

Periodically forced Fokker-Planck equation and stochastic resonance

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The time-dependent solution of the periodically forced Fokker-Planck equation is determined, following an iteration procedure. The mixing property of the system is verified. The enhancements of the autocorrelation and the signal-to-noise ratio are distinguished and localized, in different noise strength regions. The conditions for the validity of the adiabatic approximation and the way to systematically improve the adiabatic result are clarified.

I. INTRODUCTION

Macroscopic systems are subjected to a variety of stochastic forces coming from some uncontrollable sources as, for example, the random microscopic impacts of internal subsystems or the variability of external conditions. Ordinarily, these effects are small when compared to the macroscopic variables. Thus they usually provide only a small perturbation to the deterministic motion. However, in certain critical situations such small effects may play a crucial role in determining the system's evolution.¹⁻⁹ The so-called stochastic resonance (SR) is a representative example of such phenomena.¹⁰⁻²¹

The early investigations of SR were initiated by the discovery that the earth climate between the ice ages and relatively warm periods changed dramatically while the external forces acting on the climate system were very weak. To explain this unusual behavior the model of a system moving in a double-well potential in the presence of both noise and external period input has been suggested.¹⁰⁻¹³ It was found that in the presence of small noise the bistable system may, indeed, produce a periodic output much stronger than the input. This SR is expected to arise when the frequency of the external force, Ω , is about the mean hopping rate between the two potential wells.^{12,18} Recently, the study of the SR has been extended to other systems and focused on the analysis of the signal-to-noise ratio (SNR). It has been shown that in a certain parameter region the SNR may increase with increased noise. However, the peak of the SNR has no relation to the resonance between the frequency of the signal and the hopping rate. This led Fox¹⁶ to the conclusion that the SR is by no means a resonance between the modulation frequency and Kramers's rate. Fox suggests using a more appropriate term, "noise-induced signal-to-noise ratio enhancement (SNRE)," instead of the term SR. Recently, two interesting experiments have been performed to illustrate the SR and the SNRE, one

involving a Schmitt trigger circuit and the other a bi-directional ring laser.^{19,20}

The periodically forced Fokker-Planck equation (FPE) in a bistable potential

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x} [C(x) + \epsilon h(x) \cos(\Omega t + \theta)] P(x,t) + D \left[\frac{\partial^2}{\partial x^2} \right] P(x,t) \quad (1.1)$$

has been most extensively used in predicting theoretically the SR and the SNRE. Here $C(x)$ is the drift term in the absence of forcing, D the diffusion coefficient, $\epsilon h(x)$ the forcing amplitude, and Ω, θ the forcing frequency and initial phase, respectively. For simplicity, we will consider only antisymmetric drift which vanishes at $\pm c$,

$$C(-x) = -C(x), \quad C(\pm c) = 0 \quad (1.2a)$$

and take $h(x)$ to be constant,

$$h(x) = 1. \quad (1.2b)$$

It will be shown that the results obtained for this special FPE can be directly applied to the general periodically forced FPE by slightly modifying the formalism. Owing to the nonstationary nature of the system introduced by the time-dependent forcing $\epsilon \cos(\Omega t + \theta)$, no explicit solution of (1.1) is available.

Recently, Jung and Hanggi¹⁷ transformed (1.1) to a two-dimensional FPE

$$\frac{\partial W(x, \varphi, t)}{\partial t} = -\frac{\partial}{\partial x} [C(x) + \epsilon \cos(\varphi + \theta)] W(x, \varphi, t) - \Omega \left[\frac{\partial}{\partial \varphi} \right] W(x, \varphi, t) + D \left[\frac{\partial^2}{\partial x^2} \right] W(x, \varphi, t), \quad (1.3)$$

where φ is the phase of the forcing. They found that Eq. (1.3) has purely imaginary eigenvalues, indicating the nonmixing nature of the dynamics. As Presilla, Marchesoni, and Gammaitoni reported,¹⁸ because of this nonmixing property, at asymptotically large times the solution of (1.1) retains information from the initial state. Now in dealing with this nonmixing property, one invokes various average procedures with respect to the phase.^{16,18} Effectively therefore, one treats a randomly perturbed FPE,¹⁸ and this makes a systematic analysis much more involved. In the present paper, we take a substantially different attitude. We verify that the FPE (1.1) enjoys the mixing property, even though (1.3) may lack it. Thus, in the presence of a small periodic forcing the initial preparation should be forgotten exponentially just as it is in the unforced FPE. The only difference between the persistent states in both cases (i.e., the states realized as $t \rightarrow \infty$ for any initial preparations) is that it is periodic for the former while stationary for the latter. With this conclusion, we can apply the theory of eigenfunction expansion, or equivalently, the Floquet theory, to Eq. (1.1), and get systematic results in terms of the perturbation expansion. The perturbation approach is performed in such a way that one may predict characteristic features of the exact solution such as the mixing property of the system from the formal solution. Furthermore, the lowest order of the perturbation can be obtained in a considerably simpler way than before,^{16,18} and more accurate results are easily accessible by proceeding to higher orders in the perturbation.

In Sec. II we discuss the mixing property of (1.1) and review the previous studies of this problem. The persistent solution of (1.1) is obtained in terms of the perturbation theory in Sec. III. Section IV describes the transient process by applying the Floquet theory, whereas Sec. V deals with the initial-value problem, and the mechanism ensuring that the initial condition is forgotten for long times. In Sec. VI the theoretical results are illustrated and checked on an exactly solvable model, the periodically forced FPE with linear drift. In Sec. VII the SR and the SNRE are examined in the light of the linear-response theory of the periodically forced FPE. It is found that both SR and SNRE exist, but appear in different regimes of the noise strength. In the last section a comparison with recent investigations of the problem is made along with some remarks on the relevance of the results.

II. MIXING PROPERTY OF THE PERIODICALLY FORCED FPE

In Refs. 17 and 21 Jung and Hanggi conclude that the presence of purely imaginary eigenvalues in (1.3) indicates the nonmixing property of the solution. Furthermore, Presilla, Marchesoni, and Gammaitoni¹⁸ found that as $t \rightarrow \infty$ the asymptotic solution $P_\infty(x, t)$ depends on both the phase of the forcing θ and the initial condition $P(x, 0)$.

One of the objects of this paper is to show the mixing property of the solution of the original FPE (1.1). We emphasize that the two Eqs. (1.3) and (1.1) are not com-

pletely identical in their ergodic properties. In fact, the solution of (1.1) is related to the solution of (1.3) by the integral relation,

$$P(x, t) = \int_0^{2\pi} W(x, \varphi, t) d\varphi, \quad (2.1)$$

as a result of which some properties of (1.3) have been lost. Thus one cannot definitely conclude nonmixing of (1.1) directly from the properties of (1.3). For instance, Jung and Hanggi showed that the x -independent left eigenfunctions $\exp(-in\varphi)$ correspond to purely imaginary eigenvalues of (1.3). Thus the initial preparation belonging to the subspace of these eigenfunctions will be remembered. However, no such x -independent eigenfunctions with purely imaginary eigenvalues exist for (1.1). Naturally, for Eq. (1.1) there exists a subspace containing the asymptotic part of the solution, which is both x dependent and t dependent. Nevertheless, as we will see in Secs. V and VI, different initial preparations give identical contributions to the dynamics in this subspace.

The mixing property of (1.1) can also be expected on the basis of the following simple intuitive argument. Without noise, a periodically forced monostable dissipative deterministic system must approach a periodic solution depending only on the control parameters and the external force. Thus any initial preparation of the system must be forgotten in the course of time. If this system is subjected to a small noise, a periodically varied probability distribution sharply centered around the deterministic path will be built up, and no initial information can be expected to survive for large times. In the case of multiwell potential, there are two regimes of the evolution if the noise is small. In the short-time regime the system approaches quickly the metastable state, while performing periodic motions around the stable points. In this stage the evolution "forgets" the concrete position in each potential well while still remembering how much probability was initially assigned to each well. In the long-time regime the probability balance between the various potential wells is established, and subsequently, the initial condition is completely forgotten. The persistent solution realized at $t \rightarrow \infty$ is nonstationary, but includes no information on the initial preparation.

All the above intuitive arguments on the mixing feature of (1.1) will be verified more quantitatively in the following sections. Furthermore, we fix θ in the calculation procedure, and perform an average over θ only after the solution for fixed θ has been specified. The general proof of the mixing property of FPE's, including (1.1) as a special case, can be seen in Chap. 6 of Ref. 3.

III. PERSISTENT PROBABILITY DISTRIBUTION

As ϵ is small the periodic forcing can be regarded as a perturbation. The solutions of the unperturbed FPE

$$\begin{aligned} \frac{\partial P(x, t)}{\partial t} &= LP(x, t) \\ &= -\frac{\partial}{\partial x} [C(x)P(x, t)] + D \left[\frac{\partial^2}{\partial x^2} \right] P(x, t) \end{aligned} \quad (3.1)$$

constitute then a good starting point for analyzing (1.1). Actually, the complete knowledge of the solution of (3.1) is still lacking. Nevertheless, (3.1) is much better under-

stood than (1.1), and hereafter we assume that the evolution described by (3.1) is known. Specifically, we assume that we have determined the complete set of the eigenvalues λ_n and the corresponding eigenfunctions $u_n(x)$, $n = 0, 1, 2, \dots$, of the Fokker-Planck operator L ,

$$Lu_n(x) = -\lambda_n u_n(x). \tag{3.2}$$

An approach extensively used in dealing with (3.1) is to make a transformation

$$P(x, t) = N^{1/2} e^{V(x)/2D} \Phi(x, t), \tag{3.3}$$

where

$$V(x) = \int_0^x C(x) dx, \quad N = \left[\int_{-\infty}^{\infty} e^{V(x)/D} dx \right]^{-1} \tag{3.4}$$

and

$$P(x) = N e^{V(x)/D} \tag{3.5}$$

is the stationary solution of (3.1). It is well known that $\Phi(x, t)$ satisfies the Schrödinger equation

$$\frac{\partial \Phi(x, t)}{\partial t} = H \Phi(x, t), \tag{3.6}$$

in which the effective Hamiltonian H is a known Hermitian operator. Solving the eigenvalue equations

$$H \Phi_n(x) = -\lambda_n \Phi_n(x) \tag{3.7}$$

we may define a complete set of orthonormal eigenvectors

$$\int_{-\infty}^{\infty} \Phi_n(x) \Phi_m(x) dx = \delta_{n,m}. \tag{3.8}$$

It is obvious that λ_n defined in (3.7) are identical to those defined in (3.2), and that $\Phi_n(x)$ are related to $u_n(x)$ by

$$u_n(x) = N^{1/2} e^{V(x)/2D} \Phi_n(x). \tag{3.9}$$

In the following sections we prefer to use $u_n(x)$ rather than $\Phi_n(x)$ as basis. The advantage of this will soon be clear. Let us formally write $u_n(x)$ as the right eigenvectors

$$|n\rangle = u_n(x), \quad n = 0, 1, 2, \dots \tag{3.10}$$

The left eigenvectors are defined as

$$\langle m|n\rangle = \int_{-\infty}^{\infty} u'_m(x) u_n(x) dx = \delta_{m,n}. \tag{3.11}$$

From (3.3), (3.8), (3.10), and (3.11), $\langle n|$ can be expressed as

$$\langle n| = u'_n(x) = N^{-1/2} e^{-V(x)/2D} \Phi_n(x). \tag{3.12}$$

One should notice that due to the non-Hermitian property of the Fokker-Planck operator L , the left eigenvectors are not identical to the right eigenvectors. An eigenvalue and eigenvector of particular importance are

$$\begin{aligned} \lambda_0 &= 0, \quad \Phi_0(x) = N^{1/2} e^{V(x)/2D}, \\ |0\rangle &= u_0(x) \equiv P(x) = N e^{V(x)/D}, \end{aligned} \tag{3.13}$$

and

$$\langle 0| = u_0(x) = 1. \tag{3.14}$$

The following identities are obvious due to the normalization condition and the physical requirement:

$$\begin{aligned} \int_{-\infty}^{\infty} u_0(x) dx &= \langle 0|0\rangle = 1, \\ \int_{-\infty}^{\infty} u_n(x) dx &= \langle 0|n\rangle = 0, \quad n \neq 0 \\ \lim_{x \rightarrow \pm\infty} [u_n(x)] &= 0, \quad n = 0, 1, 2, \dots \end{aligned} \tag{3.15}$$

Now the essential task in solving (1.1) is to find the probability distribution $P(x, t)$, starting from an arbitrary preparation, say, $\delta(x - x_0)$. All physically interesting and observable quantities such as the autocorrelation function $\langle x(t + \tau)x(t) \rangle$ or its power spectrum can be worked out without difficulty based on $P(x, t)$. In this section we focus on the long-time behavior of the system, i.e., we consider the solution

$$\lim_{t \rightarrow \infty} P(x, t) = P_{\infty}(x, t), \tag{3.16}$$

which must be periodic with period $2\pi/\Omega$. Let us write $P_{\infty}(x, t)$ in the form of a Fourier series

$$P_{\infty}(x, t) = f_0(x) + \sum_{m=1} [f_m(x) \cos(m\Omega t + m\theta) + k_m(x) \sin(m\Omega t + m\theta)], \tag{3.17}$$

with $f_n(x)$ and $k_n(x)$ being real. At this stage, we do not know whether $f_n(x)$ and $k_n(x)$ include the information of the initial state, although we will show later that this is not the case. $f_n(x)$ and $k_n(x)$ can be further expressed in terms of the basis of (3.10) as

$$f_m = \sum_n f_{n,m} |n\rangle, \quad k_m = \sum_n k_{n,m} |n\rangle. \tag{3.18a}$$

This yields

$$\begin{aligned} P_{\infty}(x, t) &= \sum_{n=0}^{\infty} Q_n(t) |n\rangle \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q_{n,m}(t) |n\rangle, \end{aligned}$$

with

$$Q_{n,0} = f_{n,0} \tag{3.18b}$$

$$Q_{n,m}(t) = f_{n,m} \cos(m\Omega t + m\theta) + k_{n,m} \sin(m\Omega t + m\theta),$$

where $f_{n,m}$ and $k_{n,m}$ are x - and t -independent real numbers. Inserting (3.18) into (1.1) and identifying on both sides the terms corresponding to a given m and n , we have

$$0 = -\lambda_n f_{n,0} + (\epsilon/2) \sum_{v=0}^{\infty} f_{v,1} \left\langle n \left| \frac{\partial}{\partial x} \right| v \right\rangle, \quad m = 0 \tag{3.19a}$$

$$\begin{aligned} -\Omega f_{n,1} \sin(\Omega t + \theta) + \Omega k_{n,1} \cos(\Omega t + \theta) &= -\lambda_n [f_{n,1} \cos(\Omega t + \theta) + k_{n,1} \sin(\Omega t + \theta)] + \epsilon \sum_{v=0}^{\infty} f_{v,0} \left\langle n \left| \frac{\partial}{\partial x} \right| v \right\rangle \cos(\Omega t + \theta) \\ &\quad + (\epsilon/2) \sum_{v=1}^{\infty} \left[f_{v,2} \left\langle n \left| \frac{\partial}{\partial x} \right| v \right\rangle \cos(\Omega t + \theta) + k_{v,2} \left\langle n \left| \frac{\partial}{\partial x} \right| v \right\rangle \sin(\Omega t + \theta) \right], \quad m = 1 \end{aligned} \tag{3.19b}$$

$$\begin{aligned} &-m \Omega f_{n,m} \sin(m \Omega t + m \theta) + m \Omega k_{n,m} \cos(m \Omega t + m \theta) \\ &= -\lambda_n [f_{n,m} \cos(m \Omega t + m \theta) + k_{n,m} \sin(m \Omega t + m \theta)] \\ &\quad + (\epsilon/2) \sum_{v=0}^{\infty} \left[f_{v,m-1} \left\langle n \left| \frac{\partial}{\partial x} \right| v \right\rangle + f_{v,m+1} \left\langle n \left| \frac{\partial}{\partial x} \right| v \right\rangle \right] \cos(m \Omega t + m \theta) \\ &\quad + (\epsilon/2) \sum_{v=0}^{\infty} \left[k_{v,m-1} \left\langle n \left| \frac{\partial}{\partial x} \right| v \right\rangle + k_{v,m+1} \left\langle n \left| \frac{\partial}{\partial x} \right| v \right\rangle \right] \sin(m \Omega t + m \theta), \quad m \geq 1. \end{aligned} \tag{3.19c}$$

Several interesting features are already worthwhile remarking.

(1) From the structure of (3.19) it is obvious that the leading order of $f_{n,q}$ and $k_{n,q}$ is ϵ^q .

(2) After the leading order, the next to the leading order and the following orders of $f_{n,m}$ and $k_{n,m}$ are $\epsilon^{m+2}, \epsilon^{m+4}, \dots$. Therefore $Q_{n,m}(t)$ can be written as

$$Q_{n,m}(t) = \epsilon^m Q_{n,m}^{(m)}(t) + \epsilon^{m+2} Q_{n,m}^{(m+2)}(t) + \dots, \tag{3.20}$$

where

$$\begin{aligned} Q_{n,0}^{(2q)} &= f_{n,0}^{(2q)}, \\ Q_{n,m}^{(m+2q)}(t) &= f_{n,m}^{(m+2q)} \cos(m \Omega t + m \theta) \\ &\quad + k_{n,m}^{(m+2q)} \sin(m \Omega t + m \theta), \quad m \geq 1. \end{aligned}$$

(3) We have the identities

$$\left\langle 0 \left| \frac{\partial}{\partial x} \right| n \right\rangle \equiv 0 \tag{3.21}$$

for all n . This has some interesting consequences. On the one hand, it indicates that the coefficient $Q_{0,0} = f_{0,0}$ is not changed by the perturbation $\epsilon \cos(\Omega t + \theta)$,

$$Q_{0,0}^{(2q)} = 0, \quad q = 1, 2, \dots$$

On the other hand, all the time-dependent terms in (3.18) do not contain the eigenvector $|0\rangle$, namely,

$$Q_{0,m}(t) = f_{0,m} = k_{0,m} = 0, \quad m \geq 1. \tag{3.22}$$

Both aspects are consistent with the requirement that the normalization condition should be presented for all times.

Equations (3.19) can be solved order by order. In each order, the solution is exact and explicit. For instance, to the ϵ^0 order, we have [cf. (3.19a) and (3.19b)]

$$f_{0,0}^{(0)} = f_{0,0} = 1, \tag{3.23a}$$

$$f_{n,0}^{(0)} = 0, \quad n \neq 0 \tag{3.23b}$$

expressing the fact that the system should approach the stationary solution of (3.1) as the periodic force vanishes. Inserting (3.23) into (3.19b) we arrive at two independent sets of coupled linear algebraic equations

$$\begin{aligned} \lambda_n f_{n,1}^{(1)} + \Omega k_{n,1}^{(1)} &= \left\langle n \left| \frac{\partial}{\partial x} \right| 0 \right\rangle, \\ -\Omega f_{n,1}^{(1)} + \lambda_n k_{n,1}^{(1)} &= 0, \end{aligned} \tag{3.24}$$

which can be easily solved as

$$\begin{aligned} f_{n,1}^{(1)} &= R_{n,0} g_{n,1} \cos(\alpha_{n,1}), \\ k_{n,1}^{(1)} &= -R_{n,0} g_{n,1} \sin(\alpha_{n,1}), \end{aligned} \tag{3.25}$$

with

$$R_n = \left\langle n \left| \frac{\partial}{\partial x} \right| m \right\rangle, \quad g_{n,m} = [\lambda_n^2 + (m \Omega)^2]^{-1/2},$$

and

$$\cos(\alpha_{n,m}) = \lambda_n g_{n,m}, \quad \sin(\alpha_{n,m}) = -m \Omega g_{n,m},$$

leading to

$$Q_{n,1}^{(1)}(t) = R_{n,0} g_{n,1} \cos(\Omega t + \theta + \alpha_{n,1}). \tag{3.26}$$

Inserting (3.25) into (3.19a) and (3.19c) we obtain the solution $Q_{n,0}^{(2)}$ and $Q_{n,2}^{(2)}(t)$ to the second order. For the former we have

$$Q_{n,0}^{(2)} = (1/2) \sum_i R_{n,i} R_{i,0} g_{n,0} g_{i,1} \cos(\alpha_{i,1}). \tag{3.27}$$

For the latter we again get two independent sets of coupled algebraic equations

$$\begin{aligned} \lambda_n f_{n,2}^{(2)} + 2\Omega k_{n,2}^{(2)} &= (1/2) \sum_i R_{n,i} R_{i,0} g_{i,1} \cos(\alpha_{i,1}) , \\ -2\Omega f_{n,2}^{(2)} + \lambda_n k_{n,2}^{(2)} &= (1/2) \sum_i R_{n,i} R_{i,0} g_{i,1} \sin(\alpha_{i,1}) , \end{aligned} \tag{3.28}$$

$$\begin{aligned} Q_{n,2}^{(2)}(t) &= (1/2) \sum_i R_{n,i} R_{i,0} g_{i,1} g_{n,2} \\ &\times \cos(2\Omega t + 2\theta + \alpha_{i,1} + \alpha_{n,2}) . \end{aligned} \tag{3.29}$$

leading to

Up to ϵ^2 order we can now specify the persistent probability distribution as

$$\begin{aligned} P_\infty(x, t) &= |0\rangle + \sum_n \epsilon R_{n,0} g_{n,1} \cos(\Omega t + \theta + \alpha_{n,1}) |n\rangle \\ &+ (\epsilon^2/2) \sum_n \left[\sum_i R_{n,i} R_{i,0} g_{n,0} g_{i,1} \cos(\alpha_{i,1}) + \sum_i R_{n,i} R_{i,0} g_{i,1} g_{n,2} \cos(2\Omega t + 2\theta + \alpha_{i,1} + \alpha_{n,2}) \right] |n\rangle . \end{aligned} \tag{3.30}$$

The same procedure can be performed order by order to higher-order terms. We can actually specify the solution of a general term in terms of the lower ones as

$$Q_{n,m}^{(m+2q)}(t) = (1/2)^{m+2k-1} \sum_{k=0}^q \sum_{i_\nu} T \left[\prod_{\nu=1}^{m+2k-1} (R_{i_{\nu+1}, i_\nu} g_{i_{\nu+1}, j_{\nu+1}}) \times \cos \left[m\Omega t + m\theta + \sum_{\nu=0}^{m+2k-1} \alpha_{i_{\nu+1}, j_{\nu+1}} \right] \right] Q_{i_1, 0}^{(2q-2k)} , \tag{3.31}$$

where the operator T indicates the summation of all the terms for j_ν to take any integer values under the conditions

$$\begin{aligned} i_{m+2k} &= n, \quad j_1 = 1, \quad j_{m+2k} = m , \\ j_\nu &\geq 1, \quad j_{\nu+1} - j_\nu = \pm 1, \quad \nu = 0, 1, 2, \dots \end{aligned}$$

and $Q_{j,0}^{(2\mu)}$ is given by the lower-order terms as

$$(1/4\pi) \sum_i \int_0^{2\pi} R_{j,i} g_{j,0} Q_{i,1}^{(2\mu-1)}(t) \cos(\Omega t + \theta) d\theta . \tag{3.32}$$

The operator acting on $Q_{j,0}^{(2q-2k)}$ plays a role similar to a propagator in the field theory. From (3.31) all the terms in (3.20) can be reconstructed successively from the zeroth-order ones

$$Q_{0,0}^{(0)} = 1, \quad Q_{n,0}^{(0)} = 0, \quad n \geq 1 . \tag{3.33}$$

For instance, a direct calculation of (3.31) for $\epsilon^0 \rightarrow \epsilon^2$ produces (3.30). The solution of the third order can also be written as

$$\begin{aligned} Q_{n,3}^{(3)}(t) &= \frac{1}{4} \sum_i \sum_j R_{n,i} R_{i,j} R_{j,0} g_{n,3} g_{i,2} g_{j,1} \cos(3\Omega t + 3\theta + \alpha_{n,3} + \alpha_{i,2} + \alpha_{j,1}) , \\ Q_{n,1}^{(3)}(t) &= \frac{1}{4} \sum_i \sum_j R_{n,i} R_{i,j} R_{j,0} g_{n,1} g_{i,2} g_{j,1} \cos(\Omega t + \theta + \alpha_{n,1} + \alpha_{i,2} + \alpha_{j,1}) \\ &+ \frac{1}{2} \sum_i \sum_j R_{n,i} R_{i,j} R_{j,0} g_{n,1} g_{i,0} g_{j,1} \cos(\Omega t + \theta + \alpha_{n,1}) \cos(\alpha_{j,1}) . \end{aligned} \tag{3.34}$$

In short, Eq. (3.31) provides a technically simple way to calculate the persistent solution. It also makes it possible to draw some general conclusions without specifying the exact solution.

(1) In each order we get an exact solution. The iteration form is particularly convenient for the analytic investigation. With the help of the computer by utilizing the symbolic manipulation it is easy to calculate the probability distribution to higher orders.

(2) In (3.31) and (3.32), the frequency of the external force Ω and the eigenvalues λ_n appear in only two sets of

coefficients, $g_{n,m}$ and $\alpha_{n,m}$. The entire persistent probability distribution has been well organized to summations of products of three kinds of entities, the matrix elements $R_{n,i}$, the renormalization factors $g_{n,m}$, and the phase shift $\alpha_{n,m}$. This regular organization turns out to be very convenient for analytic works as well as computer simulations.

(3) The persistent solution is unique. The reason is that both the normalization constant ($Q_{0,0}^{(0)} = 1$) and the solution of (3.31) and (3.32) are unique.

(4) The persistent solution does not contain informa-

tion of the initial state since this information is included neither in the normalization constant $Q_{0,0}^{(0)}=1$ nor in the iteration relations (3.31). This conclusion is different from the results by Presilla, Marchesoni, and Gammaitoni.¹⁸

IV. TRANSIENT PROCESS, FLOQUET THEORY

To investigate the transient process towards the persistent solution, one should specify the exponentially damping terms. We assume that the probability distribution at the transient process takes the form of

$$P(x, t) = \sum_q \exp(-\lambda'_q t) P(\lambda'_q, x, t), \quad (4.1)$$

where λ'_q is a discrete quasieigenvalue of Eq. (1.1) which can be continuously reduced to λ_q as we adjust ϵ continuously to zero. The evolution of the various quasieigenfunctions are, apparently, independent of each other. Thus for each λ'_q the subdynamics $P(\lambda'_q, x, t)$ can be determined inside its own subspace.

Assuming a particular λ'_q , and expanding $P(\lambda'_q, x, t)$ in terms of ϵ , $|n\rangle$, and the various harmonics, we have

$$\begin{aligned} P(\lambda'_q, x, t) &= Z_q \sum_n Q_n(\lambda'_q, t) |n\rangle \\ &= Z_q \sum_i \epsilon^i Q^{(i)}(\lambda'_q, x, t) \\ &= Z_q \sum_i \epsilon^i \sum_n \sum_{m=0}^i Q_{n,m}^{(i)}(\lambda'_q, t) |n\rangle, \end{aligned} \quad (4.2a)$$

$$Z_q = Z_q^{(0)} + \epsilon Z_q^{(1)} + \epsilon^2 Z_q^{(2)} + \epsilon^3 Z_q^{(3)} + \dots, \quad (4.2b)$$

where Z_q , $q=1, 2, \dots$, appear to be arbitrary preconstants of quasieigenfunctions (note $Z_0=1$). For the initial condition $P(x, 0) = \delta(x - x_0)$, we obviously have

$$\begin{aligned} Q_{q,0}^{(0)} &= 1, \quad Q_{n,0}^{(0)} = 0, \quad n \neq q \\ Z_q^{(0)} &= N^{-1/2} \exp[-V(x_0)/(2D)] \Phi_q(x_0). \end{aligned} \quad (4.3)$$

For the q th subdynamics, we may get the solutions $Q_n(\lambda'_q, t)$ in terms of iteration as in Eq. (3.31) except that the eigenvalues λ_n should be replaced by $\lambda_n - \lambda'_q$. Specifically, one needs only to change $g_{n,m}$ and $\alpha_{n,m}$ for $Q_n(t)$ to $g_{n,m}(\lambda'_q)$ and $\alpha_{n,m}(\lambda'_q)$ for $Q_n(\lambda'_q, t)$, respectively, as

$$\begin{aligned} g_{n,m}(\lambda'_q) &= [(\lambda_n - \lambda'_q)^2 + (m\omega)^2]^{-1/2}, \\ \cos[\alpha_{n,m}(\lambda'_q)] &= \frac{(\lambda_n - \lambda'_q)}{[(\lambda_n - \lambda'_q)^2 + (m\omega)^2]^{1/2}}, \\ \sin[\alpha_{n,m}(\lambda'_q)] &= -\frac{m\omega}{[(\lambda_n - \lambda'_q)^2 + (m\omega)^2]^{1/2}}. \end{aligned} \quad (4.4a)$$

The iteration is performed in such a way that

$$Q_{q,0}(\lambda'_q) = Q_{q,0}^{(0)}(\lambda'_q) = 1, \quad (4.4b)$$

which is similar to the coefficient $Q_{0,0}$ for the persistent solution. Notice, however, that we have at this stage an unknown prefactor Z_q in (4.2). Now let us specify the ei-

genvalue λ'_q which deviates from λ_q due to the periodic forcing. The solution of λ'_q can be found by inserting (4.2) into (3.19a) and multiplying both sides of the equation by $\langle q|$,

$$\begin{aligned} \lambda_q - \lambda'_q &= (1/4\pi) \sum_i \epsilon^i \sum_{n \neq q} \int_0^{2\pi} Q_{n,1}^{(i)}(\lambda'_q, t) R_{q,n} \\ &\quad \times \epsilon \cos(\Omega t + \theta) d\theta, \end{aligned} \quad (4.5)$$

which is an implicit function of λ'_q . This function can be made explicit by expanding λ'_q on ϵ ,

$$\lambda'_q = \lambda_q^{(0)} + \epsilon^2 \lambda_q^{(2)} + \dots, \quad (4.6)$$

where the solutions of the first two orders read

$$\lambda_q^{(0)} = \lambda_q, \quad (4.7a)$$

$$\begin{aligned} \lambda_q^{(2)} &= (\epsilon^2/2) \sum_{n \neq q} R_{q,n} R_{n,q} g_{q,0}(\lambda_q) g_{n,1}(\lambda_q) \\ &\quad \times \cos[\alpha_{n,1}(\lambda_q)]. \end{aligned} \quad (4.7b)$$

The general form in q th subdynamics is similar to that of the persistent solution, i.e., the zeroth subdynamics. However, two major differences which are of conceptual importance should be emphasized. First, we have exactly one zero quasieigenvalue

$$\lambda'_0 = \lambda_0 = 0$$

because $\langle 0|(\partial/\partial x)|n\rangle = 0$. Thus the zero eigenvalue of (3.1) is not affected by the periodic perturbation. On the contrary, all others are modified by the signal. Furthermore, unlike the persistent solution which is completely solved in its own subspace independently of the initial condition and the dynamics in other subspaces, for the q th ($q \neq 0$) subdynamics there remains an unknown constant Z_q ($Z_0=1$) which cannot be fixed in the q th subspace though all other coefficients in Eq. (4.2) can be specified independently in this subspace. Therefore all the information of the initial state and the couplings between various subspaces are contained in the constants Z_q ($q \neq 0$) which will be determined in Sec. V.

V. INITIAL-VALUE PROBLEM, MIXING

Let us suppose that the initial probability distribution is

$$P(x, 0) = \delta(x - x_0). \quad (5.1)$$

Under the evolution (1.1), Eq. (5.1) yields a probability distribution $P(x, t; x_0, 0)$ at time t . For a general initial preparation $P(x, 0)$, the solution can be formally written as

$$P(x, t) = \int P(x, t; x_0, 0) P(x_0, 0) dx_0. \quad (5.2)$$

Thus the problem is completely solved if $P(x, t; x_0, 0)$ can be specified. In the sequel, we consider only the initial state (5.1) and simply write $P(x, t)$ instead of $P(x, t; x_0, 0)$.

Expanding $P(x, t)$ and $\delta(x, -x_0)$ in the basis $|n\rangle$, we have

$$P(x, t) = \sum_{n=0}^{\infty} b_n(t) |n\rangle, \tag{5.3}$$

with

$$b_0(t) = b_0(0) = 1 \tag{5.4}$$

and

$$b_n(0) = N^{-1/2} \exp[-V(x_0)/(2D)] \Phi_n(x_0). \tag{5.5}$$

According to the analysis of Secs. III and IV, we can further express $b_n(t)$ as

$$b_n(t) = Q_n(t) + \sum_{q=1}^{\infty} Z_q \exp(-\lambda'_q t) Q_n(\lambda'_q, t), \tag{5.6}$$

where $Q_n(t)$ and $Q_n(\lambda'_q, t)$ have been explicitly given in Eqs. (3.18), (3.31) and (4.2), (4.5), (3.31), respectively, and Z_q remains to be determined. Taking $t=0$, we have from (5.6) a set of coupled linear algebraic equations

$$Q_n(0) + \sum_{q=1}^{\infty} Z_q Q_n(\lambda'_q, 0) = b_n(0), \quad n = 1, 2, \dots \tag{5.7}$$

[Note that for $n=0$ we have the identities $b_0(t) = Q_0(t) = 1$, $Q_0(\lambda'_q, t) = 0$, $q \neq 0$, which are independent of the particular initial preparation.] In principle, Z_q , $q \neq 0$, can be found by solving (5.7).

Truncating (5.7) to a finite, possibly, very large value M , the equations for the modes retained read

$$Q_n(0) + \sum_{q=0}^M Z_q Q_n(\lambda'_q, 0) = b_n(0), \quad n = 1, 2, \dots \tag{5.8}$$

which admit solutions

$$\mathbf{Z} = \underline{\mathbf{S}}^{-1}(\mathbf{B} - \mathbf{Q}), \tag{5.9}$$

where \mathbf{Z} , \mathbf{B} , and \mathbf{Q} are M -dimensional vectors

$$\mathbf{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_M \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1(0) \\ b_2(0) \\ \vdots \\ b_M(0) \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} Q_1(0) \\ Q_2(0) \\ \vdots \\ Q_M(0) \end{bmatrix}, \tag{5.10}$$

and $\underline{\mathbf{S}}$ is a known matrix with elements

$$S_{ij} = Q_i(\lambda'_j, 0). \tag{5.11}$$

An alternative way of solving (5.7) is to expand Z_q in terms of ϵ . Inserting Eqs. (3.31) and (4.2) into (5.7), we can identify $Z_q^{(0)}, Z_q^{(1)}, \dots$ order by order. For instance, in the orders ϵ^0 and ϵ^1 we obtain

$$Z_q^{(0)} = b_q(0), \tag{5.12}$$

$$Z_q^{(1)} = -R_{q,0} g_{q,1} \cos(\theta + \alpha_{q,1}) - \sum_{m \neq 0, q} R_{q,m} g_{q,1}(\lambda_m) \cos[\theta + \alpha_{q,1}(\lambda_m)] b_m(0). \tag{5.13}$$

Thus $Q_n(\lambda'_q, t)$ can be calculated in each subspace, independently of the dynamics of other subspaces [see (3.31) and (4.5)]. The constants Z_q can be found by solving a set of coupled equations which includes the initial conditions as well as the results of all other subspaces. However, once Z_q are determined initially they will remain constant at all times, and the evolutions in various subspaces are independent of each other for $t > 0$. The situation for the persistent solution (the dynamics in the subspace $q=0$ corresponding to $\lambda'=\lambda=0$) is essentially different from all the other subdynamics. The initial condition does not enter this subspace because the identity $Z_0=1$ is valid whatever the initial state. For small ϵ , each $-\lambda'_q$, $q > 0$, must be negative. The initial condition should be forgotten exponentially. The periodically forced FPE (1.1) is therefore mixing. By taking the basis (3.10) the particular features of the persistent solution can be explicitly manifested by $\langle 0 | (\partial/\partial x) | n \rangle = 0$. It is an important advantage to use this basis.

All the analyses of Secs. III-V can be extended to a more general FPE (1.1) with an arbitrary perturbation $h(x)$. It suffices to replace $R_{n,m} = \langle n | (\partial/\partial x) | m \rangle$ by $\langle n | (\partial/\partial x) h(x) | m \rangle$. Hence, in the following we only need to consider the simplest case $h(x) = 1$.

VI. AN EXACTLY SOLVABLE MODEL: PERIODICALLY FORCED FPE WITH LINEAR DRIFT

To test our theory we consider in this section an exactly solvable model, the periodically forced FPE with linear drift

$$\frac{\partial P(x, t)}{\partial t} = -\frac{\partial}{\partial x} [-\lambda x + \epsilon \cos(\Omega t + \theta)] P(x, t) + D \left[\frac{\partial^2}{\partial x^2} \right] P(x, t). \tag{6.1}$$

The moment equations of (6.1) may be easily written down as

$$\begin{aligned} \frac{d \langle x(t) \rangle}{dt} &= -\lambda \langle x(t) \rangle + \epsilon \cos(\Omega t + \theta), \\ \frac{d \langle x(t)^2 \rangle}{dt} &= -2\lambda \langle x(t) \rangle^2 + 2\epsilon \langle x(t) \rangle \cos(\Omega t + \theta) + 2D, \\ \frac{d \langle x(t)^3 \rangle}{dt} &= -3\lambda \langle x(t) \rangle^3 + 3\epsilon \langle x(t)^2 \rangle \cos(\Omega t + \theta) + 6 \langle x(t) \rangle D, \end{aligned} \tag{6.2}$$

$$\begin{aligned} \frac{d \langle x(t)^n \rangle}{dt} &= -n\lambda \langle x(t) \rangle + n\epsilon \langle x(t)^{n-1} \rangle \cos(\Omega t + \theta) \\ &\quad + n(n-1) \langle x(t)^{n-2} \rangle. \end{aligned}$$

The first moment equation is exactly the deterministic equation of evolution which can be solved explicitly as

$$\begin{aligned} \langle x(t) \rangle &= C \exp(-\lambda t) + \epsilon g \cos(\Omega t + \theta + \alpha), \\ g &= (\lambda^2 + \Omega^2)^{-1/2}, \\ \cos \alpha &= \lambda g, \quad \sin \alpha = -\Omega g, \end{aligned} \tag{6.3}$$

where the constant C is fixed by the initial condition and will eventually be forgotten exponentially. The evolution of the higher moments can likewise be exactly solved in terms of the control parameters and the lower moments. Again, as $t \rightarrow \infty$, all the initial information will be entirely forgotten due to the dissipation $-n\lambda$. This justifies our conclusions in Secs. II and V.

Now let the initial state be a Gaussian distribution. Then solution of (6.1) keeps the Gaussian property for all times

$$P(x,t) = u(t) \exp\{-[x - y(t)]^2 / \gamma(t)\} . \tag{6.4}$$

$$P(x,t) = [2\pi D (1 - Ce^{-2\lambda t}) / \lambda]^{-1/2} \exp\{\lambda[x - x_0 e^{-2\lambda t} - g \cos(\Omega t + \theta + \alpha)]^2 / [2D(1 - Ce^{-2\lambda t})]\} , \tag{6.6}$$

where the constants C and x_0 have to be determined by the initial condition (which is, obviously, forgotten exponentially as $t \rightarrow \infty$), and g and α are given in Eq. (6.3). If the initial probability distribution is a δ function centered at x_0 , Eq. (6.6) reduces to

$$P(x,t) = [2\pi D (1 - e^{-2\lambda t}) / \lambda]^{-1/2} \exp\{\lambda[x - x_0 e^{-2\lambda t} - g \cos(\Omega t + \theta + \alpha)]^2 / [2D(1 - e^{-2\lambda t})]\} . \tag{6.7}$$

As $t \rightarrow \infty$, the persistent solution reads

$$P(x,t) = (2\pi D / \lambda)^{-1/2} \times \exp\{-\lambda[x - g \cos(\Omega t + \theta + \alpha)]^2 / 2D\} \tag{6.8}$$

which, in accordance with our earlier consideration, does not contain any information on the initial state.

We now apply the theory of Secs. III and IV to the linear drift case. The formulas in Secs. III and IV are considerably simplified by the identities

$$\lambda'_n = \lambda_n = n\lambda, \quad n = 0, 1, 2, \dots , \tag{6.9}$$

$$\begin{aligned} Q_0 &= 1, \quad Q^{(1)}(t) = g \cos(\Omega t + \theta + \alpha) , \\ Q^{(2)}(t) &= \frac{1}{4} g^2 [\cos(2\Omega t + 2\theta + 2\alpha) + 1] = [g \cos(\Omega t + \theta + \alpha)]^2 / 2 , \\ Q^{(3)}(t) &= \frac{1}{24} g \cos^3(3\Omega t + 3\theta + 3\alpha) + \frac{1}{8} g g_{3,1}^2 [\cos(\Omega t + \theta + 2\alpha + \alpha_{3,1}) + 2 \cos(\Omega t + \theta + \alpha_{3,1})] \\ &= [g \cos(\Omega t + \theta + \alpha)]^3 / 6 . \end{aligned} \tag{6.12}$$

A tedious, though elementary calculation yields

$$Q^{(n)}(t) = [g \cos(\Omega t + \theta + \alpha)]^n / n! \tag{6.13}$$

and the solution of the persistent probability distribution can be formally written as

$$\begin{aligned} P_\infty(x,t) &= \sum_{n=0}^{\infty} (1/n!) [eg \cos(\Omega t + \theta + \alpha)]^n |n\rangle \\ &= \sum_{n=0}^{\infty} (1/n!) \left[eg \cos(\Omega t + \theta + \alpha) \frac{\partial}{\partial x} \right]^n |0\rangle \\ &= \exp \left[eg \cos(\Omega t + \theta + \alpha) \frac{\partial}{\partial x} \right] |0\rangle \\ &= u_0 [x - eg \cos(\Omega t + \theta + \alpha)] . \end{aligned} \tag{6.14}$$

Inserting (6.4) into (6.1) and identifying on both sides the terms of equal powers $[x - y(t)]^0$, $[x - y(t)]^1$, and $[x - y(t)]^2$, we arrive at the three equations

$$\frac{d\gamma(t)}{dt} = -2\lambda\gamma(t) + 4D , \tag{6.5a}$$

$$\frac{dy(t)}{dt} = -\lambda y(t) + \epsilon \cos(\Omega t + \theta) , \tag{6.5b}$$

$$\frac{du(t)}{dt} = \lambda - 2Du(t) / \gamma(t) , \tag{6.5c}$$

which can be solved successively. The final solution of (6.1) reads

$$\begin{aligned} u_n(x) &= d^n u_0(x) / dx^n , \\ \left\langle n \left| \frac{\partial}{\partial x} \right| m \right\rangle &= \begin{cases} 1, & n = m + 1 \\ 0 & \text{otherwise} . \end{cases} \end{aligned} \tag{6.10}$$

[Note that the identities (6.10) are not valid if we take the basis (3.7).] Owing to (6.9) and (6.10) we may simplify (3.18b) to

$$P_\infty(x,t) = Q_0 + \sum_{n=1}^{\infty} \epsilon^n Q^{(n)}(t) |n\rangle . \tag{6.11}$$

Inserting (6.9) and (6.10) into (3.26), (3.27), (3.29), and (3.34), we have

The same procedure can be performed to manifest the transient process and get the similar result

$$P(\epsilon, x, t) = P[\epsilon = 0, x - \epsilon g \cos(\Omega t + \theta + \alpha), t] . \tag{6.15}$$

[To obtain (6.15) we use additional equalities $\lambda'_q = \lambda_q$ for all integer q .] Thus, for the exactly solvable linear drift model, our general formulas reduce to the exact solution. For the FPE with nonlinear drift, no such simple and explicit solution exists. However, the iteration approach (3.31), (3.32) still applies and can be used to produce an approximate solution to any desired degree of accuracy.

VII. STOCHASTIC RESONANCE

In the previous sections, we introduced a systematic approach in dealing with the periodically forced FPE. In this section we examine the use of this formalism for understanding the phenomena SR and SNRE. For this pur-

pose, we need only consider the linear response of the system to the signal. Keeping thus the ϵ^0 and ϵ^1 orders in the probability distribution we write

$$P(x, t) = P_\infty(x, t) + P_d(x, t), \quad (7.1)$$

where the exponentially damping part reads

$$P_d(x, t) = \sum_{q=1}^{\infty} \exp(-\lambda_q t) \left[\left(b_q(0) - \epsilon R_{q,0} g_{q,1} \cos(\theta + \alpha_{q,1}) - \epsilon \sum_{m \neq 0, q} R_{q,m} g_{q,1}(\lambda_m) \cos[\theta + \alpha_{q,1}(\lambda_m)] b_m(0) \right) |q\rangle \right. \\ \left. + \epsilon \sum_{n \neq 0} R_{n,q} g_{n,1}(\lambda_q) \cos[\Omega t + \theta + \alpha_{n,1}(\lambda_q)] b_q(0) |n\rangle \right] \quad (7.2)$$

and the persistent solution is

$$P_\infty(x, t) = |0\rangle + \sum_{n=1}^{\infty} \epsilon R_{n,0} \cos(\Omega t + \theta + \alpha_{n,1}) / (\lambda_n^2 + \Omega^2)^{1/2} |n\rangle. \quad (7.3)$$

Based on Eqs. (7.1)–(7.3) one may easily calculate the average $\langle x(t) \rangle_\infty$, and the autocorrelation function $\langle x(t+\tau)x(t) \rangle_{a,\infty}$ (\mathcal{N}_{ACF}) for long times,

$$\langle x(t) \rangle_\infty = \lim_{t \rightarrow \infty} \langle x(t) \rangle \\ = \sum_{n=1}^{\infty} \epsilon g_{n,1} \langle 0|x|n\rangle \left\langle n \left| \frac{\partial}{\partial x} \right| 0 \right\rangle \cos(\Omega t + \theta + \alpha_{n,1}), \quad (7.4)$$

$$\mathcal{N}_{\text{ACF}} = \langle x(t+\tau)x(t) \rangle_{a,\infty} \\ = \int_0^{2\pi} d\theta \lim_{t \rightarrow \infty} \int xy P(x, \tau+t; y, t) P(y, t; x_0, 0) dx dy. \quad (7.5)$$

The integral (7.5) can be easily specified by inserting (7.2) and (7.3) into (7.5). A comparison of (7.4) with the forced deterministic equation

$$\frac{dx}{dt} = -\lambda x + \epsilon \cos(\Omega t + \theta) \quad (7.6)$$

is interesting. Equation (7.6) has an exact solution

$$x(t)_\infty = \epsilon g \cos(\Omega t + \theta + \alpha), \quad (7.7)$$

with g and α being given by Eq. (6.3). Hence, for the linear-response theory, Eq. (1.1) can be regarded as an infinite number of independent overdamped particles sub-

jected to a dissipation coefficient λ_n , and forced by periodic signals with an amplitude renormalized by a factor $\langle 0|x|n\rangle \langle n|\partial/\partial x|0\rangle$, $n = 1, 3, 5, \dots$.

In the case of

$$\Omega \ll D, \quad D \ll \Delta V = V(0) - V(c), \quad (7.8)$$

where c represents the center of a potential basin [cf. (1.2a)] we have

$$\Omega, \lambda_1 \ll \lambda_i, \quad i = 2, 3, \dots \quad (7.9)$$

Therefore one needs to keep only a few relevant terms in (7.2) and (7.3),

$$P_\infty(x, t) = |0\rangle + \epsilon \left\langle 1 \left| \frac{\partial}{\partial x} \right| 0 \right\rangle \cos(\Omega t + \theta + \alpha_{1,1}) / (\lambda_1^2 + \Omega^2)^{1/2} |1\rangle, \quad (7.10)$$

$$P_d(x, t) = \exp(-\lambda_1 t) \left[b_1(0) - \epsilon \left\langle 1 \left| \frac{\partial}{\partial x} \right| 0 \right\rangle \cos(\theta + \alpha_{1,1}) / (\lambda_1^2 + \Omega^2)^{1/2} \right] |1\rangle, \quad (7.11)$$

leading to

$$\langle x(t) \rangle_\infty = \epsilon \left\langle 1 \left| \frac{\partial}{\partial x} \right| 0 \right\rangle \langle 0|x|1\rangle \cos(\Omega t + \theta + \alpha_{1,1}) / (\lambda_1^2 + \Omega^2)^{1/2}, \quad (7.12)$$

$$\mathcal{N}_{\text{ACF}} = \frac{1}{2} \epsilon^2 \left[\left\langle 1 \left| \frac{\partial}{\partial x} \right| 0 \right\rangle \right]^2 \left(\langle 0|x|1\rangle^2 \cos(\Omega \tau) / (\lambda_1^2 + \Omega^2) + (\langle 0|x|1\rangle)^2 \exp(-\lambda_1 \tau) \left[1 - \frac{1}{2} \epsilon^2 \left\langle 1 \left| \frac{\partial}{\partial x} \right| 0 \right\rangle \right]^2 / (\lambda_1^2 + \Omega^2) \right). \quad (7.13)$$

Notice that the two inequalities in (7.8) have to be valid for guaranteeing (7.10) and (7.11), and consequently, (7.12) and (7.13) as well. In particular, for a given external frequency, both too large and too small D 's may break the validity of (7.11)–(7.13).

We next calculate the Fourier transform of the ACF (7.13), i.e., the power spectrum of the system, as

$$S = S_s + S_n, \tag{7.14a}$$

$$S_s = (\pi/2)\epsilon^2 \left\langle \left\langle 1 \left| \frac{\partial}{\partial x} \right| 0 \right\rangle \right\rangle^2 (\langle 0|x|1 \rangle)^2 / (\lambda_1^2 + \Omega^2) [\delta(\omega - \Omega) + \delta(\omega + \Omega)], \tag{7.14b}$$

$$S_n = 2[\lambda_1 / (\lambda_1^2 + \omega^2)] (\langle 0|x|1 \rangle)^2 \left[1 - (1/2)\epsilon^2 \left\langle \left\langle 1 \left| \frac{\partial}{\partial x} \right| 0 \right\rangle \right\rangle^2 / (\lambda_1^2 + \Omega^2) \right]. \tag{7.14c}$$

The SNR at $\omega = \Omega$ (\mathcal{N}_{SNR}) reads (Refs. 15 and 16)

$$\mathcal{N}_{\text{SNR}} \simeq (\pi/4)\epsilon^2 \left\langle \left\langle 1 \left| \frac{\partial}{\partial x} \right| 0 \right\rangle \right\rangle^2 / \lambda_1. \tag{7.15}$$

To produce explicitly the expressions for the \mathcal{N}_{ACF} and \mathcal{N}_{SNR} we still need to specify the elements $\langle 1 | (\partial/\partial x) | 0 \rangle$ and $\langle 0|x|1 \rangle$. The left eigenvectors $\langle 0|$ and $\langle 1|$ read (Ref. 3)

$$\langle 0| = 1, \tag{7.16a}$$

$$\langle 1| = \bar{u}_1^{(0)}(x) + \bar{u}_1^{(1)}(x), \tag{7.16b}$$

with

$$\bar{u}_1^{(0)}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \tag{7.16c}$$

$$\bar{u}_1^{(1)}(x) = \begin{cases} -(\lambda_1/D) \int_x^\infty \exp[-V(x)/D] dx \int_x^\infty \exp[V(x)/D] dx, & x > 0 \\ (\lambda_1/D) \int_x^\infty \exp[-V(x)/D] dx \int_x^\infty \exp[V(x)/D] dx, & x < 0 \end{cases}$$

respectively. Multiplying the stationary solution of (1.1) to $\langle 0|$ and $\langle 1|$, we get $|0\rangle$ and $|1\rangle$. Therefore we obtain

$$\langle 0|x|1 \rangle \simeq c, \tag{7.17}$$

$$R_{1,0} = \left\langle 0 \left| \frac{\partial}{\partial x} \right| 1 \right\rangle \simeq R_1 + R_2, \tag{7.18}$$

with

$$R_1 = \int_{-\infty}^\infty \bar{u}_1^{(0)}(x) \frac{\partial}{\partial x} u_0(x) dx = -2N = -[2V''(c)/(\pi D)]^{1/2} \exp[-\Delta V/D], \tag{7.19}$$

$$R_2 = \int_{-\infty}^\infty \bar{u}_1^{(1)}(x) \frac{\partial}{\partial x} u_0(x) dx = -c\lambda_1/D, \tag{7.20}$$

where

$$V''(c) = \left. \frac{d^2 V(x)}{dx^2} \right|_{x=c}.$$

Inserting (7.17)–(7.19) into (7.13) and (7.15) and noting

$$\lambda_1 = [V''(0)V''(c)]^{1/2} \exp(-\Delta V)/\pi, \tag{7.21}$$

$$R_1 = \{ [2\pi D/V''(0)]^{1/2}/c \} R_2, \tag{7.22}$$

$$[2\pi D/V''(0)]^{1/2}/c \ll 1,$$

we obtain

$$\mathcal{N}_{\text{ACF}} = \frac{1}{2} \epsilon^2 \lambda_1^2 c^4 \cos(\Omega\tau) / [D^2(\lambda_1^2 + \Omega^2)] + c^2 \exp(-\lambda_1\tau) \{ 1 - \frac{1}{2} \epsilon^2 c^2 \lambda_1^2 / [D^2(\lambda_1^2 + \Omega^2)] \} \tag{7.23}$$

and

$$\mathcal{N}_{\text{SNR}} \simeq (\pi/4)\epsilon^2 c^2 \lambda_1 / D^2, \tag{7.24}$$

which are exactly Eqs. (3.12) and (5.9) in Ref. 15 if one specifies λ_1 for the model Eq. (5.1) of this reference. (In Ref. 15, α_1 , α_0 , and η are identified with R_2 , λ_1 , and ϵ of the present paper, respectively.) Consequently, the results of the adiabatic approximation are completely recovered by taking both limits (7.8) and (7.22). An interesting point is that one can never recover the adiabatic approximation if the vector $\langle 1|$ is replaced by the leading term $\bar{u}_1^{(0)}(x)$, i.e., if the element $R_{1,0}$ is replaced by R_1 only.

From the above calculations several conclusions can be drawn.

(1) If we fix D and change the frequency of the external force Ω , the maximum amplitude of the output is at $\Omega = 0$. One finds, obviously, a zero-frequency “resonance.” The Kramers rate plays the role of fraction rather than the internal frequency. This point has been clearly manifested in Eqs. (7.3), (7.4), (7.6), and (7.7).

(2) If we keep $\Omega \ll 1$ and increase the strength of the

noise, we find a peak of the ACF of the output at $\lambda_1 \simeq \Omega$, i.e., at the “resonance” between the external frequency and the Kramers rate, which plays the role of the “internal frequency.” However, the resonance comes from the prefactor

$$\left\langle \left\langle 1 \left| \frac{\partial}{\partial x} \right| 0 \right\rangle \right\rangle^2 / (\lambda_1^2 + \Omega^2),$$

where the numerator is crucial for the enhancement of the ACF [see Eqs. (7.18)–(7.20)]. Thus, as emphasized in the early works, the mechanism of the SR is essentially different from the usual resonance.

(3) Increasing D further, one may find an enhancement of \mathcal{N}_{SNR} at $D \simeq \Delta V$. This enhancement has been analyzed theoretically in Refs. 15 and 16 and observed experimentally in Refs. 19 and 20. This evidence led Fox to suggest the use of the term SNRE instead of the original term SR (Ref. 16). Here we stress that as $\Omega \ll 1$ both the SR and the SNRE exist. They arise in different D regions, and they are actually different concepts. The SR represents the enhancement of \mathcal{N}_{ACF} , and appears as the hopping rate is approximately equal to the external frequency.¹⁸ The SNRE represents the enhancement of the signal-to-noise ratio, and arises as the diffusion strength D is approximately equal to the potential barrier ΔV . However, it is obvious that the conditions of Eqs. (7.8) are not satisfied now, and the SNRE cannot be guaranteed, because it arises at $D \simeq \Delta V$. One should thus, at least, explain why the rest of the terms of (7.1)–(7.3) are still negligible for large D .

(4) From Eqs. (7.13) and (7.23), we can verify that the signal may take the “energy” of the noise. There is an energy shift and a kind of energy balance between the noise and the signal as stated in Ref. 15.

(5) To date, the most successful theory in dealing with the SR and the SNRE is the adiabatic theory (Refs. 12 and 15). However, it is still not quite clear which terms have been neglected by the adiabatic approximation, and how one can systematically improve the approximation. Equations (7.1)–(7.3) provide an exact result of the linear-response theory for the problem. They show that the adiabatic approximation simply amounts to replacing (7.1)–(7.3) by (7.15) wherein $\langle 1 | (\partial/\partial x) | 0 \rangle$ is given by R_2 only. In particular, it means neglecting the terms in (7.1)–(7.3) apart from those in (7.10) and (7.11) as required by the conditions (7.8). Second, it means neglecting R_1 in the element $\langle 1 | (\partial/\partial x) | 0 \rangle$ as required by the condition (7.22). The former limit has been more or less known while our results present, for the first time, an explanation of the latter condition. In fact, there are some

conflicts between these different approximations. For instance, in Ref. 15, R_1 has been neglected by the adiabatic approximation while in Ref. 18, R_2 has been neglected by replacing $\langle 1 |$ by $\bar{u}_1^{(0)}(x)$ [see (3.12) in Ref. 18]. This is the reason why the result of the adiabatic approximation cannot be recovered by Ref. 18 under the condition available for the adiabatic approximation. One may expect that the results of Refs. 15 and 18 can be valid in distinct parameter regimes indicated by the inequality (7.22).

VIII. REMARKS AND COMPARISONS

In this paper a systematic way to solve the periodically forced one-dimensional FPE was presented based on the separation of the entire dynamics into various independent subspaces (or quasidegreespaces in the Floquet theory). In each subspace the evolution of the system has been given in terms of an iterative relation by the perturbation approach. In the lowest order a rather compact linear-resonance result (7.1)–(7.3) is provided. More accurate solutions can be obtained by performing the calculation to higher orders. This iteration procedure is easier than the one attending the solution of the two-dimensional FPE.^{17,21}

The systematic analysis allows one to clarify certain ambiguities prevailing in the literature. First, we have succeeded in showing that (1.1) is mixing. A numerical verification of the mixing property of (1.1) is reported in a forthcoming paper²² in which we justify that the system asymptotically approaches the same nonstationary solution at large t when the initial preparations are different. This conclusion remains true for both x -independent and x -dependent $h(x)$.

Under the conditions (7.8) and (7.22) we recover the result of the adiabatic approximation.¹⁵ However, an essentially new point is that the early concept of the SR referring to the enhancement of \mathcal{N}_{ACF} and the recent concept of the SNRE referring to the enhancement of the ratio of the signal power spectrum to that of noise represent different aspects of the system and arise in distinct regions, and then they should thus not be confused with each other. This remark complements the comment by Fox.¹⁶

The exact linear-response solution (7.1)–(7.3) allows one to understand the status of the adiabatic approximation. Comparing (7.1)–(7.3) with (7.13), (7.15), (7.23), and (7.24) one can clearly see which terms have been neglected by the adiabatic approximation, and how to systematically improve the approximation by taking back various relevant missing terms. For instance, we expect that the following probability distribution:

$$P_\infty(x, t) = |0\rangle + \epsilon \left\langle 1 \left| \frac{\partial}{\partial x} \right| 0 \right\rangle \cos(\Omega t + \theta + \alpha_{1,1}) / (\lambda_1^2 + \Omega^2)^{1/2} |1\rangle + \epsilon \left\langle 3 \left| \frac{\partial}{\partial x} \right| 0 \right\rangle \cos(\Omega t + \theta + \alpha_{3,1}) / (\lambda_3^2 + \Omega^2)^{1/2} |3\rangle, \quad (8.1)$$

with $\langle 1 | (\partial/\partial x) | 0 \rangle$ being given by the full terms $R_1 + R_2$ and λ_3 and $\langle 3 | (\partial/\partial x) | 0 \rangle$ being given in the leading order of D , may give much better results which may be valid from $D=0$ to relatively large D . Such higher-order contributions involving λ_3 and $\langle 3 | (\partial/\partial x) | 0 \rangle$ will be analyzed in future work.

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- ¹P. Glansdorff and I. Prigogine, *Thermodynamics of Structure, Stability, and Fluctuation* (Wiley-Interscience, New York, 1971).
- ²G. Nicolis and I. Prigogine, *Self-organization of Nonequilibrium Systems* (Wiley-Interscience, New York, 1977).
- ³H. Risken, *The Fokker-Planck Equation* (Springer, New York, 1983).
- ⁴N. G. Van Kampen, *Adv. Chem. Phys.* **34**, 245 (1976).
- ⁵M. Suzuki, *Adv. Chem. Phys.* **46**, 415 (1981).
- ⁶R. Graham and T. Tel, *Phys. Rev. A* **31**, 1109 (1985).
- ⁷R. Graham and T. Tel, *Phys. Rev. A* **33**, 1322 (1986).
- ⁸G. Hu, *Phys. Lett. A* **116**, 361 (1986).
- ⁹G. Hu, *Phys. Rev. A* **38**, 3693 (1988).
- ¹⁰R. Benzi, G. Parisi, A. Suter, and A. Vulpiani, *J. Phys. A* **14**, 453 (1981).
- ¹¹C. Nicolis and G. Nicolis, *Tellus* **33**, 225 (1981).
- ¹²C. Nicolis, *Tellus* **34**, 1 (1982).
- ¹³R. Benzi, G. Parisi, A. Suter, and Vulpiani, *Tellus* **34**, 10 (1982).
- ¹⁴B. Caroli, C. Caroli, B. Roulet, and D. Saint-James, *Physica A* **108**, 233 (1981).
- ¹⁵B. McNamara and K. Wiesenfeld, *Phys. Rev. A* **39**, 4854 (1989).
- ¹⁶R. Fox, *Phys. Rev. A* **39**, 4148 (1989).
- ¹⁷P. Jung and P. Hanggi, *Europhys. Lett.* **8**, 505 (1989).
- ¹⁸C. Presilla, F. Marchesoni, and L. Gammaitoni, *Phys. Rev. A* **40**, 2105 (1989).
- ¹⁹S. Fauve and F. Heslot, *Phys. Lett.* **97A**, 5 (1983).
- ²⁰B. McNamara, K. Wiesenfeld, and R. Roy, *Phys. Rev. Lett.* **60**, 2626 (1988).
- ²¹P. Jung, *Z. Phys. B* (to be published).
- ²²C. Nicolis, G. Nicolis, and G. Hu, *Phys. Lett. A* (to be published).