha;n) + 2[\hat{N}(\alpha;n) - \hat{N}_0(\alpha;n)] \ , \tag{B2}$$

$$\hat{N}_0(\alpha;n+1) = \hat{N}_1(\alpha;n) \ , \tag{B3}$$

$$\hat{N}_j(\alpha;n+1) = \hat{N}_{j-1}(\alpha;n) + \hat{N}_{j+1}(\alpha;n), \quad 1 \leq j \leq \alpha - 2 \ , \tag{B4}$$

$$\hat{N}_{\alpha-1}\alpha;n+1) = \hat{N}_\alpha(\alpha;n) + \hat{N}_{\alpha+1}(\alpha;n) + \hat{N}_{\alpha-2}(\alpha;n) \ , \tag{B5}$$

$$\hat{N}_\alpha(\alpha;n+1) = \hat{N}_{\alpha-1}(\alpha;n) \ , \tag{B6}$$

$$\hat{N}_{\alpha+1}(\alpha;n+1) = \hat{N}_\alpha(\alpha;n) + \hat{N}_{\alpha+1}(\alpha;n) \tag{B7}$$

for $\alpha \geq 3$. When $\alpha < 3$, the above equations simplify in an obvious manner.

Equations (B3)–(B5) are based on the following observation. Each path having $(n+1)$ steps is constructed from a path containing n steps and the last step. If $X_{n+1} - M_{n+1} = \alpha - j$ ($j \leq \alpha - 2$), then the corresponding path can be constructed from any path in $G^n(\alpha)$ for which either $X_n - M_n = \alpha - j - 1$ and $\xi_{n+1} = 1$ or $X_n - M_n = \alpha - j + i$ and $\xi_{n+1} = -1$. When $j = \alpha - 1$, we consider all paths of $n+1$ steps for which $X_n - M_n = 2$ and $\xi_{n+1} = -1$ or a path for which $X_n - M_n = 0$ and $\xi_{n+1} = 1$ or paths for which $X_n - M_n = -1$ (i.e., the last step in the path produced a new minimum) and $\xi_{n+1} = 1$. Equations (B2), (B6), and (B7) are based on similar reasoning. The system of equations (B2)–(B7) with the obvious initial conditions

$$\hat{N}_j(\alpha;1) = \delta_{j,\alpha-1} + \delta_{j,\alpha+1} \tag{B8}$$

is most conveniently solved by means of generating functions. The generating function (GF) $D(z)$ associated with a quantity $\hat{D}(n)$ is defined here as (note that the sum starts at $k=1$):

$$D(z) = \sum_{k=1}^{\infty} z^k \hat{D}(k) \ . \tag{B9}$$

The following system of equations corresponding to Eqs. (B2)–(B7) and initial condition Eq. (B8), is obtained:

$$\frac{1}{z} N_0(\alpha;z) = N_1(\alpha;z) \ , \tag{B10}$$

$$\frac{1}{z} N_j(\alpha;z) = N_{j-1}(\alpha;z) + N_{j+1}(\alpha;z), \quad 1 \leq j \leq \alpha - 2 \ , \tag{B11}$$

$$\frac{1}{z} N_{\alpha-1}(\alpha;z) = N_\alpha(\alpha;z) + N_{\alpha+1}(\alpha;z) + N_{\alpha-2}(\alpha;z) + 1 \ , \tag{B12}$$

$$\frac{1}{z} N_\alpha(\alpha;z) = N_{\alpha-1}(\alpha'z) \ , \tag{B13}$$

$$\frac{1}{z} N_{\alpha+1}(\alpha;z) = N_\alpha(\alpha;z) + N_{\alpha+1}(\alpha;z) + 1 \ . \tag{B14}$$

The number of parts of n steps increases at most as 2^n. Hence, as can be seen from Eq. (B9) all the above GF's are analytic in the disc $|z| < \frac{1}{2}$. Equations (B12)–(B14) can be solved to yield

$$N_{\alpha-2}(\alpha;z) + \left[\frac{z}{1-z} - \frac{1}{z}\right] N(\alpha-1;z) = -\frac{1}{1-z} \ . \tag{B15}$$

The system of Eqs. (B11) can be solved using the ansatz

$$N_j(\alpha;z) = a\lambda_+^j + b\lambda_-^j \ , \tag{B16}$$

where

$$\lambda_\pm = \frac{1}{2z} \pm \left[\frac{1}{4z^2} - 1\right]^{1/2} \tag{B17}$$

and the constants a and b are to be determined by Eqs. (B10) and (B15). We find

$$N_0(\alpha;z) = \frac{1}{z} \frac{\lambda_+ - \lambda_-}{\lambda_-^{\alpha+1}(1-\lambda_-) - \lambda_+^{\alpha+1}(1-\lambda_+)} \tag{B18}$$

and from Eq. (B2) [noting that $N(\alpha;1)=2$ for $\alpha \geq 0$] one obtains

$$N(\alpha;z) = \frac{2z}{1-2z} - \frac{z}{1-2z} N_0(\alpha;z) \ . \tag{B19}$$

We define

$$P(\alpha;n) = \text{Prob}[\rho(S^n;n) \leq \alpha] \ . \tag{B20}$$

Since $P(\alpha;n) = 2^{-n}\hat{N}(\alpha;n)$, it is easily seen that $N(\alpha;z/2)$ in Eq. (B19) is the GF corresponding to $P(\alpha;n)$. It follows from Eq. (B18) that $N(\alpha;z/2)$ is analytic in the whole complex plane except possibly at $z = \pm 1$ which are branch points of $\lambda_\pm(z/2)$. However, it is easy to check that $N_0(\alpha;z/2)$ has no such branch points (since $\lambda_\pm \to \lambda_\mp$ upon a rotation around $z=2$, leaving $N_0(\alpha;z/2)$ invariant.) Also $N_0(z) \to \infty$ when $z \to \infty$. Defining the variable θ by $\lambda_\pm(z/2) = e^{\pm i\theta}$, Eq. (B18) yields for the poles θ_m (or z_m), of $N_0(z;2)$:

$$\theta_m = \frac{(2m+1)\pi}{2\alpha+3} \quad \text{or} \quad z_m = \frac{1}{\cos\theta_m}, \quad m = 0, 1, \ldots, \alpha+1 \ . \tag{B21}$$

Using Eq. (B19), the result is obtained from Cauchy's theorem applied to the exterior of a circle of radius $1+\epsilon$ around the origin and we find

$$P(\alpha,n) = \frac{2}{2\alpha+3}\sum_{m=0}^{\alpha+1}(-1)^m\cos^2\left[\frac{\theta_m}{2}\right]\left[\sin\frac{\theta_m}{2}\right]^{-1}\cos^n\theta_m \ , \tag{B22}$$

where

$$\theta_m=\frac{(2m+1)\pi}{2\alpha+3},\quad 0\le m\le\alpha+1 \ .$$

The limiting distribution for large n can be easily obtained. Asymptotically in n, terms for which $\ln(\cos\theta_m)\gg 1/n$ vanish exponentially. Thus, in Eq. (B22), this leaves terms for which $1-\cos^2\theta_m\le O(1/n)$, i.e., (a) $\theta_m^2\le O(1/n)$ or (b) $\theta_m^2\le\pi-O(1/n)$ and this is possible only if $\alpha>n-O(n^{1/2})$. Now, the factor $[\sin(\theta_m/2)]^{-1}$ in Eq. (B22) is $O(\alpha)$ and $O(1)$ for the terms belonging to above categories (a) or (b), respectively, and only terms belonging to (a) are to be kept asymptotically. In the same limit, the sum may be extended to ∞ (this introduces an exponentially small error) yielding the Erdös-Kac[28] formula for the maximum displacement in a random walk (in units $\sigma=1$):

$$P(\alpha;n)=\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\exp\left[-\frac{n(2m+1)^2\pi^2}{8\alpha^2}\right] . \tag{B23}$$

The general validity of the limiting distribution of $P(\alpha;n)$ [in Eq. (B23)] follows from Donsker's theorem[35] together with the corollary 1 of theorem (5.1) in Ref. 36. In short, if $X_n\to X$ in distribution, then $h(X_n)\to h(X)$ in distribution, when h is a a continuous function. Since, obviously, the extent of a walk is a continuous functional on the set of paths, the invariance principle applies here.[35,36]

The following results can be derived from Eq. (B23):

$$E(\rho^n)\simeq\frac{\sqrt{\pi}}{2}n^{1/2}+O(1)\quad\text{for } n\gg 1 \ .$$

The second moment is similarly obtained, yielding $E[(\rho^n)^2]\simeq 2Gn$, where G is Catalan's constant ($G=0.916\,966^+$). A similar result to Eq. (B23) can be obtained using the result of Golosov.[15] The same formulas apply for the original ξ_i (i.e., without rescaling) except that n (the number of steps) is to be replaced by $n\sigma^2$.

2. Extent versus span

The asymptotic average value of the extent equals $\sqrt{\pi/2}\,n^{1/2}$ whereas that of the range equals[34] $\sqrt{8/\pi}\,n^{1/2}$. The range and the extent coincide for half of the paths (those for which the range is determined by a positive value of X_n-X_m, $n>m$). For the other half, the range is given by the "negative" extent. When $\text{Prob}(\xi_j)=\text{Prob}(-\xi_j)$, it follows from symmetry considerations that in half the paths, the range corresponds to the ("positive") extent. Defining in an obvious way a negative extent, it follows that in the other half of paths, the range corresponds to the negative extent. Defining S^{++},S^{+-} to be the average extent and the average negative extent for the first half of paths and defining similarly S^{--} and S^{-+}, it is obvious that

$$E(R)=\tfrac{1}{2}S^{++}+\tfrac{1}{2}S^{--}$$

and

$$E(\rho)=\tfrac{1}{2}S^{++}+\tfrac{1}{2}S^{-+} \ .$$

Since both $E(R)$ and $E(\rho)$ are known and since $S^{++}=S^{--}$ and $S^{+-}=S^{+-}$ by symmetry, we find that

$$\frac{S^{+-}}{S^{++}}=\frac{\pi}{2}-1=0.57^+ \ .$$

Thus, the ratio of the average negative extent to that of the range is asymptotically a universal constant. Furthermore, using Donsker's theorem[35] it is obvious that the requirement of symmetry on $\text{Prob}(\xi_i)$ can be lifted.

APPENDIX C: THE GAP PROPERTY

The present appendix is devoted to proving the fact used in Sec. IV B, that $f_2/\tau^2=O(\epsilon^\gamma)$ where γ is an $O(1)$ number, i.e., Eq. (4.3). This fact is called, below, the gap property. Recall Eq. (4.22), where the index M is chosen so as to maximize the product on the r.h.s. of the above inequality. Also recall Eq. (4.20) which defines $\zeta(r,s)$ as the maximum among the products $\prod_{i=k}^{m}r_i$ where $r\le k\le m\le s$ and $r_i\equiv q_i/p_i$. It is convenient to define

$$\pi(k,m)=\prod_{i=k}^{m}r_i \ . \tag{C1}$$

In what follows, the interval $[k_0,m_0]$ for which the above product attains its maximal value will be called the generator of the extent of $[r,s]$. Let $[A,B]$ be the generator of the extent of $[0,L]$. That is, the initial and final positions of the extent of $[0,L]$ are A and B, respectively (cf. Fig. 2). In a similar manner, let $[a_1,a_2]$ and $[b_1,b_2]$ be the generators of the extent of $[0,M]$ and the extent of $[M,L]$, respectively. Obviously $a_2\le M\le b_1$. We observe that on $[a_2,b_1]$, the product $\pi(a_2,j)$ for all $j\le M$ is smaller than unity. Otherwise, one could choose an index J, say, such that upon changing a_2 to J the product $\pi(a_1,J)$ would be larger than $\pi(a_1,a_2)$, which is impossible since $[a_1,a_2]$ generates the extent of $[0,M]$. Also, $\pi(J,b_1)<1$ for $M\le J\le b_1$ for similar reasons.

Consider the various possibilities for the location of M with respect to A and B:

1. $M\le A$: In this case $[b_1,b_2]=[A,B]$ since $[A,B]$ generates the extent of $[0,L]$ and certainly also the one of $[M,L]$. Since $\zeta(0,M)\le\zeta(0,A)$ in this case, one must have $M=A$ when $\zeta(0,M)=\zeta(0,A)$. When $\zeta(0,M)<\zeta(0,A)$ one may choose $M=A$. Thus, case 1 reduces to $M=A$.

2. $M\ge B$: This case is analogous to case 1 and $[a_1,a_2]\equiv[A,B]$. In this case $M=B$.

3. $A<M<B$. Here, we have to consider several possibilities.

The possible locations of a_2 (which by definition must satisfy $a_2\le M$) are now considered. If $a_2<A$, it means that the generator of the extent of $[0,M]$ is to the left of the segment $[A,M]$. Thus $\zeta(0,M)=\zeta(0,A)$. Next, since

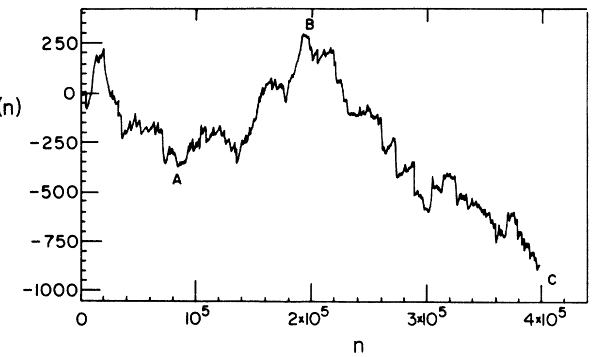

FIG. 2. The position of a random walker in a symmetric ($p=q=\frac{1}{2}$) random walk as a function of the number of steps. The extent and the span, for a walk of length 400 000 are indicated by (A,B) and (B,C), respectively. The thick line has been drawn to guide the eye.

the extent of $[0,L]$ is generated by $[A,B]$, it follows that $\zeta(A,L)=\zeta(0,L)$. Now $\zeta(0,L)$ is larger than $\zeta(M,L)$, or else A would have satisfied $M\le A$, contrary to assumption 3. Consequently, $\zeta(A,L)\zeta(0,A)$ is larger than $\zeta(0,M)\zeta(M,L)$, contrary to our assumption as to the choice of the point M. Hence $a_2\ge A$. Next, consider a_1. If $a_1<A$, one may conclude that $\pi(a_1,A)>1$, or else $\pi(a_1,a_2)$ could be increased by redefining $a_1=A$, contrary to the definition of $\zeta(0,M)$, However, $\pi(a_1,A)>1$ one has $\pi(a_1,B)>\pi(A,B)$, which is impossible since $[A,B]$ generates the extent of $[0,L]$. Thus, the assumption $a_1<A$ leads to a contradiction. If $a_1>A$, it follows that $\pi(A,a_1)<1$, or else $\pi(A,a_2)>\pi(a_1,a_2)$ contrary to the definition of $[a_1,a_2]$. However, for $\pi(A,a_1)<1$ we obtain $\pi(a_{1,B)>}\pi(A,B)$ contrary to the definition of $[A,B]$. Hence $a_1>A$ also leads to a contradiction.

We conclude that only $a_1=A$ is possible. Similar reasoning yields $b_2=B$. All in all we have to consider only three cases: case (i): $M=A$ and $\zeta(M,L)=\zeta(0,L)$; case (ii): $M=B$ and $\zeta(0,M)=\zeta(0,L)$; and case (iii): $A<M<B$; $a_1=A$ and $b_2=B$. We shall now consider each of these cases separately. Since cases (i) and (ii) are analogous, it is sufficient to consider cases (i) and (iii) only.

Case (i). In this case $\zeta(M,L)=\zeta(0,L)$. It is convenient to define

$$y=\frac{\zeta(0,M)}{\zeta(0,L)} . \tag{C2}$$

Equation (4.27) then reads

$$\ln\left[\frac{f_2}{\tau^2}\right]<\ln y+\ln\left[\frac{(L+1)^5}{4c^2}\right] . \tag{C3}$$

When $M=O(1)$, one has $\zeta(0,M)=O(1)$ and $y=O(\epsilon^{\beta})$, trivially (recall $\epsilon=e^{-\sqrt{L}}$ and $\tau\simeq e^{-\beta\sqrt{L}}$). When $M\gg1$ and $M\ll L$, i.e., $M=O(L^{1-\eta})$, $0<\eta<1$ then $y\simeq O(\exp(\beta_2\sqrt{L}-\beta_1\sqrt{M}))$ [cf. Eq. (3.11)] where β_1 and β_2 are $O(1)$ numbers and the gap property is trivially satisfied again. It remains to consider the case $M=O(L)$.

Recall that by definition [cf. Eq. (4.19)]

$$\ln\zeta(r,s)=\mathrm{Ext}(r,s) \tag{C4}$$

and that

$$\mathrm{Ext}(0,L)=\tau+O(\ln L) . \tag{C5}$$

As in the text, it is convenient to define $\xi_i=\ln r_i$ as the size of an elementary step. We also define $X_i=\sum_{j=1}^{i}\xi_j$ as the coordinate of a random walker after i steps (cf. Sec. III). The path corresponding to case (i) has the following structure (see Fig. 3): X_{a_1} is the minimal value of $\{X_j;0\le j\le a_2\}$. Also $X_M<X_{a_1}$ (or else the extent would have been generated by $[a_1,B]$). Also note that $\sum_{j=a_2}^{k}\xi_j<0$ for $a_2<k\le M$. Obviously, the extent of $[0,M]$ equals $\ln[\zeta(0,M)]$ and it does not exceed $\ln[\zeta(0,L)]$ by definition. The paths: $\{X_j0;\le j\le M\}$ which are consistent with the presently discussed case (i) are thus defined by

$$S_M=\{\{X_j;1\le j\le M\};\mathrm{Ext}(0,M)<\tau;X_M=\min_{1\le j\le M}X_j\} . \tag{C6}$$

A larger set of paths containing S_M as a subset is defined by

$$T_M=\{\{X_j;1\le j\le M\};\mathrm{Ext}(0,M)<\tau\} . \tag{C7}$$

Under the conditions of case (i), $\zeta(M,L)\simeq\tau$. We define $P_S(Q>\alpha)$ and $P_T(Q>\alpha)$ to be the probability distributions of a functional Q on the spaces of paths $S(0,M)$ and $T(0,M)$, respectively.

The probability that $\ln y$ exceeds a number $-|u|$ is now considered (obviously, $y\le1$). Clearly, for a given value of τ and when case (i) is considered

$$\mathrm{Prob}(\ln>-|u|)=\mathrm{Prob}_S[\mathrm{Ext}(0,M)>\ln\tau-|u|] . \tag{C8}$$

When one enlarges the set $S(0,M)$ to $T(0,M)$ one has

$$\mathrm{Prob}_S[\mathrm{Ext}(0,M)>\ln\tau-|u|]<\mathrm{Prob}_T[\mathrm{Ext}(0,M)>\ln\tau-|u|] . \tag{C9}$$

Also,

$$\mathrm{Prob}_T[\mathrm{Ext}(0,M)>\ln\tau-|u|]=1-\mathrm{Prob}_T[\mathrm{Ext}(0,M)<\ln\tau-|u|] . \tag{C10}$$

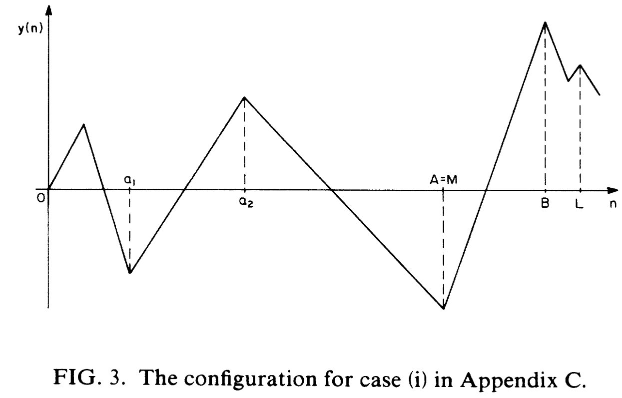

FIG. 3. The configuration for case (i) in Appendix C.

It is easy to see that (for $x < \ln\tau$)

$$\text{Prob}_T[\text{Ext}(0,M)<x]=\frac{\text{Prob}[\text{Ext}(0,M)<x]}{\text{Prob}[\text{Ext}(0,M)<\ln\tau]} \quad \text{(C11)}$$

where the Prob on the r.h.s. of the above denotes the probability distribution in the space of *all* possible paths from 0 to M (i.e., without restriction on their extent). This probability distribution is the one given by Eq. (3.10) with L replaced by M and will be denoted, hereafter, by "A" for clarity of notation. Thus

$$\text{Prob}(\ln y > -|u|) < \frac{A(M,\ln\tau)-A(M,\ln\tau-|u|)}{A(M,\ln\tau)} \quad \text{(C12)}$$

for given M and τ. Choosing now $|u|=L^{\eta}$ with $0<\eta<\frac{1}{2}$ we have

$$\text{Prob}(\ln y > -L^{\eta})=\frac{A(M,\ln\tau)-A(M,\ln\tau-L^{\eta})}{A(M,\ln\tau)} . \quad \text{(C13)}$$

Using Eq. (3.10), the numerator, denoted by Num, appearing on the r.h.s. of Eq. (C13) equals

$$\text{Num}=\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\left(e^{-K(m)}-e^{-K(m)B(m)}\right) , \quad \text{(C14)}$$

where

$$K(m)=\frac{M\sigma^2(2m+1)^2\pi^2}{8\ln^2\tau} \quad \text{(C15)}$$

and

$$B(m)=\left[1-\frac{L^{\eta}}{\ln\tau}\right]^{-2} . \quad \text{(C16)}$$

For typical values, $\ln\tau=O(\sqrt{L})$ and

$$B(m)\simeq 1+\frac{2(L^{\eta})}{\ln\tau}+O(L^{2\eta-1}) . \quad \text{(C17)}$$

Using (C17), the mth term in (C14) thus equals

$$\frac{4}{\pi}\frac{(-1)^m}{2m+1}e^{-K(m)}\left[1-\exp\left[\frac{-2K(m)L^{\eta}}{\ln\tau}\right]\right] . \quad \text{(C18)}$$

Since M is assumed to be $O(L)$ and $\ln\tau=O(\sqrt{L})$, one finds that for sufficiently small m, $K(m)$ in Eq. (C15) is $O(1)$ and the term appearing in the exponent in the square brackets of Eq. (C18) satisfies

$$2K(m)\frac{L^{\eta}}{\ln\tau}=O(L^{\eta-1/2}) \quad \text{(C19)}$$

and this quantity vanishes for large L. For large m, each term in (C14) vanishes exponentially. Now, $A(M,\ln\tau)$, the denominator in Eq. (C12), is the probability that in a path of length M the extent is smaller than $\ln\tau$. This probability is $O(1)$ for $M=O(L)$. Finally, the following bound is obtained:

$$\text{Prob}(\ln y > -L^{\eta}) < \frac{R}{L^{1/2-\eta}} \quad \text{(C20)}$$

for any choice of M and typical β; R in Eq. (C20) is an $O(1)$ number. Since $y > f_2/\tau^2$ we have finally obtained

$$\text{Prob}\left[\frac{f_2}{\tau^2} > \exp -L^{\eta}\right] < \frac{Q}{L^{1/2-\eta}}, \quad 0<\eta<\frac{1}{2} \quad \text{(C21)}$$

and since the factor $(L+1)^5/4c^2$ is much smaller than $\exp(L^{\eta})$ it may be ignored. The latter inequality expresses the fact that with probability approaching one for large L, f_2/τ^2 is smaller than $\exp(-L^{\eta})$ for any η smaller than $\frac{1}{2}$. Hence

$$\ln\left[\frac{f_2}{\tau^2}\right]=O(-\sqrt{L}) \quad \text{(C22)}$$

for typical systems, as claimed.

Case (iii). In this case, $A<M<B$; $a_1=A$ and $a_2=B$. It is convenient to define the following quantities (see Fig. 4):

$$h_1=\sum_{j=A}^{a_2}\xi_j,\quad h_2=\sum_{j=b_1}^{B}\xi_j$$
$$h=\sum_{j=A}^{B}\xi_j,\quad \Delta=-\sum_{j=a_2}^{b_1}\xi_j \quad \text{(C23)}$$

where $\xi_j \equiv \ln r_j$ Obviously, $\Delta>0$ else $\zeta(0,M)\zeta(0,L)$ could have been increased by having $a_2=b_1$. We wish to estimate the logarithm of the r.h.s. of Eq. (4.27):

$$\ln\zeta(0,M)+\ln\zeta(M,L)-2\ln\tau=h_1+h_2-2h .$$

Clearly (see Fig. 4)

$$h=\ln[\zeta(0,L)]=h_1+h_2-\Delta . \quad \text{(C24)}$$

As mentioned, $\ln\tau\simeq h+O((\ln L))$. Since the logarithmic correction is clearly negligible, it will be ignored below. It is convenient to define the (positive) variable x:

$$x=2h-(h_1+h_2)=h-\Delta . \quad \text{(C25)}$$

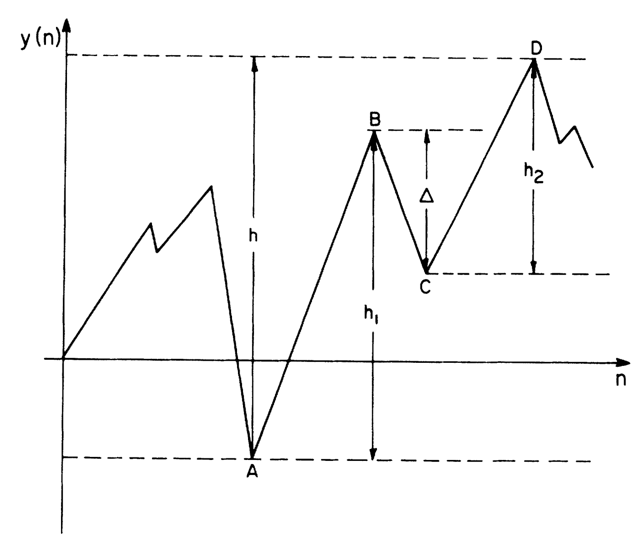

FIG. 4. The variables h, h_1, h_2, and Δ as defined in Appendix C. Note that the quantity Δ corresponding to the extent (A,B) in Fig. 2 is very small compared to the extent.

Thus, Eq. (4.27) becomes (omitting an irrelevant prefactor):

$$\frac{f_2}{\tau^2} < e^{-x} . \tag{C26}$$

It is shown below that $x = O(\sqrt{L})$ with probability tending to one in the limit of large L. In what follows, we consider the rescaled variable $\xi_j \to \sigma^{-1}\xi_j$ and assume that the (rescaled) ξ_j's are binomally distributed. The case of a general distribution is obtained by invoking the invariance principle[35,36] (cf. Appendix B).

The problem can be posed as follows. Given that (a) the extent equals h and (b) the size of the generator of the extent does not exceed L, it is required to prove that the *largest negative extent* Δ in the segment $[A,B]$ is much smaller than h. Since $h = O(\sqrt{L})$ and $x \equiv h - \Delta$, it is equivalent to prove that $x = O(\sqrt{L})$. This problem can be set as a one-dimensional symmetrical random-walk problem, as follows.

A walk is defined as a collection of paths satisfying some restrictions. First, we define an x-type walk. A random symmetric walker starts at site 0 and reaches site h_1 for the first time. In the process he may return to 0 or h_1 as many times as he wishes but is restricted to the segment $[0,h_1]$. After his *last* visit to h_1, he leaves h_1 back to point $h_1 - \Delta$. The walker then proceeds towards h without ever reaching $h_1 - \Delta - 1$ (repeated visits to $h_1 - \Delta$ are allowed). Once h has been reached, the walk is stopped. Furthermore, an x-type walk is a walk for which the values of x and h are fixed [hence by Eq. (C25) of Δ as well], yet $h_1 \geq 0$ and $h_2 \geq 0$ can take any value). Such a walk is schematically depicted in Fig. 4.

Next, an R-type walk is defined. Consider an n-step path starting at site 0 and ending at site g (on a lattice) which is restricted to the segment $[0,g]$. In the process, the walker is not allowed to return to 0. Once g has been reached for the first time the walker may return to g as many times as he wishes. After n steps, the walker must be found at g where the walk is stopped. Let $\hat{R}(g,n)$ and $R(g,z)$ be its PDF and corresponding GF, respectively. Obviously, $\hat{R}(g,n)$ and $R(g,z)$ depend only on the length, g, of the segment. The generating function corresponding to this walk is easily constructed in terms of the elementary GF's defined in Appendix A. A walk containing r "departures" from g contributes a term $q_g^r t_g$ to $R(g,z)$. We recall that t_g is the GF for leading 0 for good and reach g for the first time (cf. Appendix A); q_g is the GF for leaving g and returning to it for the first time without reaching 0 in the process. All allowed paths are inside $[0,g]$. Thus, summing over all possible departures from g:

$$R(g,z) = \sum_{r=0}^{\infty} q_g^r t_g = \frac{1}{1-q_g} t_g . \tag{C27}$$

Clearly,

$$t_{g+1}(z) = \frac{z}{2} R(g,z) . \tag{C28}$$

Denote by $F(h,x,z)$ the generating function corresponding to an x-type walk. Since a walk is composed of three successive R-type walks: the first is in a segment of length h_1, the second is in a segment of length Δ and the third is in a segment of length h_2, where h_1, Δ, h_2 are related to h and x by Eq. (C24) and (C25). Summing over all such paths we have

$$F(h,x,z) = \sum_{g=0}^{x} R(g+h-x,z) R(h-x,z) R(h-g,z) \tag{C29}$$

which by using Eq. (C28) becomes:

$$F(h,x,z) = \left[\frac{2}{z}\right]^3 t_{h-x+1} \sum_{g=0}^{x} t_{g+h-x+1} t_{h-g+1} \tag{C30}$$

The corresponding PDF $\hat{F}(h,x,n)$ is obtained by inverting $F(x,h,z)$. It yields the probability of performing an x-type walk in n steps for a given h. In the definition of $\hat{F}(x,h,n)$, there is no restriction on the number of steps. Consider, now, all x-type paths, restricted to $n \leq L$ steps. Within this set of paths, define $\hat{Q}(h,x;L)$ to be the PDF of x for given h. To find $\hat{Q}(h,x;L)$ the quantity $\hat{F}(h,x,n)$ has to be normalized to the number of paths in the restricted set

$$\hat{Q}(h,x;L) = \frac{\sum_{n=0}^{L} \hat{F}(h,x,n)}{\sum_{n=0}^{L} R(h,n)} . \tag{C31}$$

Since the total number of steps is not restricted to $[A,B]$ but is assumed to be less than L, $Q(h,x;L)$ is an upper bound for the PDF of paths corresponding to case (iii).

By the general properties of generating functions [cf. Eq. (3.1)] we have the probability of (performing an x-type walk)

$$\mathrm{Prob}(x) = \sum_{n=0}^{\infty} \hat{F}(h,x,n) = F(h,x,1) . \tag{C32}$$

Since [Eq. (A5)]

$$t_N(1) = \frac{1}{2N} , \tag{C33}$$

substituting in Eq. (C30) one obtains

$$F(h,x,1) = \frac{1}{h-x+1} \sum_{g=0}^{x} \frac{1}{(g+h-x+1)(h-g+1)} . \tag{C34}$$

The sum may be replaced by an integral in the limit of large h (which is the case here)

$$F(h,x,1) = \frac{1}{(h-x+1)(h-x/2+1)} \ln \frac{h+1}{h-x+1} . \tag{C35}$$

Clearly

$$\sum_{n=0}^{L} \hat{F}(h,x,n) \leq \sum_{n=0}^{\infty} \hat{F}(h,x,n) = F(h,x,1) . \tag{C36}$$

Thus, $F(h,x,1)$ is an upper bound for the numerator in Eq. (C31). Next, the denominator of Eq. (C31) is estimated. Using Eqs. (3.1), (C28), and (C33):

$$\sum_{n=0}^{L}\hat{R}(h,n)=\sum_{n=0}^{\infty}R(h,n)-\sum_{n=L}^{\infty}\hat{R}(h,n)$$
$$=R(h,1)-\sum_{n=L}^{\infty}\hat{R}(h,n)$$
$$=\frac{1}{h+1}-\sum_{n=L}^{\infty}\hat{R}(h,n)\ . \qquad \text{(C37)}$$

In order to perform the last sum in Eq. (C37) we use Eq. (C28) which yields

$$\hat{R}(h,n)=2\hat{t}_{h+1}(n-1)\ . \qquad \text{(C38)}$$

By using Eq. (A6) we have

$$\hat{t}_{n+1}(n-1)=\frac{2}{2(h+1)}\sum_{p=1}^{h/2}(-)^{p+1}\tan^2\left[\frac{\pi p}{h+1}\right]\times\exp\left[(n-1)\ln\cos\left[\frac{\pi p}{h+1}\right]\right]$$

(the factor of 2 comes from the symmetry in the sum: $p\to h-p$). The range of values of interest is $h=O(\sqrt{L})$ and $n\geq L$ and $L\gg 1$. Clearly, due to the exponential factor only terms for which $p=O(1)$ contribute. The exponential factor can be approximated by for large h and n:

$$\exp\left[(n-1)\ln\cos\left[\frac{\pi p}{h+1}\right]\right]=\exp\left[-\frac{\pi^2p^2}{2}\frac{n}{h^2}\right]\left[1-O\left[\frac{n}{h^4}\right]\right]\ .$$

Finally, we have asymptotically, for large h and n

$$\hat{R}(h,n)\simeq\frac{2}{h^3}W\left[\frac{n}{h^2}\right]\ , \qquad \text{(C39)}$$

where

$$W(u)=\pi^2\sum_{p=1}^{\infty}(-1)^{p+1}p^2\exp\left[-\frac{\pi^2p^2}{2}u\right]\ . \qquad \text{(C40)}$$

The sum on the r.h.s. of Eq. (C37) may be replaced by an integral

$$\sum_{n=L}^{\infty}\hat{R}(h,n)\simeq\frac{1}{h^3}\int_L^{\infty}W\left[\frac{n}{h^2}\right]dn\ .$$

In the problem at hand, there is an $O(1)$ number γ for which $h=\gamma\sqrt{L}$, thus $L=h^2/\gamma^2$. Substituting these values in the above integral, one obtains

$$\sum_{n=L}^{\infty}\hat{R}(h,n)\simeq\frac{1}{h}\int_{1/\gamma^2}^{\infty}W(u)du\ , \qquad \text{(C41)}$$

the integral being $O(1)$. It follows from Eqs. (C37) and (C41) that

$$\sum_{n=0}^{\infty}\hat{R}(h,n)\simeq\frac{1}{h+1}-\frac{1}{h}\int_{1/\gamma^2}^{\infty}W(u)du=\frac{1}{\nu h}\ , \qquad \text{(C42)}$$

where ν is an $O(1)$ constant. Using the upper bound for the numerator of Eq. (C31) given by Eq. (C36) and the estimate of the denominator appearing in Eq. (C31) given by Eq. (C42), we find the following upper bound for $\hat{Q}(h,x;L)$:

$$\hat{Q}(x,h;L)<\nu\frac{h}{(h-x+1)(h-x/2+1)}\ln\frac{h+1}{h-x+1}\ . \qquad \text{(C43)}$$

Recalling that $\hat{Q}(h,x;L)$ is the upper bound for the PDF for the paths corresponding to case (iii), it remains to find a bound for the probable values of x. It follows from the bound Eq. (C43) that for $0<\eta<\frac{1}{2}$

$$\text{Prob}(x<L^{1/2-\eta})\leq\sum_{x=0}^{L^{1/2-\eta}}\hat{Q}(h,x;L)\leq\nu\sum_{x=0}^{L^{1/2-\eta}}\frac{h}{(h-x+1)(h-x/2+1)}\ln\frac{h+1}{h-x+1}=O\left[\frac{1}{L^{\eta/2}}\right]\ . \qquad \text{(C44)}$$

Since x is smaller than $h=O(\sqrt{L})$ and thus has, by Eq. (C44) vanishing probability to be $O(\sqrt{L})$, we conclude that $x=O(\sqrt{L})$ with a probability that tends to 1 for large L. This completes the proof.

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